A WEAK LAW WITH RANDOM INDICES FOR RANDOMLY WEIGHTED SUMS OF ROWWISE INDEPENDENT RANDOM ELEMENTS IN RADEMACHER TYPE p BANACH SPACES

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ABSTRACT: For randomly weighted and randomly indexed sums of the form $\sum_{j=1}^{T_n} A_{nj}\Big(V_{nj} - E(V_{nj}I(||V_{nj}|| \le c_n))\Big)$ where $\{A_{nj}, j \ge 1, n \ge 1\}$ is an array of rowwise independent random variables, $\{V_{nj}, j \ge 1, n \ge 1\}$ is an array of rowwise independent random elements in a separable real Rademacher type p Banach space, $\{c_n, n \ge 1\}$ is a sequence of positive constants, and $\{T_n, n \ge 1\}$ is a sequence of positive integer-valued random variables, we present conditions under which the general weak law of large numbers $\sum_{j=1}^{T_n} A_{nj}\Big(V_{nj} - E(V_{nj}I(||V_{nj}|| \le c_n))\Big) \xrightarrow{P} 0$ holds. It is not assumed that the $\{V_{nj}, j \ge 1, n \ge 1\}$ have expected values or absolute moments. The sequences $\{A_{nj}, j \ge 1\}$ and $\{V_{nj}, j \ge 1\}$ are assumed to be independent for all $n \ge 1$. However, no conditions are imposed on the joint distributions of the random indices $\{T_n, n \ge 1\}$ and no independence conditions are imposed between $\{T_n, n \ge 1\}$ and $\{A_{nj}, V_{nj}, j \ge 1, n \ge 1\}$. The sharpness of the weak law is illustrated by examples.

Key words and phrases: Separable real Rademacher type p Banach space; array of rowwise independent random elements; weighted sums; random weights; random indices; weak law of large numbers; convergence in probability.

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1. INTRODUCTION

Consider an array $\{V_{nj}, j \geq 1, n \geq 1\}$ of rowwise independent random elements defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a separable real Banach space $\mathcal{X}$ with norm $||\cdot||$. Let $\{A_{nj}, j \geq 1, n \geq 1\}$ be an array of rowwise independent (real-valued) random variables (called *random weights*) and let $\{T_n, n \geq 1\}$ be a sequence of positive integer-valued random variables (called *random indices*). In the current work, a general weak law of large numbers (WLLN) will be established for the randomly indexed *weighted sums* $\sum_{j=1}^{T_n} A_{nj}V_{nj}$. This result takes the form

$$\sum_{j=1}^{T_n} A_{nj}\Big(V_{nj} - E(V_{nj}I(||V_{nj}|| \leq c_n))\Big) \xrightarrow{P} 0$$

where $\{c_n, n \geq 1\}$ is a sequence of positive constants. It should be noted that each $A_{nj}V_{nj}$ is automatically a random element (see, e.g., Taylor, 1978, p. 24). The sequences $\{A_{nj}, j \geq 1\}$ and $\{V_{nj}, j \geq 1\}$ are assumed to be independent for all $n \geq 1$ but it is not assumed that $\{A_{nj}, V_{nj}, j \geq 1\}$ and $\{A_{n'j}, V_{n'j}, j \geq 1\}$ are independent for $n \neq n'$. No conditions are imposed on the joint distributions of the $\{T_n, n \geq 1\}$ whose marginal distributions are constrained solely by (3.1), and no independence conditions are imposed between $\{T_n, n \geq 1\}$ and $\{A_{nj}, V_{nj}, j \geq 1, n \geq 1\}$. It is also not being assumed that the $\{V_{nj}, j \geq 1, n \geq 1\}$ have expected values or absolute moments. The Banach space $\mathcal{X}$ is assumed to be of *Rademacher type* p $(1 \leq p \leq 2)$. (Technical definitions such as this will be discussed in Section 2.)

