Intersection theorems for Uniform Partitions

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Slides available at: https://uregina.ca/∼meagherk/Australia.pdf

The Motivating Problem

What is the largest collection of subsets from $\{1, 2, \ldots, n\}$, so that any two subsets contain a common element?

Restrict to k-subsets, subsets with exactly k elements

Example

This is an example of 3-subsets from $\{1, 2, \ldots, 6\}$ with size 10:

$$\begin{array}{ccccc} \{1,2,3\} & \{1,2,4\} & \{1,2,5\} & \{1,2,6\} & \{1,3,4\} \\ \{1,3,5\} & \{1,3,6\} & \{2,3,4\} & \{2,3,5\} & \{2,3,6\} \end{array}$$

All sets contain at least two elements of $\{1, 2, 3\}$.

This is an example of 3-subsets from $\{1, 2, \ldots, 7\}$ with size 15:

$$\begin{array}{ccccc} \{1,2,3\} & \{1,2,4\} & \{1,2,5\} & \{1,2,6\} & \{1,2,7\} \\ \{1,3,4\} & \{1,3,5\} & \{1,3,6\} & \{1,3,7\} & \{1,4,5\} \\ \{1,4,6\} & \{1,4,7\} & \{1,5,6\} & \{1,5,7\} & \{1,6,7\} \end{array}$$

All sets contain 1.

Erdős-Ko-Rado Theorem

An **intersecting k-set system** is a collection of subsets of $[1..n]$, each of size k, so that any two have at least one element in common.

Theorem (Erdős-Ko-Rado Theorem - 1961)

Let $\mathcal{A}$ be an intersecting k-set system on an n-set. If $n \geq 2k$, then $|\mathcal{A}| \leq \binom{n-1}{k-1}$.

1. A largest collection of intersecting k-sets is one in which all the sets that contain a common point.
2. These collections are called **trivially intersecting** or **canonically** intersecting.

Generalizations Erdős-Ko-Rado Theorem

For any object that has an "intersection" we can ask:

What is the size of largest set of intersecting objects?

Which intersecting set attain the maximum size?

In general:

- Each **object** is made of k *atoms*.
- Two objects **intersect** if they contain a common atom.
- A **canonically intersecting system** is the set of all objects that contain a fixed atom.

Object	Atoms
k-Subsets of $[1..n]$	elements from $\{1, \ldots, n\}$
Integer sequences	pairs (i, a) (entry a is in position i)
Permutations	pairs (i, j) (the permutation maps i to j)
Perfect matchings in a graph	edges in the graph
Set partition	subsets (cells in the partition)

Partitions

Definition

A **uniform** (k,ℓ)***-partition*** is a set partition of $\{1,2,\ldots,k\ell\}$ with exactly ℓ blocks each of size k.

$$P = \{P_1, P_2, \ldots, P_\ell\}$$

These are also called *uniform set partitions*.

The number of uniform (k,ℓ)-partitions is

$$U(k,\ell) = \frac{1}{\ell!}\binom{k\ell}{k}\binom{k(\ell-1)}{k}\cdots\binom{k}{k}$$

Example

A uniform $(3,4)$-partition:

$$1\ 4\ 7 \mid 2\ 5\ 10 \mid 3\ 8\ 10 \mid 6\ 9\ 12$$

Intersecting sets

Two set partitions

$$P = \{P_1, P_2, \ldots, P_\ell\} \quad \text{and} \quad Q = \{Q_1, Q_2, \ldots, Q_\ell\}$$

1. are **intersecting** if $P_i = Q_j$ for some i and j; (contain a common part)
2. are t**-intersecting** if

   $$P_{i_1} = Q_{j_1},\ P_{i_2} = Q_{j_2},\ \ldots\ P_{i_t} = Q_{j_t}$$

   for distinct $i_1, \ldots, i_t$ and distinct $j_1, \ldots, j_t$; (contain t common parts)
   1-intersecting is intersecting.
3. are **partially-**t **intersecting** if $|P_i \cap Q_j| \geq t$ for some i and j. (a pair of parts contain t common elements).

