Math 305 Fall 2011
The Density of $\mathbb{Q}$ in $\mathbb{R}$

The following two theorems tell us what happens when we add and multiply by rational numbers. For the first one, we see that if we add or multiply two rational numbers together, then the result is necessarily a rational number. The proof follows directly from the definition of what it means for a number to be rational.

Theorem 1. *If $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$, then*

(a) $x + y \in \mathbb{Q}$, *and*

(b) $xy \in \mathbb{Q}$.

Proof. Since $x \in \mathbb{Q}$, we know that we can write $x = a/b$ where $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, and $b \neq 0$. Similarly, since $y \in \mathbb{Q}$, we know that we can write $y = c/d$ where $c \in \mathbb{Z}$, $d \in \mathbb{Z}$, and $d \neq 0$. Therefore,

$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bc} \quad \text{and} \quad xy = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

The fact that a, b, c, and d are all integers implies that each of $ad + bc$, ac, and bd is an integer. Moreover, the assumptions that $b \neq 0$ and $d \neq 0$ imply that $bd \neq 0$. Thus, we have expressed both $x + y$ and xy as ratios of an integer to a non-negative integers. By definition, this means that $x + y \in \mathbb{Q}$ and $xy \in \mathbb{Q}$ as required. □

For the second one we see that if we add a rational number to an irrational number, the result is necessarily irrational, whereas if we multiply a non-zero rational number by an irrational number, the result is necessarily irrational. The theorem can be proved by contradiction.

Theorem 2. *If $x \in \mathbb{Q}$ and $y \in \mathbb{R} \setminus \mathbb{Q}$, then*

(a) $x + y \in \mathbb{R} \setminus \mathbb{Q}$, *and*

(b) $xy \in \mathbb{R} \setminus \mathbb{Q}$ *provided* $x \neq 0$.

Proof. We begin by observing that if $y \in \mathbb{R} \setminus \mathbb{Q}$ and $x = 0$, then $x + y = 0 + y = y \in \mathbb{R} \setminus \mathbb{Q}$. Therefore, in order to prove **(a)** and **(b)**, assume that $x \in \mathbb{Q}$ with $x \neq 0$. By definition of rational, this means that we can write $x = a/b$ with $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ with both $a \neq 0$ and $b \neq 0$. If we assume that $x + y$ is rational so that $x + y = c/d$ where $c \in \mathbb{Z}$ and $d \in \mathbb{Z}$ with $d \neq 0$, then we find

$$\frac{a}{b} + y = \frac{c}{d} \quad \text{implying that} \quad y = \frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd}.$$

The fact that a, b, c, and d are all integers implies that each of $bc + ad$ and bd is an integer. Moreover, the assumptions that $b \neq 0$ and $d \neq 0$ imply that $bd \neq 0$. Thus, we have expressed y as the ratio of an integer to a non-negative integer. This contradicts the assumption that

$y \in \mathbb{R} \setminus \mathbb{Q}$ and so we conclude that we must have $x + y \in \mathbb{R} \setminus \mathbb{Q}$. Similarly, if we assume that $xy \in \mathbb{Q}$ so that $xy = e/f$ where $e \in \mathbb{Z}$, $f \in \mathbb{Z}$ and $f \neq 0$, then we find

$$\frac{a}{b} \cdot y = \frac{e}{f} \quad \text{implying that} \quad y = \frac{be}{af}.$$

The fact that a, b, e, and f are all integers implies that each of be and af is an integer. Moreover, the assumptions that $b \neq 0$ and $f \neq 0$ imply that $af \neq 0$. Thus, we have expressed y as the ratio of an integer to a non-negative integer. This contradicts the assumption that $y \in \mathbb{R} \setminus \mathbb{Q}$ and so we conclude that we must have $xy \in \mathbb{R} \setminus \mathbb{Q}$. □

How are the rational numbers distributed throughout the real numbers? We suspect that there are a lot of them, but how can we describe this in a mathematically meaningful way? The first result we'll prove shows that there is a rational number between any two other rational numbers. This means that no matter how close together two rational numbers might be, we can always squeeze a third rational number between the first two.

Theorem 3. *If $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$ with $x < y$, then there exists a $z \in \mathbb{Q}$ such that*

$$x < z < y.$$

Proof. Let $z = (x+y)/2$ so that $z \in \mathbb{Q}$ by Theorem 1. Note we are using both facts contained in that theorem, namely the fact that a sum of rationals $(x + y)$ is rational, as well as the fact that a product of rationals ($x + y$ and $1/2$) is rational. Also observe that $x < y$ so that adding x to both sides implies $2x < x+y$, whereas adding y to both sides implies $x+y < 2y$. Combining these two statements yields $2x < x + y < 2y$ which implies that

$$x < \frac{x + y}{2} < y$$

as required. □

The previous result might suggest that there are no gaps in the rational numbers since we just showed that we can always squeeze a rational number between any two other rational numbers. However, we will now show that we can also squeeze an irrational number between those two rationals; this means that there just might be "gaps" in the rational numbers.