Hong (1996) studied the WLLN problem for randomly indexed partial sums $\sum_{j=1}^{T_n} X_{nj}$ from an array of random variables $\{X_{nj}, j \geq 1, n \geq 1\}$. Except for some recent work of Rosalsky and Sreehari (2001a, 2001b) concerning mean convergence and the WLLN, respectively, in martingale type p Banach spaces, we are unaware of any literature of investigation on limit laws for *both* randomly weighted and randomly indexed sums $\sum_{j=1}^{T_n} A_{nj}V_{nj}$ even when the V_{nj} are random variables. In the current work, we establish a Rademacher type p Banach space version of the main result of Rosalsky and Sreehari (2001b).

Let θ_{nj} be a generic symbol for a (suitably selected) conditional expectation, $j \geq 1, n \geq 1$. Assuming $\mathcal{X}$ is of martingale type p, Adler, Rosalsky, and Volodin (1997a) proved under a Cesàro type condition of Hong and Oh (1995) (which is weaker than Cesàro uniform integrability as introduced by Chandra (1989)) the WLLN $\sum_{j=1}^{k_n} a_{nj}(V_{nj} - \theta_{nj}) \xrightarrow{P} 0$ where $\{a_{nj}, 1 \leq j \leq k_n < \infty, n \geq 1\}$ is an array of constants with $k_n \to \infty$. In a martingale type p Banach space setting, Hong, Ordóñez Cabrera, Sung, and Volodin (1999, 2000) gave conditions for the WLLN (with random indices) $\sum_{j=1}^{T_n} a_{nj}(V_{nj} - \theta_{nj}) \xrightarrow{P} 0$ to hold. Also in a martingale type p Banach space setting, Adler, Rosalsky, and Volodin (1997a) proved the $\mathcal{L}_r$ convergence result $||\sum_{j=1}^{k_n} a_{nj}(V_{nj} - \theta_{nj})|| \xrightarrow{\mathcal{L}_r} 0$ where $1 \leq r \leq p$ under a uniform integrability type condition introduced by Ordóñez Cabrera (1994) which contains Cesàro uniform integrability as a special case.

When $\mathcal{X}$ is of Rademacher type p $(1 \leq p \leq 2)$ and the array $\{V_{nj}, j \geq 1, n \geq 1\}$ is comprised of rowwise independent random elements, Adler, Rosalsky, and Volodin (1997b) established a WLLN (with random indices) of the form $\sum_{j=1}^{T_n} a_j(V_{nj} - \mu_{nj})/b_n \xrightarrow{P} 0$ where $\{\mu_{nj}, j \geq 1, n \geq 1\}$ is an array of suitable elements of $\mathcal{X}$ and $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ are sequences of constants with $0 < b_n \to \infty$.

Taylor and Padgett (1974), Wei and Taylor (1978a, 1978b), Taylor and Calhoun (1983), Taylor, Raina, and Daffer (1984), Ordóñez Cabrera (1988), and Adler, Rosalsky, and Taylor (1992) studied either the weak or the almost sure (a.s.) limiting behavior of randomly weighted partial sums of Banach space valued random elements. However, in all of those articles, the number of terms in the partial sums is deterministic.

The plan of the paper is as follows. Some technical definitions will be given in Section 2. The main result will be presented in Section 3. In Section 4, some examples will be offered which illustrate the sharpness of the result.

Finally, the symbol C denotes throughout a generic constant $(0 < C < \infty)$ which is not necessarily the same one in each appearance, and for $x > 0$ the symbol $[x]$ denotes the greatest integer in x.

2. PRELIMINARIES

Technical definitions relevant to the current work will be discussed in this section.

The *expected value* or *mean* of a random element V, denoted by EV, is defined to be the *Pettis integral* provided it exists. That is, V has expected value $EV \in \mathcal{X}$ if $f(EV) = E(f(V))$ for every $f \in \mathcal{X}^*$ where $\mathcal{X}^*$ denotes the *(dual)* space of all continuous linear functionals on $\mathcal{X}$. A sufficient condition for EV to exist is that $E||V|| < \infty$ (see, e.g., Taylor, 1978, p. 4). Thus, $E(VI(||V|| < c))$ exists for every random element V and constant $c < \infty$.

