UNIVERSITY OF REGINA
DEPARTMENT OF MATHEMATICS & STATISTICS
Mathematics 217-040
Midterm Examination
Spring 2013

Time: 75 Min.
Instructor: Bahman Ahmadi

NAME: ______________________
STUDENT NO.: ________________

This exam consists of 4 problems. Show all your work and explain your answers. The examination will be graded out of a total of 30 points.

(5 marks) 1. Solve the given differential equation by the method of undetermined coefficients.

$$y'' - y = \sinh x$$

First we solve $y'' - y = 0$.

Aux. Eq. $m^2 - 1 = 0 \to m_1 = 1,\ m_2 = -1 \to y_c = c_1 e^{x} + c_2 e^{-x}$

Now we find y_p. Since $g(x) = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$,

we choose $y_p = Ae^{x} + Be^{-x}$; but since e^{x} is in the fundamental set, we choose $y_p = Axe^{x} + Be^{-x}$.

Also since e^{-x} shows up in the fundamental set, we should choose $y_p = Axe^{x} + Bxe^{-x}$

Now we calculate

$$y_p' = Ae^{x} + Axe^{x} + Be^{-x} - Bxe^{-x}$$

$$\to y_p'' = Ae^{x} + Ae^{x} + Axe^{x} - Be^{-x} - Be^{-x} + Bxe^{-x}$$

$$= 2Ae^{x} + Axe^{x} - 2Be^{-x} + Bxe^{-x}$$

We must have

$$y_p'' - y_p = \sinh x = \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$$

From this, we get $A = 1/4,\ B = -1/4 \to \boxed{y_p = \frac{1}{4}xe^{x} + \frac{1}{4}xe^{-x}}$

(10 marks) 2. Solve the given differential equation by the method of variation of parameters.

$$y''' + y' = \tan x$$

First we solve $y''' + y' = 0 \to$ Aux. Eq.: $m^3 + m = 0 \to m(m^2+1) = 0$

$\to m_1 = 0,\ m_2 = i,\ m_3 = -i \to$

$$y_c = \underset{\substack{\uparrow \\ y_1}}{C_1 1} + \underset{\substack{\uparrow \\ y_2}}{C_2 \cos x} + \underset{\substack{\uparrow \\ y_3}}{C_3 \sin x}$$

Now we find y_p:

$$W = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x = 1$$

$$W_1 = \begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \tan x & -\cos x & -\sin x \end{vmatrix} = \tan x\left(\cos^2 x + \sin^2 x\right) = \tan x$$

$$W_2 = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \tan x & -\sin x \end{vmatrix} = -\cos x \tan x = -\sin x$$

$$W_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \tan x \end{vmatrix} = -\sin x \tan x = -\frac{\sin^2 x}{\cos x} = \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x$$

From these we get

$$u_1' = \frac{W_1}{W} = \tan x \to u_1 = -\ln|\cos x|$$

$$u_2' = \frac{W_2}{W} = -\sin x \to u_2 = +\cos x$$

$$u_3' = \frac{W_3}{W} = \cos x - \sec x \to u_3 = \sin x - \ln|\sec x + \tan x|$$

$$\to y_p = u_1 y_1 + u_2 y_2 + u_3 y_3 = \boxed{-\ln|\cos x| + \cos^2 x + \sin^2 x - \sin x \ln|\sec x + \tan x|}$$

(5 marks) 3. Solve the given differential equation by using the substitution $u = y'$.

$$(y+1)y'' = (y')^2$$

Let $u = y' \rightarrow y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy}$

$\rightarrow (y+1)u\frac{du}{dy} = u^2 \rightarrow (y+1)\frac{du}{dy} = u$

$\rightarrow \frac{du}{u} = \frac{dy}{y+1} \rightarrow \ln|u| = \ln|y+1| + c_1'$

$\rightarrow |u| = |y+1|e^{c_1'} \rightarrow u = \pm e^{c_1'}|y+1|$

Choose $\pm e^{c_1'} = c_1 \rightarrow u = c_1(y+1)$

Thus, we have

$$\frac{dy}{dx} = c_1(y+1) \rightarrow \frac{dy}{y+1} = c_1 dx$$

$\rightarrow \ln|y+1| = c_1x + c_2'$

$\rightarrow |y+1| = e^{c_1x} \cdot e^{c_2'}$

$\rightarrow y+1 = \pm e^{c_2'} e^{c_1x}$

either one is acceptable

Choose $\pm e^{c_2'} = c_2 \rightarrow y+1 = c_2e^{c_1x}$

$\rightarrow \boxed{y = c_2e^{c_1x} - 1}$

(10 marks) 4. Solve the following IVP using the Laplace transform.

$$y'' - 4y' + 4y = t^3 e^{2t}, \qquad y(0) = 0, \quad y'(0) = 1$$

$$\mathcal{L}(y'') - 4\mathcal{L}(y') + 4\mathcal{L}(y) = \mathcal{L}(t^3 e^{2t})$$

$$\rightarrow \left[s^2 Y(s) - s y(0) - y'(0)\right] - 4\left[s Y(s) - y(0)\right] + 4Y(s) = \frac{3!}{(s-2)^4}$$

$$\rightarrow s^2 Y(s) - 1 - 4s Y(s) + 4Y(s) = \frac{6}{(s-2)^4}$$

$$\rightarrow (s^2 - 4s - 4) Y(s) = \frac{6}{(s-2)^4} + 1$$

$$\rightarrow (s-2)^2 Y(s) = \frac{6}{(s-2)^4} + 1$$

$$\rightarrow Y(s) = \frac{6}{(s-2)^6} + \frac{1}{(s-2)^2}$$

$$\rightarrow y(t) = \mathcal{L}^{-1}\left(\frac{6}{(s-2)^6}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s-2)^2}\right)$$

$$= \frac{6}{5!}\mathcal{L}^{-1}\left(\frac{5!}{(s-2)^6}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s-2)^2}\right)$$

$$= \frac{6}{5!} e^{2t} t^5 + e^{2t} t$$

$$\rightarrow \boxed{y(t) = \frac{e^{2t}}{20}(t^5 + t)}$$