Example

1 2 3 | 4 5 6 | 7 8 9 | 10 11 12   1 2 3 | 4 5 6 | 7 8 10 | 9 11 12   1 4 7 | 2 5 10 | 3 8 11 | 6 9 12

Intersecting Partitions

Definition

A set of partitions is t**-intersecting** if the partitions are **pairwise** t**-intersecting**.

What is a maximum set of t-intersecting partitions?

Definition

A set of partitions is t**-partially intersecting**, if the partitions are **pairwise** t**-partially intersecting**.

What is a maximum set of t-partially intersecting partitions?

Canonical Intersecting Sets of Partitions

Definition

Fix t disjoint k-subsets $T_1, T_2, \ldots, T_t$. The set of all partitions that have $T_1, T_2, \ldots, T_t$ as parts is a **canonical t-intersecting set of partitions**.

Example

A canonical partially 1-intersecting set of $(3,3)$-partitions with $T = \{1,2,3\}$ and size 10:

$123\|456\|789$	$123\|457\|689$	$123\|458\|679$	$123\|459\|678$	$123\|467\|589$
$123\|468\|579$	$123\|469\|578$	$123\|478\|569$	$123\|479\|568$	$123\|489\|567$

The size of a canonical t-intersecting set of (k,ℓ)-partitions is

$$\frac{1}{(\ell-t)!}\binom{k(\ell-t)}{k}\binom{k(\ell-t-1)}{k}\cdots\binom{k}{k} = U(k,\ell-t)$$

Canonical t-partially Intersecting Sets of Partitions

Definition

For $t \leq k$, fix a t-subset $T \subset \{1, \ldots, k\ell\}$.
The set of all partitions that have a part containing T is a **canonical partially t-intersecting set of partitions**.

Example

A canonical partially 2-intersecting set of $(3,3)$-partitions with $T = \{1,2\}$ and size 70:

$123\|456\|789$	$123\|457\|689$	$123\|458\|679$	$123\|459\|678$
$123\|467\|589$	$123\|468\|579$	$123\|469\|578$	$123\|567\|489$
$\ldots$			
$129\|356\|478$	$129\|367\|458$	$129\|368\|457$	$129\|378\|456$

The size of a canonical t-intersecting set of (k,ℓ)-partitions is

$$\binom{k\ell - t}{k - t} \frac{1}{(\ell-1)!} \binom{k(\ell-1)}{k} \cdots \binom{k}{k} = \binom{k\ell - t}{k - t} U(k, \ell - 1)$$

Easy Asymptotic Proof for t-intersecting

Theorem (M. and Moura)

If $n = k\ell$ is sufficiently large, a t-intersecting uniform partition system is no larger than a canonical system.

Use a counting method to find, bound the number of partitions in a non-canonical system. If n is large, this bound is smaller than the size of a canonical set.

Proof.

1. Let $\mathcal{A}$ be a non-canonical intersecting set of partitions.
2. Assume $P = \{P_1, P_2, \ldots, P_\ell\} \in \mathcal{A}$.
3. Let $\mathcal{A}_i$ be all the partitions in $\mathcal{A}$ that contain P_i.
   since the system is intersecting every partitions will be in at least one $\mathcal{A}_i$
4. Bound the size of each $\mathcal{A}_i$,
   Bound uses that fact that it must intersect the partitions in $\mathcal{A}_j$.

□

Easy Asymptotic Proof

Proof.

