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On the Weak Limiting Behavior of Almost Surely Convergent Row Sums from Infinite Arrays of Rowwise Independent Random Elements in Banach Spaces

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For an array $\{V_{nk}, k \geqslant 1, n \geqslant 1\}$ of rowwise independent random elements in a real separable Banach space $\mathscr{X}$ with almost surely convergent row sums $S_n = \sum_{k=1}^{\infty} V_{nk}, n \geqslant 1$, we provide criteria for $S_n - A_n$ to be stochastically bounded or for the weak law of large numbers $S_n - A_n \xrightarrow{P} 0$ to hold where $\{A_n, n \geqslant 1\}$ is a (nonrandom) sequence in $\mathscr{X}$.

KEY WORDS: Real separable Banach space; infinite array of rowwise independent random elements; almost surely convergent row sums; infinitesimal array; φ-suitable array; $\{A_n, n \geqslant 1\}$-limitable array; stochastically bounded; Δ_2-condition; weak law of large numbers; convergence in probability.

1. INTRODUCTION

Let $\mathscr{V} = \{V_{nk}, k \geqslant 1, n \geqslant 1\}$ be an infinite array of rowwise independent random elements defined on a probability space $(\Omega, \mathscr{F}, P)$ and taking values in a real separable Banach space $\mathscr{X}$ with norm $\|\cdot\|$. No geometric conditions will be imposed on $\mathscr{X}$. We assume throughout that for all $n \geqslant 1$, the row sum $S_n \equiv \sum_{k=1}^{\infty} V_{nk}$ converges almost surely (a.s.) (to a random element). This of course implies that for all $t > 0$ and $n \geqslant 1$, the row sum

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$S_n^{(t)} \equiv \sum_{k=1}^{\infty} V_{nk}^{(t)}$ converges a.s. (to a random element) where $V_{nk}^{(t)} = V_{nk} I(\|V_{nk}\| \leqslant t)$, $k \geqslant 1$. Woyczyński,[11, 12] Jain,[7] Giné *et al.*,[4] Etemadi,[3] and Hu and Wang[6] among others have provided conditions under which series of independent random elements converge a.s.

Note that if for all $n \geqslant 1$, $V_{nk} = 0$ a.s. for all $k >$ some integer k_n, then for all $n \geqslant 1$ the convergence of $S_n = \sum_{k=1}^{\infty} V_{nk}$ to a random element is of course automatic. Thus, in the current work, we can also consider a double array of random elements $\{V_{nk}, 1 \leqslant k \leqslant k_n, n \geqslant 1\}$ where $\{k_n, n \geqslant 1\}$ is a sequence of positive integers. Such an array can be viewed as the special case of an infinite array with $V_{nk} = 0$ a.s. for all $k \geqslant k_n + 1$, $n \geqslant 1$.

In the current work, we study the weak limiting behavior of $S_n - A_n$ where $\mathscr{A} = \{A_n, n \geqslant 1\}$ is a (nonrandom) sequence in $\mathscr{X}$. More specifically, for a given function φ satisfying appropriate conditions, we provide:

(i) criteria for $\{S_n - A_n, n \geqslant 1\}$ to be *stochastically bounded* (that is,

$$\lim_{t \to \infty} \sup_{n \geqslant 1} P\{\|S_n - A_n\| > t\} = 0)$$

in terms of the limiting behavior (as t and n approach infinity) of

$$\sup_{u \geqslant 0} \varphi(u)\, P\{\|S_n^{(t)} - A_n\| > u\} \quad \text{or of} \quad \sup_{u \geqslant 0} \varphi(u)\, P\{\|S_n - A_n\| > tu\},$$

(ii) a criterion for the *weak law of large numbers* $S_n - A_n \xrightarrow{P} 0$ to hold in terms of the limiting behavior (as $n \to \infty$) of

$$\sup_{u \geqslant 0} \varphi(u)\, P\{\|S_n - A_n\| > tu\} \qquad \text{where} \quad t > 0.$$

2. DEFINITIONS AND NOTATION

In this section we present some definitions and we establish some notation.

A nonnegative nondecreasing function φ defined on $[0, \infty)$ is said to satisfy the *Δ_2-condition* if there exists a number $B \in [1, \infty)$ such that $\varphi(2t) \leqslant B\varphi(t)$ for all $t \geqslant 0$. The Δ_2-condition is equivalent (see Krasnolel'skiĭ and Rutickiĭ,[9] p. 23) to the condition: for all $A \geqslant 1$, there exists a number $B = B(A) \in [1, \infty)$ such that

$$\varphi(At) \leqslant B\varphi(t) \qquad \text{for all} \quad t \geqslant 0. \tag{2.1}$$

We note that if φ satisfies the Δ_2-condition with $\varphi(t) > 0$ for some $t > 0$, then $\varphi(t) > 0$ for all $t > 0$. To see this, suppose that $\varphi(t_0) > 0$ where $t_0 > 0$. Then $\varphi(t_1) > 0$ for all $t_1 \geqslant t_0$. Let $0 < t_1 < t_0$. Then by (2.1)

$$0 < \varphi(t_0) = \varphi\left(\frac{t_0}{t_1} t_1\right) \leqslant B\varphi(t_1)$$

for some constant $B \geqslant 1$ and so $\varphi(t_1) > 0$.