Theorem 4. *If $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$ with $x < y$, then there exists a $z \in \mathbb{R} \setminus \mathbb{Q}$ such that*

$$x < z < y.$$

Proof. Since $x + y$ is necessarily rational and since $\sqrt{2}$ is irrational, we might try the trick in the proof of the previous theorem by setting

$$z = \frac{x + y}{\sqrt{2}}.$$

Unfortunately, this need not be irrational because if $x = -y$, then $z = 0$. However, this is the only way that z can fail to be irrational. Therefore, we need to consider two cases. If either $x \geq 0$ or if $y \leq 0$, then we can take

$$z = \frac{x+y}{\sqrt{2}},$$

whereas if $x < 0 < y$, then we can take

$$z = \frac{y}{\sqrt{2}}.$$

In either case, z is necessarily irrational by Theorem 2 and the proof is complete. □

So now that we've seen that there is always "room" to squeeze both a rational number and an irrational number between any two rational numbers, it is natural to ask whether or not we can squeeze either a rational number or an irrational number (or both) between two irrational numbers. The answer is YES, although the proof is rather tricky and relies crucially on an inconspicuous assumption. Before proving this result, we will attempt to prove a simpler result, namely that we can always squeeze a rational number between a rational number and an irrational number.

Theorem 5. *If $x \in \mathbb{R} \setminus \mathbb{Q}$ with $x > 0$, then there exists a $z \in \mathbb{Q}$ such that*

$$0 < z < x.$$

Proof. There are two cases to consider, namely $x > 1$ and $0 < x < 1$. If $x > 1$, then we can simply take $z = 1$. On the other hand, if $0 < x < 1$, then the trick is to consider $1/x$. Since $x > 0$, we know that $1/x$ is well-defined. Moreover, since $x < 1$, we know that $1/x > 1$. Finally, since $x \in \mathbb{R} \setminus \mathbb{Q}$, we know that $1/x \in \mathbb{R} \setminus \mathbb{Q}$. Let $x' = 1/x$ to simplify notation. We know that $x' > 1$ is an irrational number. Here is the crucial claim. For any irrational number $x' > 0$, there exists a natural number n such that $n > x'$. (In other words, we can always round up an irrational number to the "next" natural number.) Hence, with such n, we have $x' < n$ so that $x = 1/x' > 1/n$. Clearly $1/n \in \mathbb{Q}$ so taking $z = 1/n$ completes the proof. □

Let's see. In order to prove the previous theorem, we made the assumption that for any irrational number $r > 0$, there exists a natural number n such that $n > r$. This seems like a pretty reasonable assumption. However, let's try to prove it just for fun. Actually, there is nothing special about r being irrational, and so we might consider the stronger assumption that for any real number $r > 0$, there exists a natural number n such that $n > r$.

Theorem 6. *For every $r \in \mathbb{R}$ there exists an $n \in \mathbb{N}$ such that $n > r$.*

Proof. Suppose, to the contrary, that there did not exist such an n. This implies that r must be an *upper bound* for the set $\mathbb{N}$. In other words, every element n in the set

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

must satisfy $n \leq r$. This is (clearly?) not true since $\mathbb{N}$ is unbounded and contradicts the assumption that no such n exists. □

Well, we seem to be at the end. It seems pretty obvious that $\mathbb{N}$ is unbounded. Actually, it turns out that this "obvious" fact is only true is we make an assumption. In fact, this assumption is so fundamental that it is one of our *axioms* of mathematics.

Formally, an axiom is a statement that is assumed to be true, but cannot be proved to be true. This is in contrast to a theorem which is a statement that is proved to be true using other true statements. While axioms might seem like "obvious" facts, they turn out to be axioms precisely because we would "like" them to be true yet we cannot logically prove them to be true. Moreover, not every statement can be an axiom.[1] A statement becomes an axiom if it is generally agreed upon by the mathematical community that it is fundamental to mathematics and that it cannot be deduced from other, more fundamental, statements. If you are interested in the axiomatic foundations of mathematics, you might want to consider reading Section 9. Although not directly specified in that Section, one of our axioms for arithmetic is that there exist two numbers 0 and 1 with $0 \neq 1$. (Basically, you can never prove that 0 does not equal 1. You need to "accept it" as an axiom.)