Let $\{Y_n, n \geq 1\}$ be a symmetric *Bernoulli sequence*, that is, $\{Y_n, n \geq 1\}$ are independent and identically distributed (i.i.d.) random variables with $P\{Y_1 = 1\} = P\{Y_1 = -1\} = 1/2$. Let $\mathcal{X}^\infty = \mathcal{X} \times \mathcal{X} \times \mathcal{X} \times \cdots$ and define $\mathcal{C}(\mathcal{X}) = \{(v_1, v_2, \cdots) \in \mathcal{X}^\infty : \sum_{n=1}^\infty Y_n v_n$ converges in probability$\}$. Let $1 \leq p \leq 2$. Then the separable real Banach space $\mathcal{X}$ is said to be of *Rademacher type* p if there exists a constant $0 < C < \infty$ such that $E||\sum_{n=1}^\infty Y_n v_n||^p \leq C \sum_{n=1}^\infty ||v_n||^p$ for all $(v_1, v_2, \cdots) \in \mathcal{C}(\mathcal{X})$. Hoffmann-Jørgensen and Pisier (1976) proved for $1 \leq p \leq 2$ that a real separable Banach space is of Rademacher type p if and only if there exists a constant $0 < C < \infty$ such that

$$E\Big|\Big|\sum_{j=1}^n V_j\Big|\Big|^p \leq C \sum_{j=1}^n E||V_j||^p$$

for every finite collection $\{V_1, \cdots, V_n\}$ of independent mean 0 random elements.

If a separable real Banach space is of Rademacher type p for some $1 < p \leq 2$, than it is of Rademacher type q for all $1 \leq q < p$. Every separable real Banach space is of Rademacher type (at least) 1 while the $\mathcal{L}_p$-spaces and ℓ_p-spaces are of Rademacher type $2 \wedge p$ for $p \geq 1$. Every separable real Hilbert space and separable real finite-dimensional Banach space is of Rademacher type 2.

The following lemma of Adler, Rosalsky, and Volodin (1997b) is a Kolmogorov type maximal inequality for random elements in Rademacher type p $(1 \leq p \leq 2)$ Banach spaces and it will be used to prove the main result. Lemma 2.1 is immediate when $p = 1$ and it had been obtained by Jain (1976) and Woyczyński (1978) when $p = 2$.

Lemma 2.1 (Adler, Rosalsky, and Volodin, 1997b). Let $\{V_n, n \geq 1\}$ be a sequence of independent mean 0 random elements in a separable real Rademacher type p $(1 \leq p \leq 2)$ Banach space. Then for all $n \geq 1$

$$P\left\{\max_{1\leq k\leq n}\left|\left|\sum_{j=1}^{k} V_j\right|\right| > t\right\} \leq \frac{C}{t^p}\sum_{j=1}^{n} E||V_j||^p, t > 0$$

where C is a constant which does not depend on n.

3. MAINSTREAM

With the preliminaries accounted for, the main result may now be presented. It should be noted that if $T_n/i_n \xrightarrow{P} c$ for some $c \in [0, 1)$, then the condition (3.1) of Theorem 3.1 holds. An example wherein the converse fails is given by a positive integer sequence $i_n \to \infty$ with i_n divisible by $4, n \geq 1$ and a sequence of random variables $\{T_n, n \geq 1\}$ with $P\{T_n = \frac{1}{2}i_n\} = P\{T_n = \frac{1}{4}i_n\} = \frac{1}{2}, n \geq 1$. Also, observe that the condition (3.5) of Theorem 3.1 is of the spirit of the condition $nP\{|X_1| > n\} = o(1)$ of the classical WLLN with random indices for sums of i.i.d. random variables $\{X_n, n \geq 1\}$ (see, e.g., Chow and Teicher, 1997, p. 133). Finally, note in Theorem 3.1 that the slower $i_n \to \infty$ can be taken to satisfy (3.1), the weaker are the conditions (3.4), (3.5), and (3.6).

