Intersecting Sets of Uniform Partitions

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Partitions

Definition

A (k,ℓ)*-partition* is a set partition of $\{1,2,\ldots,k\ell\}$ with exactly ℓ blocks each of size k.

$$P = \{P_1, P_2, \ldots, P_\ell\}$$

These are also called *uniform set partitions*.
The set of all uniform-(k,ℓ) partitions is denoted by $\mathcal{U}(k,\ell)$.

Two set partitions

$$P = \{P_1, P_2, \ldots, P_\ell\} \quad \text{and} \quad Q = \{Q_1, Q_2, \ldots, Q_\ell\}$$

1. are **intersecting** if $P_i = Q_j$ for some i and j.
2. are **t-intersecting** if $P_{i_1} = Q_{j_1},\ P_{i_2} = Q_{j_2},\ \ldots,\ P_{i_t} = Q_{j_t}$ for distinct $i_1,\ldots,i_t$ and distinct $j_1,\ldots,j_t$.
3. are **partially-t intersecting** if $|P_i \cap Q_j| \geq t$ for some i and j.

Intersecting sets

Example

$$P1: \quad 1\ 2\ 3 \mid 4\ 5\ 6 \mid 7\ 8\ 9 \mid 10\ 11\ 12$$

$$P2: \quad 1\ 2\ 3 \mid 4\ 5\ 6 \mid 7\ 8\ 10 \mid 9\ 11\ 12$$

$$P3: \quad 1\ 4\ 7 \mid 2\ 5\ 10 \mid 3\ 8\ 11 \mid 6\ 9\ 12$$

Definition

A **set** of partitions is (intersecting / t-intersecting / t-partially intersecting) if the partitions are pairwise (intersecting / t-intersecting / t-partially intersecting).

What is a maximum set of (intersecting / t-intersecting / t-partially intersecting) partitions?

Previous Results for t-intersection

The total number of (k, ℓ)-partitions is

$$U(k,\ell) = \frac{1}{\ell!}\binom{k\ell}{k}\binom{k\ell-k}{k}\cdots\binom{k}{k}.$$

Definition

A **canonical** t-intersecting set is all partitions containing a fixed set of t disjoint parts.

Example (1-intersecting $(3,3)$-partitions)

$1\,2\,3 \mid 4\,5\,6 \mid 7\,8\,9$	$1\,2\,3 \mid 4\,5\,7 \mid 6\,8\,9$	$1\,2\,3 \mid 4\,5\,8 \mid 6\,8\,9$
$1\,2\,3 \mid 4\,5\,9 \mid 6\,7\,8$	$1\,2\,3 \mid 4\,6\,7 \mid 5\,8\,9$	$1\,2\,3 \mid 4\,6\,8 \mid 5\,7\,9$
$1\,2\,3 \mid 4\,6\,9 \mid 5\,7\,8$	$1\,2\,3 \mid 4\,7\,8 \mid 5\,6\,9$	$1\,2\,3 \mid 4\,7\,9 \mid 5\,6\,8$
$1\,2\,3 \mid 4\,8\,9 \mid 5\,6\,7$		

The size of a canonical t-intersecting (k, ℓ)-partition is

$$U(k,\ell-t) = \frac{1}{(\ell-t)!}\binom{k\ell-kt}{k}\binom{k\ell-k(t+1)}{k}\cdots\binom{k}{k}.$$

Previous Results for t-intersection

*I*f $k = 2$, the partitions are **perfect matchings.**

Theorem

The largest t-intersecting set of perfect matchings are the canonical sets in the following cases:

- for $t = 1$ (Godsil and Meagher, 2017)
- for $t = 2$ (Fallat, Meagher and Shirazi, 2021; Lindzey, 2023).
- for all t and n sufficiently large (Lindzey, 2017)

Proofs are algebraic, using eigenvalues of a related graph.

Proposition

If n is small relative to t there are examples of t-intersecting sets larger than the canonical sets.

Conjecture

If $k \geq 3t/2 + 1$ then the largest set of t-intersecting perfect matchings are the canonical.

Trivial Cases for Partially Intersection

1. Any two (k, ℓ)-uniform partially 1-intersecting.

$$1\ 2\ 3 \mid 4\ 5\ 6 \mid 7\ 8\ 9 \mid 10\ 11\ 12 \qquad 1\ 4\ 7 \mid 3\ 5\ 10 \mid 2\ 8\ 11 \mid 6\ 9\ 12$$

2. If $t = k$, then partially-t intersecting and intersecting are the same.
3. If $k > \ell(t-1)$, then any two (k, ℓ)-partitions are partially t-intersecting.

$$1\ 2\ 3\ 4\ 5\ 6 \mid 7\ 8\ 9\ \ 10\ 11\ 12 \qquad 1\ 2\ 3\ 7\ 8\ 9 \mid 4\ 5\ 6\ 10\ 11\ 12$$

if the parts are really big, any two partitions will be partially t-intersecting.

