Math 217

Assignment 2

Solution Manual

Problem 1.(a) $\mu(x) = e^{\int p(x)dx} = e^{\int dx} = e^{x}$

$\rightarrow e^{x}(y'+y) = e^{x}.e^{3x} \rightarrow e^{x}y' + e^{x}y = e^{4x}$

$\rightarrow \frac{d}{dx}[e^{x}y] = e^{4x} \rightarrow \int \frac{d}{dx}[e^{x}y]dx = \int e^{4x}dx$

$\rightarrow e^{x}y = \frac{1}{4}e^{4x} + c \rightarrow \boxed{y = \frac{1}{4}e^{3x} + ce^{-x}}$

(b) $y' + \frac{x+2}{x}y = \frac{e^{x}}{x^2}$

$\rightarrow \mu(x) = e^{\int \frac{x+2}{x}dx} = e^{x+\ln x^2} = e^{x}.x^{2}$

$\rightarrow x^{2}e^{x}[y' + \frac{x+2}{x}y] = x^{2}e^{x}[\frac{e^{x}}{x^{2}}]$

$\rightarrow x^{2}e^{x}y' + (x^{2}+2x)ye^{x} = e^{2x} \rightarrow \frac{d}{dx}[x^{2}e^{x}y] = e^{2x}$

$\rightarrow x^{2}e^{x}y = \int e^{2x}dx \rightarrow x^{2}e^{x}y = \frac{1}{2}e^{2x} + c$

$\rightarrow \boxed{y = \frac{e^{x}}{2x^{2}} + \frac{ce^{-x}}{x^{2}}}$

(c) $\mu(\theta) = e^{\int p(\theta)d\theta} = e^{\int \sec\theta \, d\theta}$

$= e^{\ln(\sec\theta + \tan\theta)} = \sec\theta + \tan\theta$

(assuming $\sec\theta + \tan\theta > 0$)

$\rightarrow (\sec\theta + \tan\theta)\left[\frac{dr}{d\theta} + r\sec\theta\right] = (\sec\theta + \tan\theta)\cdot\cos\theta$

$\rightarrow \frac{d}{d\theta}\left[(\sec\theta + \tan\theta)\cdot r\right] = 1 + \sin\theta$

$\rightarrow r.(\sec\theta + \tan\theta) = \int(1 + \sin\theta)\,d\theta$

$\rightarrow r.(\sec\theta + \tan\theta) = \theta - \cos\theta + c \rightarrow \boxed{r = \frac{\theta - \cos\theta + c}{\sec\theta + \tan\theta}}$

Problem 2. (b) $y' + \frac{1}{x}y = \frac{e^x}{x}$

$\rightarrow \mu(x) = e^{\int p(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$\rightarrow x\left(y' + \frac{1}{x}y\right) = x.\left(\frac{e^x}{x}\right) \rightarrow \frac{d}{dx}(xy) = e^x$

$\rightarrow xy = \int e^x dx \rightarrow xy = e^x + c$

$y(1) = 2 \rightarrow 2 = e + c \rightarrow c = 2 - e \rightarrow \boxed{y = \frac{e^x + 2 - e}{x}}$

Problem 3. (d)

$$\underbrace{(e^x+y)}_{M}dx+(\underbrace{2+x+ye^y}_{N})dy=0 \qquad (1)$$

First we check if (1) is exact:

$$\begin{cases} \dfrac{\partial M}{\partial y}=1 \\ \dfrac{\partial N}{\partial x}=1 \end{cases} \longrightarrow \quad (1) \text{ is exact.}$$

Then we solve it using the method for exact ODEs:

There is a function $f(x,y)$ such that

$$\frac{\partial f(x,y)}{\partial x}=M=e^x+y$$

Hence

$$f(x,y)=\int(e^x+y)\,dx=e^x+xy+g(y)$$

We also have

$$\frac{\partial f(x,y)}{\partial y}=N=2+x+ye^y$$

Thus

$$x+g'(y)=2+x+ye^y$$

$$\rightarrow g'(y)=2+ye^y$$

$\rightarrow \quad g(y) = \int (2 + y e^y) dy = 2y + ye^y - e^y$

Therefore

$$f(x,y) = e^x + xy + 2y + ye^y - e^y$$

This implies that

$$e^x + xy + 2y + ye^y - e^y = c$$

is the general solution for (1).