Theorem 3.1. Let $\{V_{nj}, j \geq 1, n \geq 1\}$ be an array of rowwise independent random elements in a separable real Rademacher type p $(1 \leq p \leq 2)$ Banach space and let $\{A_{nj}, j \geq 1, n \geq 1\}$ be an array of rowwise independent random variables. Suppose that the sequences $\{A_{nj}, j \geq 1\}$ and $\{V_{nj}, j \geq 1\}$ are independent for all $n \geq 1$. Let $\{T_n, n \geq 1\}$ be a sequence of positive integer-valued random variables such that

$$P\{T_n > i_n\} = o(1) \tag{1}$$

for some positive integer sequence $i_n \to \infty$. Let g be a strictly increasing, continuous function defined on $[0, \infty)$ such that

$$g(0) = 0 \text{ and } \frac{g^p(x)}{x} \to \infty \text{ as } x \to \infty. \tag{2}$$

Suppose that

$$\sum_{k=1}^{n} \frac{g^p(k)}{k^2} = \mathcal{O}\left(\frac{g^p(n)}{n}\right), \tag{3}$$

$$\sum_{j=1}^{i_n} E|A_{nj}|^p = o(1), \tag{4}$$

and that there exists a sequence of positive constants $c_n \to \infty$ such that

$$\sum_{j=1}^{i_n} P\{||V_{nj}|| > c_n\} = o(1) \tag{5}$$

and

$$\lim_{k\to\infty} \sup_{n\geq 1} \frac{c_n^p}{g^{-1}(c_n)} \sum_{j=1}^{i_n} E|A_{nj}|^p k P\{||V_{nj}|| > g(k)\} = 0. \tag{6}$$

Then the WLLN

$$\sum_{j=1}^{T_n} A_{nj}\Big(V_{nj} - E(V_{nj} I(||V_{nj}|| \leq c_n))\Big) \xrightarrow{P} 0 \tag{7}$$

obtains.

Proof: Let $W_{nj} = V_{nj} I(||V_{nj}|| \leq c_n), j \geq 1, n \geq 1$. Note that for arbitrary $\varepsilon > 0$

$$\begin{aligned} P\left\{\Big|\Big|\sum_{j=1}^{T_n} A_{nj}V_{nj} - \sum_{j=1}^{T_n} A_{nj}W_{nj}\Big|\Big| > \varepsilon\right\} &\leq P\left\{T_n > i_n\right\} + P\left\{\bigcup_{j=1}^{i_n} [V_{nj} \neq W_{nj}]\right\} \\ \leq&\ o(1) + \sum_{j=1}^{i_n} P\{||V_{nj}|| > c_n\} \text{ (by (3.1))} \\ =&\ o(1) \text{ (by (3.5))}. \end{aligned}$$

Thus $\sum_{j=1}^{T_n} A_{nj}V_{nj} - \sum_{j=1}^{T_n} A_{nj}W_{nj} \xrightarrow{P} 0$, and to prove (3.7) it suffices to show that

$$\sum_{j=1}^{T_n} A_{nj}W_{nj} - \sum_{j=1}^{T_n} A_{nj}EW_{nj} \xrightarrow{P} 0. \tag{8}$$

To prove (3.8), let $\varepsilon > 0$ be arbitrary and let

$$B_n = \left[\max_{1\leq r\leq i_n} \Big|\Big|\sum_{j=1}^{r} A_{nj}\left(W_{nj} - EW_{nj}\right)\Big|\Big| > \varepsilon\right], n \geq 1.$$