Focus on (k, ℓ)-uniform **partially 2-intersecting** sets with $k > 2$.

$P = \{P_1, P_2, \ldots, P_\ell\}$ and $Q = \{Q_1, Q_2, \ldots, Q_\ell\}$ have some i, j with $|P_i \cap Q_j| \geq 2$.

Canonical Intersecting Sets of Partitions

Definition

For $t \leq k$, fix a t-subset $T \subset \{1, \ldots, k\ell\}$. The set of all partitions that have a part containing T is a **canonical t-intersecting set of partitions**.

Example

Canonical partially 2-intersecting set of $(3,3)$-partitions:

$123\|456\|789$	$123\|457\|689$	$123\|458\|679$	$123\|459\|678$
$123\|467\|589$	$123\|468\|579$	$123\|469\|578$	$123\|567\|489$
$\ldots$			
$129\|356\|478$	$129\|367\|458$	$129\|368\|457$	$129\|378\|456$

The size of a canonical partially t-intersecting set of partitions is

$$\frac{1}{(\ell-1)!}\binom{k\ell-t}{k-t}\binom{k\ell-k}{k}\cdots\binom{k}{k}.$$

Canonical Intersecting Sets of Partitions

Conjecture

If $k \leq \ell(t-1)$, then the largest set of intersecting (k,ℓ)-partitions is a canonical set of t-intersecting partitions.

Derangement Graph

Definition

For any k, ℓ define the **partition derangement graph**, $\Gamma_{G_{(k,\ell)}}$.

- The vertices are the uniform (k, ℓ)-partitions.
- Vertices $P, Q \in \mathcal{U}(k, \ell)$ are adjacent if and only if P and Q are **not** partially t-intersecting.

A set of uniform partitions is a coclique (independent set) in $\Gamma_{G_{(k,\ell)}}$, exactly if the set is a partially t-intersecting set of partitions.

Graph Properties:

1. The graph $\Gamma_{G_{(k,\ell)}}$ is regular, denote the degree by $d_{k,\ell}$
2. The derangement graph is vertex-transitive, the group $\mathrm{Sym}(k\ell)$ acts transitively on the vertices of $\Gamma_{G_{(k,\ell)}}$.
3. A resolvable balanced incomplete block t-design on $k\ell$ points with blocksize k and index $\lambda = 1$, if it exists, is a maximum clique.

Resolvable Designs

Definition

Suppose $\mathcal{B}$ is a t-(n, k, λ) design.

- A *parallel class* in $\mathcal{B}$ is a collection of disjoint sets whose union is the n-set.
- A partition of $\mathcal{B}$ into $r = \lambda \frac{\binom{n-1}{t-1}}{\binom{k-1}{t-1}}$ parallel classes is called a *resolution*.
- A t-(n, k, λ) design is *resolvable* if a resolution exists.

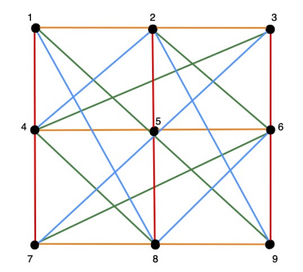
Example (Resolvable $2-(9, 3, 1)$ Design)

123 | 456 | 789 (orange)

147 | 258 | 369 (red)

159 | 267 | 348 (green)

168 | 249 | 357 (blue)



Clique/Coclique Bound

Theorem

If X is a vertex-transitive graph and $\alpha(X)$ is the size of the maximum coclique and $\omega(X)$ the size of the maximum clique, then

$$\alpha(X)\,\omega(X) = |V(X)|.$$

Theorem

If there is a resolvable balanced incomplete block t-design on $k\ell$ points with blocksize k, then a canonical partially t-intersecting partitions is a largest intersecting set.

Proof. By the clique/clique bound, a coclique is no larger than

$$\frac{\text{number vertices}}{\text{clique size}} = \frac{U(k,\ell)}{\frac{1}{\ell}\frac{\binom{k\ell}{t}}{\binom{k}{t}}} = \frac{1}{(\ell-1)!}\binom{k\ell-t}{k-t}\binom{k\ell-k}{k}\binom{k\ell-2k}{k}\cdots\binom{k}{k}$$

$$= \text{the size of a canonical partially } t\text{-intersecting set}$$

□

Ratio Bound

Theorem (Delsarte-Hoffman bound)

Let A be the adjacency matrix for a d-regular graph X on vertex set $V(X)$. If the least eigenvalue of A is τ, then

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}.$$

If equality holds for some coclique S with characteristic vector ν_S, then

$$\nu_S - \frac{|S|}{|V(X)|}\mathbf{1}$$

is an eigenvector with eigenvalue τ.