P	123 | 456 | 789
$\mathcal{A}_1$	123 | * * * | * * *
	123 | * * * | * * *
$\mathcal{A}_2$	456 | * * * | * * *
	456 | * * * | * * *
$\mathcal{A}_3$	789 | * * * | * * *
	789 | * * * | * * *

Since $\mathcal{A}$ is intersecting,

$$|\mathcal{A}_i| \leq \binom{\ell-2}{t} U(k, \ell-(t+1))$$

If (k is large relative to ℓ and t) or ($k \geq t+2$ and ℓ large relative to k and t), then

$$|\mathcal{A}| \leq \ell \binom{\ell-2}{t} U(k, \ell-(t+1)) < \frac{1}{\ell-t} \binom{(\ell-t)k}{k} U(k, \ell-(t+1)) = U(k, \ell-t)$$

Proof is only difficult if $k=2$ (perfect matching) □

Perfect Matchings

If $k = 2$, the uniform $(2, \ell)$-partitions are **perfect matchings** of 2ℓ.

$1\,2\,|\,3\,4\,|\,5\,6,\quad 1\,2\,|\,3\,5\,|\,4\,6,\quad 1\,2\,|\,3\,6\,|\,4\,5,\quad 1\,3\,|\,2\,4\,|\,5\,6,\quad 1\,3\,|\,2\,5\,|\,4\,6,\quad 1\,3\,|\,2\,6\,|\,4\,5$

The number of perfect matchings is

$$(2\ell - 1)!! = (2\ell - 1)(2\ell - 3)(2\ell - 5) \cdots 3 \cdot 1$$

and the number of perfect matchings containing a fixed edge is

$$(2\ell - 3)!! = (2\ell - 3)(2\ell - 5) \cdots 3 \cdot 1.$$

The number of perfect matchings containing a t-fixed edges is $(2\ell - 2t - 1)!!$.

Theorem

The size of the largest intersecting set of perfect matchings is $(2\ell - 3)!!$.

We will see a totally different proof method for this.

Non-canonical Intersecting Perfect Matchings

Conjecture

For $\ell \geq 3t/2 + 1$ the size of the largest t-intersecting set of perfect matchings is $(2(\ell - t) - 1)!!$.

Recall the 3-sets, each contains at least two elements from $\{1, 2, 3\}$.

Example

If $t + 3 \leq \ell < 3t/2 + 1$ then there is a larger set of intersecting perfect matchings.

- Fix a set of $t + 2$ disjoint edges $\{e_1, e_2, \ldots, e_{t+2}\}$.
- Take all the perfect matchings that have at least $t + 1$ of these $t + 2$ edges.
- This set has size

$$\binom{t+2}{t+2}(2\ell - 2(t+2) - 1)!! + \binom{t+2}{t+1}(2\ell - 2(t+2))(2\ell - 2(t+2) - 1)!!$$

  which is larger than the canonical, if $\ell < 3t/2 + 1$

This construction works for the uniform (k, ℓ)-partitions too, so the lower bound is needed.

Trivial Cases for Partial Intersection

1. Any two (k, ℓ)-uniform partitions are partially 1-intersecting.
2. If $t = k$, then partially t-intersecting and intersecting are the same.

Proposition

If $k > \ell(t-1)$, then any two (k, ℓ)-partitions are partially t-intersecting.

If the parts are really big, any two partitions will be partially t-intersecting.

Example

Consider $k = 5$, $t = 3$, $\ell = 2$ and $n = 10$:

$$0\,1\,2\,3\,4 \,|\, 5\,6\,7\,8\,9 \qquad 0\,1\,2\,4\,5 \,|\, 3\,6\,7\,8\,9 \qquad 0\,1\,5\,6\,9 \,|\, 2\,3\,4\,7\,8\,9$$

The Big Conjecture

Conjecture

If $k \leq \ell(t-1)$, then the largest set of t-partially intersecting (k, ℓ)-partitions is a canonical set of partially t-intersecting partitions

Focus on partially 2-intersecting uniform partitions.

1. If $k = 2$, this is the intersecting perfect matchings.
2. Counting doesn't work, we use a totally different method.

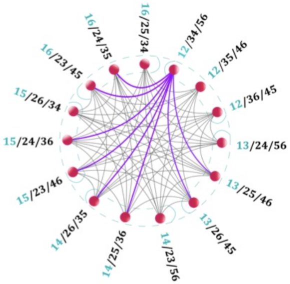
Derangement Graph

Definition

For any k, ℓ define the **partition derangement graph**, $\Gamma_{(k,\ell)}$.