Let $\mathscr{X}$ denote the class of continuous, nondecreasing functions φ defined on $[0, \infty)$ with $\varphi(0) = 0$ and $\lim_{t \to \infty} \varphi(t) = \infty$. For a random variable X and function $\varphi \in \Theta$, let

$$\Lambda_\varphi(X) = \sup_{u \geqslant 0} \varphi(u)\, P\{X > u\}.$$

Any median of a random variable X will be denoted by med(X).

For a random element V, its *symmetrized version* V^s is defined by $V^s = V - \tilde{V}$ where $\tilde{V}$ is an independent copy of V. The array $\mathscr{V}$ is said to be *symmetric* if all the random elements V_{nk} comprising the array are symmetric (that is, V_{nk} and $-V_{nk}$ are identically distributed for all $n \geqslant 1$ and $k \geqslant 1$).

Three properties of an array $\mathscr{V}$ will now be defined. It proves convenient to introduce subscripts (i, s, or ℓ) in the $T(\varepsilon)$ and $N(\varepsilon)$ below to signal the property that they pertain to.

The array $\mathscr{V}$ is said to be *infinitesimal* if

$$\lim_{\substack{t \to \infty \\ n \to \infty}} \sup_{k \geqslant 1} P\{\|V_{nk}\| > t\} = 0.$$

This is equivalent to: for all $\varepsilon > 0$, there exists a number $T = T_i(\varepsilon) > 0$ and an integer $N = N_i(\varepsilon) \geqslant 1$ such that for all $t \geqslant T$ and $n \geqslant N$

$$\sup_{k \geqslant 1} P\{\|V_{nk}\| > t\} < \varepsilon. \tag{2.2}$$

Let φ be a function in Θ satisfying the Δ_2-condition. The array $\mathscr{V}$ is said to be *φ-suitable* if

$$\lim_{\substack{t \to \infty \\ n \to \infty}} \sup_{u \geqslant 1} \varphi(u) \sum_{k=1}^{\infty} P\{\|V_{nk}\| > tu\} = 0.$$

This is equivalent to: for all $\varepsilon > 0$, there exists a number $T = T_s(\varepsilon) > 0$ and an integer $N = N_s(\varepsilon) \geqslant 1$ such that for all $t \geqslant T$ and $n \geqslant N$

$$\sup_{u \geqslant 1} \varphi(u) \sum_{k=1}^{\infty} P\{\|V_{nk}\| > tu\} < \varepsilon. \tag{2.3}$$

Let $\{A_n, n \geqslant 1\}$ be a (nonrandom) sequence in $\mathscr{X}$. The array $\mathscr{V}$ is said to be $\{A_n, n \geqslant 1\}$-*limitable* if

$$\lim_{\substack{t \to \infty \\ n \to \infty}} P\{\|S_n - A_n\| > t\} = 0.$$

This is equivalent to: for all $\varepsilon > 0$, there exists a number $T = T_\ell(\varepsilon) > 0$ and an integer $N = N_\ell(\varepsilon) \geqslant 1$ such that for all $t \geqslant T$ and $n \geqslant N$

$$P\{\|S_n - A_n\| > t\} < \varepsilon. \tag{2.4}$$

Remarks 1.

(i) Apropos of the preceding three definitions, we of course only need to verify that each inequality (2.2), (2.3), or (2.4) holds for $t = T$ because it would then automatically hold for $t > T$.

(ii) It is clear that if the array $\mathscr{V}$ is φ-suitable, then it is infinitesimal. The converse is not true.

Proposition 1. The array $\mathscr{V}$ is $\{A_n, n \geqslant 1\}$-limitable if and only if $S_n - A_n$ is stochastically bounded.