Thus, with the assumption that $0 \neq 1$, we will want to conclude that the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ consists of distinct elements and that if $n \in \mathbb{N}$, then $n+1$ is also an element of $\mathbb{N}$ with $n+1$ strictly greater than n. Although it might seem a little strange to be using an axiom about the *real* numbers in order to conclude something about the *natural* numbers, it turns out that this is a nice way to axiomatize things.

Axiom (Completeness Axiom)**.** Every nonempty subset S of $\mathbb{R}$ that is bounded above has a least upper bound; that is, $\sup S$ exists and is a real number.

Using the completeness axiom, we can prove that $\mathbb{N}$ is actually unbounded above.

Theorem 7 (The Archimedean Property of $\mathbb{R}$)**.** *The set $\mathbb{N}$ of natural numbers is unbounded above in $\mathbb{R}$.*

Proof. If $\mathbb{N}$ were bounded above, then by the completeness axiom it would have a least upper bound, say $\sup \mathbb{N} = m$. The fact that m is the *least* upper bound implies that $m-1$ is not an upper bound for $\mathbb{N}$. This implies that there exists an $n_0 \in \mathbb{N}$ such that $n_0 > m-1$. However, this would imply that $m < n_0 + 1$. Since $n_0 + 1 \in \mathbb{N}$, this contradicts the fact that m is an upper bound for $\mathbb{N}$. □

Note that now that we have proved Theorem 7 it follows that the proof of Theorem 6 is correct. This means that the proof of Theorem 5 as written is actually correct and complete. We are now able to prove our desired results, namely that there always exists rational and irrational numbers between any two other irrational numbers.

Theorem 8 (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *If $x \in \mathbb{R}$ and $y \in \mathbb{R}$ with $x < y$, then there exists a $z \in \mathbb{Q}$ such that*

$$x < z < y.$$

[1] You, the student, cannot make a statement into an axiom just because you cannot prove the statement!

Note that this theorem supersedes Theorem 3 since the hypothesis allows for x and y to be any real numbers. We are particularly interested in x and y being irrational, but that does not affect the proof. The proof is similar to the proof of Theorem 5.

Proof. We want to use the Archimedean property, namely the fact that for any real number there is always a larger natural number. Thus, we need to suppose first that $x > 0$. Since $0 < x < y$ we know that $y - x > 0$. Therefore, consider $1/(y - x)$ so that $1/(y - x) > 0$. By Theorem 7, there exists $n \in \mathbb{N}$ such that $n > 1/(y - x)$. Multiplying by $(y - x)$ and simplifying shows this is equivalent to

$$n(y - x) > 1 \quad \text{or} \quad nx + 1 < ny.$$

Since $nx > 0$ it is not too difficult to show that there exists a natural number $m \in \mathbb{N}$ such that

$$m - 1 \leq nx < m.$$

Now, the first inequality in the previous statement implies that $m \leq nx + 1$ and we showed earlier that $nx + 1 < ny$. This means that $m < ny$. However, the second inequality in the previous statement implies that $nx < m$. Combined, we have

$$nx < m < ny \quad \text{or, equivalently,} \quad x < \frac{m}{n} < y.$$

Choosing $z = m/n$ completes the proof if $x > 0$.

On the other hand, suppose that $x \leq 0$. Choose an integer k such that $k > |x|$. If we then consider $x + k$ and $y + k$ which are both positive numbers, then we can use the previous argument to conclude that there exists a rational number q with $x + k < q < y + k$. From Theorem 1 it follows that $z = q - k \in \mathbb{Q}$. □

Exercise. The one step that you might question is to show that there exists a natural number $m \in \mathbb{N}$ with $m - 1 \leq nx < m$. This is Exercise 12.9 on page 127.

Having shown that there is always room between any two numbers to squeeze a rational number, we finally prove that between any two numbers there is also room to squeeze an irrational number.

Theorem 9. *If $x \in \mathbb{R}$ and $y \in \mathbb{R}$ with $x < y$, then there exists a $z \in \mathbb{R} \setminus \mathbb{Q}$ such that*

$$x < z < y.$$

Proof. If we consider $x/\sqrt{2}$ and $y/\sqrt{2}$ then it follows from Theorem 8 that there exists $w \in \mathbb{Q}$ with $w \neq 0$ such that

$$\frac{x}{\sqrt{2}} < w < \frac{y}{\sqrt{2}}$$

implying that

$$x < w\sqrt{2} < y.$$

If we set $z = w\sqrt{2}$, then it follows from Theorem 2 that $z \in \mathbb{R} \setminus \mathbb{Q}$. □

In conclusion, we have shown that there are no gaps in the real numbers, but that there are arbitrarily small gaps in the rational numbers. The word to describe this is *dense*. Thus, we say that $\mathbb{Q}$ is dense in $\mathbb{R}$.