Then for all $n \geq 1$,

$$\begin{aligned}
&P\left\{\Big|\Big|\sum_{j=1}^{T_n} A_{nj}W_{nj} - \sum_{j=1}^{T_n} A_{nj}EW_{nj}\Big|\Big| > \varepsilon\right\} \\
&= P\left\{\Big|\Big|\sum_{j=1}^{T_n} A_{nj}(W_{nj} - EW_{nj})\Big|\Big| > \varepsilon\right\} \\
&\leq P\{T_n > i_n\} + P\{B_n \cap [T_n \leq i_n]\} \\
&\leq o(1) + P\{B_n\} \text{ (by (3.1))}. \qquad (9)
\end{aligned}$$

Note that for all $n \geq 1, \{A_{nj}(W_{nj} - EW_{nj}), j \geq 1\}$ is a sequence of independent random elements and by (3.4)

$$E||A_{nj}(W_{nj} - EW_{nj})|| \leq E(|A_{nj}|(||W_{nj}|| + E||W_{nj}||)) \leq 2c_n E|A_{nj}| < \infty, j \geq 1$$

implying that $E(A_{nj}(W_{nj} - EW_{nj}))$ exists, $j \geq 1$. Then since A_{nj} and V_{nj} are independent, $j \geq 1, n \geq 1$, we have

$$E(A_{nj}(W_{nj} - EW_{nj})) = 0, j \geq 1, n \geq 1.$$

Applying Lemma 2.1, it follows from (3.9) that

$$\begin{aligned}
&P\left\{\Big|\Big|\sum_{j=1}^{T_n} A_{nj}W_{nj} - \sum_{j=1}^{T_n} A_{nj}EW_{nj}\Big|\Big| > \varepsilon\right\} \\
\leq\ & o(1) + \frac{C}{\varepsilon^p}\sum_{j=1}^{i_n} E||A_{nj}(W_{nj} - EW_{nj})||^p \\
\leq\ & o(1) + C\sum_{j=1}^{i_n}\Big(E||A_{nj}W_{nj}||^p + E||A_{nj}EW_{nj}||^p\Big) \\
\leq\ & o(1) + C\sum_{j=1}^{i_n}\Big(E(|A_{nj}|^p||W_{nj}||^p) + E(|A_{nj}|^p(E||W_{nj}||)^p)\Big)
\end{aligned}$$

$$
\begin{aligned}
&\le \ o(1) + C\sum_{j=1}^{i_n}\Big(E(|A_{nj}|^p||W_{nj}||^p) + E(|A_{nj}|^p E||W_{nj}||^p)\Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Jensen's inequality)} \\
&= \ o(1) + C\sum_{j=1}^{i_n} E|A_{nj}|^p E||W_{nj}||^p \ \text{(since } A_{nj} \text{ and } V_{nj} \text{ are independent)} \\
&= \ o(1)
\end{aligned}
$$

by (3.3), (3.4), and (3.6) and arguing as in the proof of Theorem 3.1 of Rosalsky and Sreehari (2001b) thereby proving (3.8). □

Corollary 3.1. Let $\{V_{nj}, 1 \le j \le k_n < \infty, n \ge 1, k_n \to \infty\}$ be an array of rowwise independent random elements in a separable real Rademacher type p ($1 \le p \le 2$) Banach space. Suppose that the uniform Cesàro type condition

$$\lim_{a\to\infty} \sup_{n\ge 1} \frac{1}{k_n}\sum_{j=1}^{k_n} aP\{||V_{nj}||^r > a\} = 0$$

holds for some $0 < r < p$. Let $\{a_{nj}, 1 \le j \le k_n, n \ge 1\}$ be an array of constants such that

$$\max_{1\le j\le k_n} |a_{nj}| = \mathcal{O}(k_n^{-1/r}).$$

Then the WLLN

$$\sum_{j=1}^{k_n} a_{nj}\Big(V_{nj} - E(V_{nj}I(||V_{nj}|| \le k_n^{1/r}))\Big) \xrightarrow{P} 0$$

obtains.