- The vertices are the uniform (k, ℓ)-partitions.
- Vertices $P, Q \in \mathcal{U}(k, \ell)$ are adjacent if and only if P and Q are **not** partially 2-intersecting.

Can use any definition of intersection.



The graph $\Gamma_{(2,3)}$ (Image by Mahsa Shirazi).

Properties of Derangement Graphs

A **coclique** (independent set or stable set) in $\Gamma_{(k,\ell)}$, exactly if the set is a **partially** t**-intersecting set** of partitions.

Graph Properties:

1. The graph $\Gamma_{(k,\ell)}$ is **regular**; denote the degree by $d_{k,\ell}$
2. The derangement graph is **vertex-transitive**, the group $\mathrm{Sym}(k\ell)$ acts transitively on the vertices of $\Gamma_{(k,\ell)}$.
   $\mathrm{Sym}(k\ell)$ permutes the elements in the partitions

What are the maximum coclique in $\Gamma_{k,\ell}$?

Resolvable Designs

Definition

Suppose $\mathcal{B}$ is a t-(n, k, λ) design.

- Collection of k-sets from $\{1, \dots, n\}$, every t-subset is in exactly λ sets.
- A *parallel class* in $\mathcal{B}$ is a collection of disjoint sets whose union is the n-set.
- A partition of $\mathcal{B}$ into $r = \lambda \frac{\binom{n-1}{t-1}}{\binom{k-1}{t-1}}$ parallel classes is called a *resolution*.
- A t-(n, k, λ) design is *resolvable* if a resolution exists.

Example

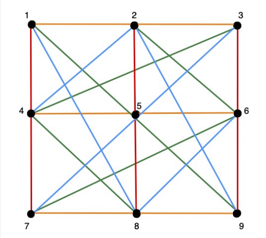
Resolvable 2-(9, 3, 1) Design:

123 | 456 | 789 (orange)

147 | 258 | 369 (red)

159 | 267 | 348 (green)

168 | 249 | 357 (blue)



Lemma

A resolvable t-$(k\ell, k, 1)$ design is a maximum clique in $\Gamma_{k,\ell}$ with size

$$\frac{1}{\ell}\frac{\binom{n}{t}}{\binom{k}{t}}$$

Theorem (Clique-Coclique Bound)

If X is a vertex-transitive graph and $\alpha(X)$ is the size of the maximum coclique and $\omega(X)$ the size of the maximum clique, then

$$\alpha(X)\,\omega(X) \leq |V(X)|.$$

Clique/Coclique Bound

Theorem

If there is a resolvable t-$(k\ell, k, 1)$ design, then the canonical partially t-intersecting partitions are the largest intersecting sets.

Proof.

By the clique/clique bound, a coclique is no larger than

$$\frac{U(k,\ell)}{\frac{1}{\ell}\frac{\binom{k\ell}{t}}{\binom{k}{t}}}\frac{1}{(\ell-1)!}\binom{k\ell-t}{k-t}\binom{k\ell-k}{k}\binom{k}{k} = U(k,\ell-1)$$

□

Lemma (Meagher)

For $n = 3k$ and k odd, partially 2-intersecting uniform k-partition system is no larger than a canonical system.

Motivating Problem

Example

What is the largest set uniform (3,3)-partitions so that no two are 2-partially intersecting?