Master Plan:

1. find 3 specific eigenvalues of the graph,
2. show all other eigenvalues are smaller, (in absolute value)
3. apply the ratio bound.

Eigenspaces of the Derangement graphs

1. The eigenspaces of $X_{k,\ell}$ are invariant under the action of $\mathrm{Sym}(k\ell)$ and thus a union of irreducible modules in the decomposition of

$$\mathrm{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}.$$

   Every irreducible representation of $\mathrm{Sym}(n)$ is labelled by an integer partition of n.

2. For each irreducible representation in $\mathrm{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$ has a corresponding eigenvalue:

$$\eta_\phi(A_\ell) = \frac{d_\ell}{|H|} \sum_{x_\ell} \sum_{h\in H} \phi(x_\ell h),$$

   This includes a sum of characters over group cosets which is hard.

3. The irreducible representations $[k\ell]$, $[k\ell-2,2]$ and $[k\ell-3,3]$ are included in $\mathrm{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$

4. The irreducible representations $[1^{k\ell}]$, $[k\ell-1,1]$, $[2,1^{k\ell-2}]$, $[2,2,1^{k\ell-4}]$, $[k\ell-2,1,1]$, $[3,1^{k\ell-3}]$, $[2,2,2,1^{k\ell-6}]$ are not.
   Proof by orbit counting.

Eigenvalues of the Derangement graphs

Theorem

For any k, ℓ,

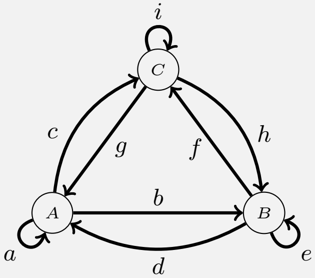
1. *The eigenvalue belonging to $[k\ell]$ is the degree $d = d(k,\ell)$.*
2. *The eigenvalue belonging to $[k\ell - 2, 2]$ is*
$$\tau = -\frac{(k-1)d}{k(\ell-1)}.$$
3. *The eigenvalue belonging to $[k\ell - 3, 3]$ is*
$$\theta = \frac{2(k-1)(k-2)d}{k^2(\ell-1)(\ell-2)}.$$
4. *For any other representation, the eigenvalue is smaller in absolute value τ.*

By the ratio bound, the maximum size of coclique in $X_{k,\ell}$ is

$$\frac{|V(X_{k,\ell})|}{1-\frac{d}{\tau}} = \frac{v}{1-\frac{d}{-\frac{(k-1)d}{k(\ell-1)}}} = \frac{v}{1+\frac{k(\ell-1)}{k-1}} = \frac{v(k-1)}{k\ell-1} = \binom{k\ell-2}{k-2}U(k,\ell-1).$$

Quotient Graphs

1. The action of a subgroup of the automorphism group on the partitions forms orbits.
2. These orbits can be used to build a **quotient graph**.



Partition the vertices in the graph into orbits.

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} \text{a} & \text{b} & \text{c} \\ \text{d} & \text{e} & \text{f} \\ \text{g} & \text{h} & \text{i} \end{pmatrix} \end{array}$$

The **quotient matrix**

Quotient Graphs

Young's subgroup $\mathrm{Sym}(k\ell)$:

- Has one orbit, so the quotient graph is

$$(d)$$

- This means d is the eigenvalue corresponding to $[k\ell]$.

Young's subgroup $\mathrm{Sym}([k\ell-2,2]) = \mathrm{Sym}(k\ell-2) \times \mathrm{Sym}(2)$

- Has two orbits: the partitions with 1 and 2 together in one part and the partitions where they are in two parts.
- The quotient matrix is the 2×2 matrix

$$\begin{pmatrix} 0 & d \\ -\tau & d+\tau \end{pmatrix}$$

- The eigenvalues are d and $\tau = -\frac{d(k-1)}{k(\ell-1)}$.
- This means $-\tau$ is the eigenvalue corresponding to $[k\ell-2,2]$.