Proof: It is easy to check that the hypotheses of Theorem 3.1 are satisfied with $T_n = k_n, n \ge 1, A_{nj} = a_{nj}, 1 \le j \le k_n, n \ge 1, i_n = k_n, n \ge 1, g(x) = x^{1/r}, x \ge 0$ where $0 < r < p$, and $c_n = k_n^{1/r}, n \ge 1$. The details are left to the reader. □

4. SOME INTERESTING EXAMPLES

Four illustrative examples will now be presented. For $1 \le r \le 2$, let ℓ_r denote the Banach space of absolute r^{th} power summable real sequences $v = \{v_i, i \ge 1\}$ with norm $||v|| = (\sum_{i=1}^{\infty} |v_i|^r)^{1/r}$. The element having 1 in its j^{th} position and 0 elsewhere will be denoted by $v^{(j)}, j \ge 1$. Define a sequence $\{V_j, j \ge 1\}$ of independent random elements in ℓ_1 by requiring the $\{V_j, j \ge 1\}$ to be independent with $P\{V_j = v^{(j)}\} = P\{V_j = -v^{(j)}\} = 1/2, j \ge 1$. Set $V_{nj} = V_j, j \ge 1, n \ge 1$. Note that for all $n \ge 1$ and $j \ge 1, V_{nj}$ is a random element in ℓ_r for all $1 \le r \le 2$.

Let $\alpha > 0$, and for each $n \geq 1$, let $\{A_{nj}, j \geq 1\}$ be a sequence of independent and identically distributed random variables with

$$P\Big\{A_{n1} = \frac{1}{n^\alpha}\Big\} = P\Big\{A_{n1} = -\frac{1}{n^\alpha}\Big\} = \frac{1}{2}$$

such that the sequences $\{A_{nj}, j \geq 1\}$ and $\{V_j, j \geq 1\}$ are independent.

Let $\theta > 0$ and let $T_n = N_n + 1, n \geq 1$ where $\{N_n, n \geq 1\}$ is a sequence of Poisson distributed random variables with $EN_n = n^\theta, n \geq 1$. Let $1 \leq d_n \to \infty$ be a numerical sequence such that

$$d_n = o(n^\theta) \text{ and } n^{\theta/2} = o(d_n). \tag{1}$$

Let $i_n = [n^\theta + d_n], n \geq 1, g(x) = x^{2/p}, x \geq 0$ where $1 \leq p \leq 2$, and $c_n = n^\gamma, n \geq 1$ where $\gamma > 0$.

The random elements $\{V_{nj}, j \geq 1, n \geq 1\}$, random variables $\{A_{nj}, j \geq 1, n \geq 1\}$ and $\{T_n, n \geq 1\}$, sequences $\{d_n, n \geq 1\}, \{i_n, n \geq 1\}$, and $\{c_n, n \geq 1\}$, and function $g(\cdot)$ will be used in the examples. The constants $\alpha > 0, \theta > 0, p \in [1, 2]$, and $\gamma > 0$ may have additional constraints imposed on them in the different examples.

The first example illustrates the essential role that condition (3.4) plays in Theorem 3.1.

Example 1. Consider the Rademacher type p Banach space ℓ_p. Now (3.2) and (3.3) hold and since $||V_{nj}|| = 1$ a.s., $j \geq 1, n \geq 1$, the conditions (3.5) and (3.6) are also satisfied. Now (3.1) holds (see Example 4.1 of Rosalsky and Sreehari (2001b)).

Next, note that by (4.1)

$$\sum_{j=1}^{i_n} E|A_{nj}|^p = \frac{i_n}{n^{\alpha p}} = (1 + o(1))\frac{n^\theta}{n^{\alpha p}}. \tag{2}$$

Thus if $\theta < \alpha p$, then (3.4) holds, and hence by Theorem 3.1, $\sum_{j=1}^{T_n} A_{nj}V_{nj} \xrightarrow{P} 0$.