Since $y(0) = 1$, we have

$$e^0 + 2 + e - e = c \rightarrow c = 3$$

Hence

$$\boxed{e^x + xy + 2y + ye^y - e^y = 3}$$

Problem 4. (a) $\quad y_1 = \cosh 2x, \; y_2 = \sinh 2x$

$y_1' = 2\sinh 2x \rightarrow y_1'' = 4\cosh 2x$

$\rightarrow y'' - 4y = 4\cosh 2x - 4\cosh 2x = 0$ ✓

y_2 is similar. Therefore we need only to check if $\{y_1, y_2\}$ is linearly independent:

$$W(y_1, y_2) = \begin{vmatrix} \cosh 2x & \sinh 2x \\ 2\sinh 2x & 2\cosh 2x \end{vmatrix} = 2(\cosh^2 2x - \sinh^2 2x) = 2 \neq 0$$

Hence $\{y_1, y_2\}$ is a fundamental set of sulotions.

Problem 5. (b) It is easy to check $y_1 = \ln x$ is a solution for the ODE. We have

$$y'' + \frac{1}{x} y' = 0$$

Hence

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}}{y_1^2} dx$$

$$= \ln x \int \frac{e^{-\int \frac{1}{x} dx}}{(\ln x)^2} dx = \ln x \int \frac{1/x}{(\ln x)^2} dx$$

$$= \ln x \cdot \frac{1}{\ln x} = 1$$

(c) It is easy to verify y_1 is a solution. We have

$$y'' + \frac{2+2x}{1-2x-x^2} y' - \frac{2}{1-2x-x^2} y = 0$$

Hence

$$y_2 = (x+1)\int \frac{e^{-\int \frac{2+2x}{1-2x-x^2}dx}}{(x+1)^2}\,dx$$

$$= (x+1)\int \frac{e^{\ln(x^2+2x-1)}}{(x+1)^2}\,dx = (x+1)\int \frac{x^2+2x-1}{(x+1)^2}\,dx$$

$$= (x+1)\int \frac{x^2+2x+1-2}{(x+1)^2}\,dx = (x+1)\int\left(1 - \frac{2}{(x+1)^2}\right)dx$$

$$= (x+1)\left[x + \frac{2}{x+1}\right] = x^2 + x + 2$$

Problem 6.(b) Aux. Eq.: $m^2 - 10m + 25 = 0$

$\rightarrow (m-5)^2 = 0 \rightarrow m_1 = m_2 = 5$

Thus $y_c = c_1 e^{5x} + c_2 x e^{5x}$

(d) Aux. Eq.: $m^3 + 3m^2 - 4m - 12 = 0$

We can guess that 2 is a root of Aux. Eq.

By long division, then, we see that

$m^3 + 3m^2 - 4m - 12 = (m-2)(m+2)(m+3) \rightarrow m_1 = 2, m_2 = -2, m_3 = -3$

Thus $$\boxed{y_c = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{-3x}}$$

(e) Aux. Eq.: $m^4 - 1 = 0$

$$\rightarrow (m^2-1)(m^2+1) = 0$$

$$\rightarrow \begin{cases} m_1 = 1 \\ m_2 = -1 \\ m_3 = i \\ m_4 = -i \end{cases} \rightarrow \begin{cases} \alpha = 0 \\ \beta = 1 \end{cases}$$

Hence

$$y_c = c_1 e^{x} + c_2 e^{-x} + e^{0}\left[c_3 \cos x + c_4 \sin x\right]$$

$$\rightarrow \boxed{y_c = c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x}$$

Problem 7. (b) First we solve

$$y'' + 3y = 0$$

$\rightarrow$ Aux. Eq.: $m^2 + 3 = 0 \rightarrow \begin{cases} m_1 = \sqrt{3}\, i \\ m_2 = -\sqrt{3}\, i \end{cases} \rightarrow \begin{cases} \alpha = 0 \\ \beta = \sqrt{3} \end{cases}$

Hence $y_c = c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)$.