- This is the size of the largest clique in $\Gamma_{3,3}$.
- The number of uniform (3,3)-partitions is

$$\frac{1}{3!}\binom{9}{3}\binom{6}{3}\binom{3}{3} = 280.$$

- A canonical 2-partially intersecting is a coclique of size $7 * \frac{1}{2}\binom{6}{3} = 70$.

$$1\ 2\ *\ |\ *\ *\ *\ |\ *\ *\ *$$

- There is a clique of size 4.

$$1\,2\,3\,|\,4\,5\,6\,|\,7\,8\,9, \quad 1\,4\,7\,|\,2\,5\,8\,|\,3\,6\,9, \quad 1\,5\,9\,|\,2\,6\,7\,|\,3\,4\,8, \quad 1\,6\,8\,|\,2\,4\,9\,|\,3\,5\,7$$

- This is maximum since

$$70 * 4 \leq \alpha(\Gamma_{3,3})\,\omega(\Gamma_{3,3}) \leq |V(\Gamma_{3,3})| = 280.$$

A set in which no two are 2-partially intersecting corresponds to an **Orthogonal array.**

Ratio Bound

The eigenvalues of a graph are the eigenvalues of the adjacency matrix.

The largest eigenvalue of a d-regular graph is d.

Theorem (Delsarte-Hoffman Ratio bound)

Let A be the adjacency matrix for a d-regular graph X on vertex set $V(X)$. If the least eigenvalue of A is τ, then

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{d}{\tau}}.$$

If equality holds for some coclique S with characteristic vector ν_S, then

$$\nu_S - \frac{|S|}{|V(X)|}\mathbf{1}$$

is an eigenvector with eigenvalue τ.

See: “Hoffman’s ratio bound" by Haemers (https://arxiv.org/abs/2102.05529)

Eigenspaces of the Derangement graphs

1. The eigenspaces of $\Gamma_{k,\ell}$ are invariant under the action of $\mathrm{Sym}(k\ell)$ and thus are a union of irreducible modules in the decomposition of

$$\mathrm{ind}\,(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)})^{\mathrm{Sym}(k\ell)}.$$

2. We say that an eigenvalue θ **belongs to a module** if the module is a subspace of the θ-eigenspace.
3. If $\lambda \vdash n$ is an integer partition of n, then

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j)$$

   (and $\lambda_1 + \lambda_2 + \cdots + \lambda_j = n$).
4. Each irreducible module of the symmetric group corresponds to an integer partition

What this looks like for the Perfect Matchings

1. The stabilizer of a single perfect matching is

$$\mathrm{Sym}(2) \wr \mathrm{Sym}(\ell).$$

2. Since

$$\mathrm{ind}\left(1_{\mathrm{Sym}(2)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(2\ell)} = \sum 2\lambda$$

   where $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_j]$ is an integer partition of ℓ and $2\lambda = [2\lambda_1, 2\lambda_2, \ldots, 2\lambda_j]$.

3. The eigenvalues of $\Gamma_{k,\ell}$ can be found from the irreducible representations of $\mathrm{Sym}(2\ell)$, with the formula:

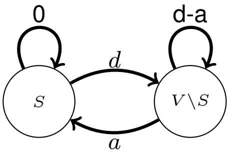
$$\eta_\phi(A_\ell) = \frac{d_\ell}{|H|} \sum_{x_\ell} \sum_{h \in H} \phi(x_\ell h),$$

   $H = \mathrm{Sym}(2) \wr \mathrm{Sym}(\ell)$, d_ℓ is the degree, x_ℓ is a permutation in $\mathrm{Sym}(2\ell)$ so that the cosets H and x_ℓ are intersecting partitions.
   This formula is hard to use.

4. Srinivasan in "The perfect matching association scheme", has a better recursive algorithm.

Finding Eigenvalues with Equitable Partitions

Let S be a coclique in a graph, and $V - S$ the remaining vertices.



If S is a coclique in a d-regular graph

$$\begin{array}{cc} & \begin{array}{cc} S & V\backslash S \end{array} \\ \begin{array}{c} S \\ V\backslash S \end{array} & \begin{pmatrix} 0 & d \\ a & d-a \end{pmatrix} \end{array}$$

This is the quotient graph.