On the other hand if $\theta \geq \alpha p$, then (3.4) fails in view of (4.2). Note that with probability 1,

$$\Big|\Big|\sum_{j=1}^{T_n} A_{nj}V_{nj}\Big|\Big| = \frac{||\sum_{j=1}^{T_n} V_{nj}||}{n^\alpha} = \frac{T_n^{1/p}}{n^\alpha}, n \geq 1.$$

Then for $0 < \varepsilon < 1$,

$$P\Big\{\Big|\Big|\sum_{j=1}^{T_n} A_{nj}V_{nj}\Big|\Big| > \varepsilon\Big\} = P\{T_n \geq \varepsilon^p n^{\alpha p}\} \geq P\{T_n > \varepsilon^p n^\theta\} \to 1$$

by the same argument as in Example 4.1 of Rosalsky and Sreehari (2001b). Thus the conclusion (3.7) of Theorem 3.1 fails.

The second example shows that in Theorem 3.1 the condition (3.5) cannot be dispensed with.

Example 2. Consider the Rademacher type p Banach space ℓ_p. Suppose $\alpha > 1$ and $\theta < p(\alpha - 1)$ are such that $2\alpha - 1 - 2\theta p^{-1} > 1$. Take $\gamma = 1$. Note that $\theta < p(\alpha - 1)$ ensures that $(\alpha - (2\alpha - 1 - 2\theta p^{-1}))p < \theta$, and thus we can choose $\delta \in (1, 2\alpha - 1 - 2\theta p^{-1})$ so that $(\alpha - \delta)p < \theta$. Consider the array $\mathcal{V}^* = \{V^*_{nj}, j \geq 1, n \geq 1\}$ of random elements in ℓ_p defined by $V^*_{nj} = n^\delta V_{nj}, j \geq 1, n \geq 1$. We will verify that the array $\mathcal{V}^*$ (in place of $\{V_{nj}, j \geq 1, n \geq 1\}$) satisfies all the assumptions of Theorem 3.1 (except (3.5)) and that the conclusion (3.7) fails. As in Example 1, the conditions (3.1), (3.2), and (3.3) hold. Now (3.4) holds in view of (4.2) and $\theta < \alpha p$. Since for all $n \geq 2$ and $j \geq 1$, with probability 1, $||V^*_{nj}|| = n^\delta > c_n$, the condition (3.5) fails. Now $\mathcal{V}^*$ satisfies (3.6) (see Example 4.2 of Rosalsky and Sreehari (2001b)). Finally, observe that with probability 1,

$$\Big|\Big|\sum_{i=1}^{T_n} A_{nj} V^*_{nj}\Big|\Big| = \frac{n^\delta T_n^{1/p}}{n^\alpha}, n \geq 1.$$

Thus, for $\varepsilon > 0$

$$P\Big\{\Big|\Big|\sum_{j=1}^{T_n} A_{nj} V^*_{nj}\Big|\Big| > \varepsilon\Big\} = P\{T_n > \varepsilon^p n^{(\alpha-\delta)p}\} = P\Big\{\frac{T_n - n^\theta}{n^{\theta/2}} > \frac{\varepsilon^p n^{(\alpha-\delta)p} - n^\theta}{n^{\theta/2}}\Big\} \to 1$$

arguing as in Example 4.1 of Rosalsky and Sreehari (2001b) and recalling that $(\alpha - \delta)\ p\ <\ \theta$. Thus, the conclusion (3.7) of Theorem 3.1 fails for the array $\mathcal{V}^*$.

The third example shows that in Theorem 3.1 the condition (3.6) cannot be dispensed with.

Example 3. Consider the Rademacher type p Banach space ℓ_p. Suppose that $\gamma \leq \alpha$, choose $\delta \in \left(\frac{\gamma}{2}, \gamma\right)$, and suppose that $\theta \in (p(\alpha - \delta), p(\alpha - \frac{\gamma}{2})]$. Consider the array $\mathcal{V}^* = \{V^*_{nj}, j \geq 1, n \geq 1\}$ of random elements in ℓ_p defined by $V^*_{nj} = n^\delta V_{nj}, j \geq 1, n \geq 1$. We will verify that the array $\mathcal{V}^*$ (in place of $\{V_{nj}, j \geq 1, n \geq 1\}$) satisfies all the assumptions of Theorem 3.1 (except (3.6)) and that the conclusion (3.7) fails. As in Example 1, the conditions (3.1), (3.2), and (3.3) hold. Now (3.4) holds in view of (4.2) and $\theta < \alpha p$. Moreover, for all $n \geq 1$ and $j \geq 1$, with probability $1, ||V^*_{nj}|| = n^\delta \leq c_n$ and hence (3.5) holds. To verify that (3.6) fails for $\mathcal{V}^*$, note that since $g(k) = k^{2/p}, k \geq 1$ we have for all fixed $k \geq 1$