Eigenvalues of the Derangement graphs

Young's subgroup $\mathrm{Sym}([k\ell-3,3]) = \mathrm{Sym}(k\ell-3)\times \mathrm{Sym}(3)$

- For $[k\ell-3,3]$, the Young's subgroup is $\mathrm{Sym}(k\ell-3,3)$ has three orbits: partitions where 1,2,3 are in one part, two parts or three parts.
- The quotient matrix is the 3×3 matrix

$$M = \begin{pmatrix} 0 & 0 & d \\ 0 & a & d-a \\ b & c & d-b-c \end{pmatrix}$$

- The eigenvalues are d, $-\tau$, θ.
- $tr(M) = d-b-c+a = d-\tau+\theta$
- Counting edges between the orbits gives equations for a, b, c, then

$$\theta = \frac{2(k-1)(k-2)d}{k^2(\ell-1)(\ell-2)}.$$

Theorem

Assume $k\ell \geq 13$ and $k \geq 3$. Then the only partitions in the decomposition of $\operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$ with dimension less than or equal to $\binom{k\ell}{3} - \binom{k\ell}{2}$ are

$$\chi_{[k\ell]}, \quad \chi_{[k\ell-2,2]}, \quad \chi_{[k\ell-3,3]}.$$

Proof. Use induction and the "branching rule". □

Bound on the Multiplicity

Theorem

The eigenvalues d, τ and θ are the three largest, in absolute value, in the derangement graph. The smallest eigenvalue for $\Gamma_{(k,\ell)}$ is τ.

Proof. By squaring the adjacency matrix and taking the trace, we have

$$vd = d^2 + m_\tau \tau^2 + m_\theta \theta^2 + \sum_{i=2}^{j} m_i \lambda_i^2.$$

Hence for every $2 \leq i \leq j$ we have

$$vd - d^2 - m_\tau \tau^2 - m_\theta \theta^2 \geq m_i \lambda_i^2.$$

This gives an upper bound on $|\lambda_i|$ in terms of d. □

Bound on Degree

Theorem (Bender)

Let $\mathcal{M}_{k,\ell}$ be the number of all $\ell \times \ell$ matrices with entries either 0 or 1, and row and columns sums equal to k. For positive integers k, ℓ

$$\lim_{\ell \to \infty} \frac{(k!)^{2\ell}}{(k\ell)!} |\mathcal{M}_{k,\ell}| = e^{-\frac{(k-1)^2}{2}}.$$

Lemma

For positive integers k, ℓ with $k \leq \ell$, let d be the degree of $\Gamma_{(k,\ell)}$. Then

$$d = \frac{k!^{\ell}}{\ell!} |\mathcal{M}_{k,\ell}|.$$

Further, for a fixed integer k with $k \geq 2$,

$$\lim_{\ell \to \infty} \frac{U(k,\ell)}{d} = e^{\frac{(k-1)^2}{2}}.$$

Bound on Degree

We had

$$\left(\frac{vd - d^2 - m_\tau \left(\frac{d(k-1)}{k(\ell-1)}\right)^2 - m_\theta \left(\frac{2(k-1)(k-2)d}{k^2(\ell-1)(\ell-2)}\right)^2}{m_i} \right)^{\frac{1}{2}} \geq |\lambda_i|$$

1. $\lim_{\ell\to\infty} \frac{U(k,\ell)}{d} = e^{\frac{(k-1)^2}{2}}$ gives an upper bound for λ_i.
2. If another eigenvalue is larger than τ, in absolute value, that eigenvalue has to have a multiplicity smaller than $\binom{k\ell}{3} - \binom{k\ell}{2}$,
3. Only d, τ or θ have multiplicity this small.
4. So τ is the least eigenvalue of $\Gamma_{k,\ell}$.
5. Apply the ratio bound.

Final Summary

Theorem

Fix an integer $k \geq 3$. For ℓ sufficiently large, the largest set of partially 2-intersecting uniform (k, ℓ)-partitions has size $\binom{k\ell-2}{k-2} U_{k,\ell-1}$.

Conjecture

For $k \geq 3$ and ℓ sufficiently large, the only sets of partially 2-intersecting (k, ℓ)-partitions with size $\binom{k\ell-2}{k-2} U_{k,\ell-1}$ are the sets $S_{i,j}$.

With a more tedious calculation on the degree approximation we get the follow:

Theorem

For $k = 3$ and all $\ell \geq 3$ the largest set of partially 2-intersecting uniform partitions has size

$$(3\ell - 2)U_{3,\ell-1}.$$

Future Work

There are two obvious questions:

Question

Can a non-canonical 2-partially intersecting set also have the maximum size?

Question

Can this method be extended t-partially intersecting uniform partitions for large values of t?