1. Counting edges gives $a = \frac{d|S|}{|V|-|S|}$.
2. The eigenvalues of the quotient graph are d and $-a = -\frac{d|S|}{|V|-|S|}$.
3. Eigenvalues of the quotient matrix **interlace eigenvalues** of the adjacency matrix.
4. If $\{S, V\backslash S\}$ form an equitable partition, the d and $-a$ are also eigenvalues of the adjacency matrix.

Perfect Matchings

1. Let S be the set of all the perfect matchings with the edge 12, and $V - S$ all the perfect matchings without the edge 12.
2. Every vertex in S is adjacent to no other vertices in S, and $d_{2,\ell}$ vertices in $V - S$.
3. Every vertex in $V - S$ is adjacent to $a = \frac{d}{2\ell-2}$ vertices in S, and $d - a$ other vertices in $V - S$.

The quotient matrix is

$$\begin{pmatrix} 0 & d \\ a & d-a \end{pmatrix}$$

d belongs to the representation $[2\ell]$ and $-a$ belongs to $[2\ell - 2, 2]$.

Lemma

In $\Gamma_{2,\ell}$, d is an eigenvalue of with multiplicity 1, (dimension of $[2\ell]$), and $-\frac{d}{2\ell-2}$ is an eigenvalue with multiplicity at least $\binom{2\ell}{2} - \binom{2\ell}{1}$ (dimension of $[2\ell - 2, 2]$).

Ratio bound

1. The trace of $A(\Gamma_{2,\ell})^2$ is equal to the sum of the eigenvalues squared:

$$vd = \sum \lambda_i^2 m_i = d^2(1) + \left(-\frac{d}{2\ell-2}\right)^2 \left(\binom{2\ell}{2} - \binom{2\ell}{1}\right) + \sum_{i \geq 2} \lambda_i^2 m_i$$

2. So for any other eigenvalue λ_i,

$$\frac{vd - d^2(1) + (-\frac{d}{2\ell-2})^2(\binom{2\ell}{2} - \binom{2\ell}{1})}{m_i} \geq \lambda_i^2$$

3. By considering the degrees of the irreducible representations of $\mathrm{Sym}(2\ell)$, we can show that $-\frac{d}{2\ell-2}$ is the least eigenvalue.
4. Apply the ratio bound:

$$\alpha(\Gamma_{2,\ell}) \leq \frac{(2k-1)!!}{1 - \frac{d}{-\frac{d}{2k-2}}} = (2k-3)!!$$

Perfect Matchings - List of Results

Theorem (Godsil and Meagher)

The size of the largest intersecting set of perfect matchings is $(2\ell - 3)!!$.

Characterization by the perfect matching polytope.

Theorem (Fallat, Meagher, Shirazi)

For $\ell \geq 4$*, the size of the largest 2-intersecting set of perfect matchings is* $(2\ell - 5)!!$.

Theorem (Chase, Dafni, Filmus and Lindzey)

The only sets of 2-intersecting set of perfect matchings with size $(2\ell - 5)!!$ *are the canonical intersecting sets.*

Theorem (Lindzey)

For ℓ *sufficiently large, the size of the largest* t*-intersecting set of perfect matchings is* $(2(\ell - t) - 1)!!$.

Eigenvalues for 2-Partially Intersecting (k, ℓ) Partitions

Master Plan:

1. find 3 specific eigenvalues of the graph,
2. show all other eigenvalues are smaller, (in absolute value)
3. apply the ratio bound.

Details:

1. The eigenspaces of $\Gamma_{k,\ell}$ are invariant under the action of $\mathrm{Sym}(k\ell)$ and thus a union of irreducible modules in the decomposition of
$$\mathrm{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}.$$
2. For each irreducible representation in $\mathrm{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$ has a corresponding eigenvalue.
3. The irreducible representations $[k\ell]$, $[k\ell-2,2]$ and $[k\ell-3,3]$ are included in $\mathrm{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$
4. The irreducible representations $[1^{k\ell}]$, $[k\ell-1,1]$, $[2,1^{k\ell-2}]$, $[2,2,1^{k\ell-4}]$, $[k\ell-2,1,1]$, $[3,1^{k\ell-3}]$, $[2,2,2,1^{k\ell-6}]$ are not.
Proof by orbit counting.