$$P\{||V^*_{nj}|| > g(k)\} = \begin{cases} 0, 1 \leq j \leq i_n, 1 \leq n \leq k^{\frac{2}{p\delta}} \\ 1, 1 \leq j \leq i_n, n > k^{\frac{2}{p\delta}} \end{cases}$$

and hence

$$\begin{aligned} &\sup_{n \geq 1} \frac{c_n^p}{g^{-1}(c_n)} \sum_{j=1}^{i_n} E|A_{nj}|^p k P\{||V^*_{nj}|| > g(k)\} \\ &= \sup_{n > k^{\frac{2}{p\delta}}} n^{(\gamma p - 2\alpha p)/2} i_n k \end{aligned}$$

$$
\begin{aligned}
&\geq \sup_{n > k^{\frac{2}{p\delta}}} n^{(\gamma p - 2\alpha p + 2\theta)/2} k \text{ (since } i_n \geq n^\theta, n \geq 1) \\
&\geq k^{(\gamma p - 2\alpha p + 2\theta)/p\delta} k \left(\text{since } \theta \leq p\Big(\alpha - \frac{\gamma}{2}\Big)\right) \\
&= k^{(\gamma p - 2\alpha p + 2\theta + p\delta)/p\delta} \\
&\geq k^{(\gamma p - 2\alpha p + 2p(\alpha - \delta) + p\delta)/p\delta} \text{ (since } \theta > p(\alpha - \delta)) \\
&= k^{(\gamma - \delta)/\delta} \\
&\to \infty \text{ (since } \gamma > \delta).
\end{aligned}
$$

Thus, (3.6) fails for $\mathcal{V}^*$. Finally, since $p(\alpha - \delta) < \theta$ and arguing as in Example 2, we have for $\varepsilon > 0$ that

$$P\Big\{\Big|\Big|\sum_{j=1}^{T_n} A_{nj} V_{nj}^*\Big|\Big| > \varepsilon\Big\} \to 1$$

and thus the conclusion (3.7) of Theorem 3.1 fails for the array $\mathcal{V}^*$.

The fourth example shows that the Rademacher type p hypothesis cannot be dispensed with in Theorem 3.1.

Example 4. Suppose $p > 1$ and consider the Banach space ℓ_r where $1 \leq r < p$. It is well known that ℓ_r is not of Rademacher type p. Suppose that $\theta = \alpha r$. Thus, (4.2) and $p > r$ ensure that (3.4) holds. As in Example 1, the conditions (3.1), (3.2), (3.3), (3.5), and (3.6) hold. All of the hypotheses of Theorem 3.1 are satisfied except for the underlying Banach space being of Rademacher type p. Note that with probability 1,

$$\Big|\Big|\sum_{j=1}^{T_n} A_{nj} V_{nj}\Big|\Big| = \frac{||\sum_{j=1}^{T_n} V_{nj}||}{n^\alpha} = \frac{T_n^{1/r}}{n^\alpha}, n \geq 1.$$

Thus for $0 < \varepsilon < 1$, arguing as in Example 1 (or Example 4.1 of Rosalsky and Sreehari (2001b)),

$$P\Big\{\Big|\Big|\sum_{j=1}^{T_n} A_{nj} V_{nj}\Big|\Big| > \varepsilon\Big\} = P\{T_n > \varepsilon^r n^{\alpha r}\} \to 1$$

and so the conclusion (3.7) of Theorem 3.1 fails.

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