Now we start with $y_p = (Ax^2+Bx+c)e^{3x}$

(Since this is not in the fundamental set, we can proceed)

$\rightarrow y_p' = (2Ax+B)e^{3x} + 3(Ax^2+Bx+c)e^{3x}$

$\rightarrow y_p'' = 2Ae^{3x} + 3(2Ax+B)e^{3x} + 3(2Ax+B)e^{3x} + 9(Ax^2+Bx+c)e^{3x}$

Plugging these in the ODE, we get:

$$\left[2Ae^{3x} + (6Ax+3B)e^{3x} + (6Ax+3B)e^{3x} + 9(Ax^2+Bx+c)e^{3x}\right]$$

$$+ 3\left[(Ax^2+Bx+c)e^{3x}\right] = 48x^2e^{3x}$$

$\xrightarrow[\text{from both sides}]{\text{Cross } e^{3x}}$

$$2A + 6Ax + 3B + 6Ax + 3B + 9Ax^2 + 9Bx + 9C$$

$$+ 3Ax^2 + 3Bx + 3C = 48x^2$$

$$\rightarrow (12A)x^2 + (6A+12B)x + (2A+6B+12C) = 48x^2$$

$$\rightarrow \begin{cases} 12A = 48 \rightarrow \boxed{A=4} \\ 6A+12B=0 \rightarrow 12B = -24 \rightarrow \boxed{B=-2} \\ 2A+6B+12C=0 \rightarrow 12C = -8+12 = 4 \rightarrow \boxed{C=1/3} \end{cases}$$

$$\rightarrow \boxed{y_p = (4x^2-2x+1/3)e^{3x}}$$

(C) First we solve

$$y'' - 2y' + y = 0$$

$\rightarrow$ Aux. Eq.: $m^2 - 2m + 1 = 0 \rightarrow (m-1)^2 = 0 \rightarrow m_1 = m_2 = 1$

Thus

$$y_c = c_1 e^x + c_2 x e^x$$

Now since $2\cosh x = e^x + e^{-x}$, we start with $y_p = Ae^x + Be^{-x}$. But since e^x is in the fundamental set, we replace Ae^x by Axe^x. Again since xe^x is in the fundamental set, we replace Axe^x by Ax^2e^x. Thus $y_p = Ax^2e^x + Be^{-x}$.

Then we have

$$y_p' = 2Axe^x + Ax^2e^x - Be^{-x}$$

$$\rightarrow y_p'' = 2Ae^x + 2Axe^x + 2Axe^x + Ax^2e^x + Be^{-x}$$

$$= 2Ae^x + 4Axe^x + Ax^2e^x + Be^{-x}$$

Plugging these in the ODE, we get

$$\left[2Ae^{x} + 4Axe^{x} + Ax^{2}e^{x} + Be^{-x}\right]$$

$$-2\left[2Axe^{x} + Ax^{2}e^{x} - Be^{-x}\right] + \left[Ax^{2}e^{x} + Be^{-x}\right] = e^{x} + e^{-x}$$

$$\rightarrow (2A)e^{x} + (4B)e^{-x} = e^{x} + e^{-x} \rightarrow A = 1/2,\ B = 1/4$$

$$\rightarrow \boxed{y_p = \frac{x^{2}}{2}e^{x} + \frac{1}{4}e^{-x}}$$