Eigenvalues for 2-Partially Intersecting (k, ℓ) Partitions

Theorem

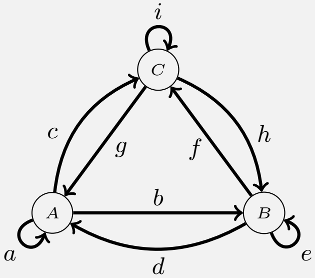
For any k, ℓ,

1. *The eigenvalue belonging to $[k\ell]$ is the degree $d = d_{k,\ell}$.*
2. *The eigenvalue belonging to $[k\ell - 2, 2]$ is*
$$\tau = -\frac{(k-1)d}{k(\ell-1)}.$$
3. *The eigenvalue belonging to $[k\ell - 3, 3]$ is*
$$\theta = \frac{2(k-1)(k-2)d}{k^2(\ell-1)(\ell-2)}.$$
4. *For any other representation, the eigenvalue is smaller in absolute value τ.*

By the ratio bound, the maximum size of coclique in $X_{k,\ell}$ is

$$\frac{|V(\Gamma_{k,\ell})|}{1-\frac{d}{\tau}} = \frac{U_{k,\ell}}{1-\frac{d}{-\frac{(k-1)d}{k(\ell-1)}}} = \frac{U_{k,\ell}}{1+\frac{k(\ell-1)}{k-1}} = \frac{U_{k,\ell}\ (k-1)}{k\ell-1} = \binom{k\ell-2}{k-2} U_{k,\ell-1}.$$

1. The action of a subgroup of the automorphism group on the partitions forms orbits.
2. These orbits can be used to build a **quotient graph**.



Partition the vertices in the graph into orbits.

$$\begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} \text{a} & \text{b} & \text{c} \\ \text{d} & \text{e} & \text{f} \\ \text{g} & \text{h} & \text{i} \end{pmatrix} \end{array}$$

The **quotient matrix**

Quotient Graphs

Young's subgroup $\mathrm{Sym}(k\ell)$:

- Has one orbit, so the quotient graph is

$$(d)$$

- This means d is the eigenvalue corresponding to $[k\ell]$.

Young's subgroup $\mathrm{Sym}([k\ell-2,2]) = \mathrm{Sym}(k\ell-2) \times \mathrm{Sym}(2)$

- Has two orbits: the partitions with 1 and 2 together in one part and the partitions where they are in two parts.
- The quotient matrix is the 2×2 matrix

$$\begin{pmatrix} 0 & d \\ -\tau & d+\tau \end{pmatrix}$$

- The eigenvalues are d and $\tau = -\frac{d(k-1)}{k(\ell-1)}$.
- This means $-\tau$ is the eigenvalue corresponding to $[k\ell-2,2]$.

Eigenvalues of the Derangement graphs

Young's subgroup $\mathrm{Sym}([k\ell-3,3]) = \mathrm{Sym}(k\ell-3)\times\mathrm{Sym}(3)$

- This group has three orbits: partitions where 1,2,3 are in one part, two parts or three parts.
- The quotient matrix is the 3×3 matrix

$$M = \begin{pmatrix} 0 & 0 & d \\ 0 & a & d-a \\ b & c & d-b-c \end{pmatrix}$$

- The eigenvalues are d, $-\tau$, θ.
- $tr(M) = d-b-c+a = d-\tau+\theta$
- Counting edges between the orbits gives equations for a, b, c, then

$$\theta = \frac{2(k-1)(k-2)d}{k^2(\ell-1)(\ell-2)}.$$

Bound on the Multiplicity

Theorem

Assume $k\ell \geq 13$ *and* $k \geq 3$*. Then the only partitions in the decomposition of* $\operatorname{ind}\left(1_{\mathrm{Sym}(k)\wr\mathrm{Sym}(\ell)}\right)^{\mathrm{Sym}(k\ell)}$ *with dimension less than or equal to* $\binom{k\ell}{3} - \binom{k\ell}{2}$ *are*

$$\chi_{[k\ell]}, \quad \chi_{[k\ell-2,2]}, \quad \chi_{[k\ell-3,3]}.$$

Proof.

Use induction and the "branching rule". □

Bound on the Multiplicity

Theorem

The eigenvalues d, τ and θ are the three largest, in absolute value, in the derangement graph. The smallest eigenvalue for $\Gamma_{(k,\ell)}$ is τ.

Proof.

By squaring the adjacency matrix and taking the trace, we have

$$vd = d^2 + m_\tau \tau^2 + m_\theta \theta^2 + \sum_{i=2}^{j} m_i \lambda_i^2.$$

Hence for every $2 \leq i \leq j$ we have

$$vd - d^2 - m_\tau \tau^2 - m_\theta \theta^2 \geq m_i \lambda_i^2.$$

This gives an upper bound on $|\lambda_i|$ in terms of d. □

Bound on Degree

Theorem (Bender)

Let $\mathcal{M}_{k,\ell}$ be the number of all $\ell \times \ell$ matrices with entries either 0 or 1, and row and columns sums equal to k. For positive integers k, ℓ

$$\lim_{\ell \to \infty} \frac{(k!)^{2\ell}}{(k\ell)!} |\mathcal{M}_{k,\ell}| = e^{-\frac{(k-1)^2}{2}}.$$

Lemma

For positive integers k, ℓ with $k \leq \ell$, let d be the degree of $\Gamma_{(k,\ell)}$. Then

$$d = \frac{k!^{\ell}}{\ell!} |\mathcal{M}_{k,\ell}|.$$

Further, for a fixed integer k with $k \geq 2$,

$$\lim_{\ell \to \infty} \frac{U(k,\ell)}{d} = e^{\frac{(k-1)^2}{2}}.$$

Bound on Degree

We had

$$\left(\frac{vd - d^2 - m_\tau \left(\frac{d(k-1)}{k(\ell-1)}\right)^2 - m_\theta \left(\frac{2(k-1)(k-2)d}{k^2(\ell-1)(\ell-2)}\right)^2}{m_i} \right)^{\frac{1}{2}} \geq |\lambda_i|$$

1. $\lim_{\ell\to\infty} \frac{U(k,\ell)}{d} = e^{\frac{(k-1)^2}{2}}$ gives an upper bound for λ_i.
2. If another eigenvalue is larger than τ, in absolute value, that eigenvalue has to have a multiplicity smaller than $\binom{k\ell}{3} - \binom{k\ell}{2}$,
3. Only d, τ or θ have multiplicity this small.
4. So τ is the least eigenvalue of $\Gamma_{k,\ell}$.
5. Apply the ratio bound.

Final Summary

Theorem

Fix an integer $k \geq 3$. For ℓ sufficiently large, the largest set of partially 2-intersecting uniform (k, ℓ)-partitions has size $\binom{k\ell-2}{k-2} U_{k,\ell-1}$.

Conjecture

For $k \geq 3$ and ℓ sufficiently large, the only sets of partially 2-intersecting (k, ℓ)-partitions with size $\binom{k\ell-2}{k-2} U_{k,\ell-1}$ are the sets $S_{i,j}$.

With a more tedious calculation on the degree approximation we get the follow:

Theorem

For $k = 3$ and all $\ell \geq 3$ the largest set of partially 2-intersecting uniform partitions has size

$$(3\ell - 2)U_{3,\ell-1}.$$

Future Work

There are two obvious questions:

Question

Can a non-canonical 2-partially intersecting set also have the maximum size?

Question

Can this method be extended t-partially intersecting uniform partitions for large values of t?