Proof. Sufficiency is obvious. To prove necessity, let $\varepsilon > 0$. Clearly it may be assumed that $N_\ell(\frac{\varepsilon}{2}) \geqslant 2$. Let $T' \geqslant T_\ell(\frac{\varepsilon}{2})$ be such that

$$P\{\|S_n - A_n\| > T'\} < \frac{\varepsilon}{2} \qquad \text{for} \quad 1 \leqslant n \leqslant N_\ell\left(\frac{\varepsilon}{2}\right) - 1.$$

Then for all $t \geqslant T'$,

$$\begin{aligned} &\sup_{n \geqslant 1} P\{\|S_n - A_n\| > t\} \\ &\quad \leqslant \sup_{1 \leqslant n \leqslant N_\ell(\frac{\varepsilon}{2}) - 1} P\{\|S_n - A_n\| > t\} + \sup_{n \geqslant N_\ell(\frac{\varepsilon}{2})} P\{\|S_n - A_n\| > t\} \\ &\quad < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

□

Remark 2. In view of Proposition 1, for an array $\mathscr{V}$ which is $\{A_n, n \geqslant 1\}$-limitable, the integer $N_\ell(\varepsilon)$ can be taken to be 1 for all $\varepsilon > 0$ (provided $T_\ell(\varepsilon)$ is taken to be sufficiently large).

3. PRELIMINARY LEMMAS

Some lemmas needed to establish the main results are presented in this section.

The following two lemmas are well known and are stated for the convenience of the reader. See Etemadi[3] for some results relating to Lemma 1. Lemma 2 is originally due to Hoffmann-Jørgensen[5] and may also be found in Jain and Marcus[8].

Lemma 1 (Ref. 2). Let the array $\mathscr{V}$ be symmetric. Then for all $t > 0$ and $n \geqslant 1$

$$\sum_{k=1}^{\infty} P\{\|V_{nk}\| > t\} \leqslant \frac{2P\{\|S_n\| > t\}}{1 - 2P\{\|S_n\| > t\}}$$

provided $P\{\|S_n\| > t\} \leqslant \frac{1}{2}$.

Lemma 2 (Ref. 5). Let the array $\mathscr{V}$ be symmetric. Then for all $t > 0$ and $n \geqslant 1$,

$$P\{\|S_n\| > 3t\} \leqslant \sum_{k=1}^{\infty} P\{\|V_{nk}\| > t\} + 4(P\{\|S_n\| > t\})^2.$$

Lemma 3. Let $\{A_n, n \geqslant 1\}$ be a (nonrandom) sequence in $\mathscr{X}$. Then for all $t > 0$, $\varepsilon > 0$, and $n \geqslant 1$,

$$|P\{\|S_n - A_n\| > \varepsilon\} - P\{\|S_n^{(t)} - A_n\| > \varepsilon\}| \leqslant P\{\sup_{n \geqslant 1} \|V_{nk}\| > t\}$$

$$\leqslant \sum_{k=1}^{\infty} P\{\|V_{nk}\| > t\}. \tag{3.1}$$

Proof. The second inequality of (3.1) is obvious. To prove the first inequality, let $t > 0$, $\varepsilon > 0$, and $n \geqslant 1$ and note that

$$\begin{aligned}P\{\|S_n-A_n\|>\varepsilon\} &\geqslant P\{\|S_n-A_n\|>\varepsilon, \sup_{k\geqslant 1}\|V_{nk}\|\leqslant t\}\\ &= P\{\|S_n^{(t)}-A_n\|>\varepsilon, \sup_{k\geqslant 1}\|V_{nk}\|\leqslant t\}\\ &\geqslant P\{\|S_n^{(t)}-A_n\|>\varepsilon\}-P\{\sup_{k\geqslant 1}\|V_{nk}\|>t\}\end{aligned} \tag{3.2}$$

and so

$$P\{\|S_n-A_n\|>\varepsilon\}-P\{\|S_n^{(t)}-A_n\|>\varepsilon\}\geqslant -P\{\sup_{k\geqslant 1}\|V_{nk}\|>t\}. \tag{3.3}$$

Next, note that (3.2) also holds with S_n and $S_n^{(t)}$ interchanged and so

$$P\{\|S_n^{(t)}-A_n\|>\varepsilon\}-P\{\|S_n-A_n\|>\varepsilon\}\geqslant -P\{\sup_{k\geqslant 1}\|V_{nk}\|>t\}. \tag{3.4}$$

Combining (3.3) and (3.4) yields the first inequality of (3.1). □

Some symmetrization inequalities are provided by the next lemma.

Lemma 4.

(i) If the array $\mathscr{V}$ is infinitesimal, then for all $t\geqslant 2T_i(\frac{1}{2})$, $n\geqslant N_i(\frac{1}{2})$, and $k\geqslant 1$

$$P\{\|V_{nk}\|>t\}\leqslant 2P\left\{\|V_{nk}^s\|>\frac{t}{2}\right\}.$$

(ii) If the array $\mathscr{V}$ is $\{A_n, n\geqslant 1\}$-limitable, then for all $n\geqslant N_\ell(\frac{1}{2})=1$ and $t\geqslant 2T_\ell(\frac{1}{2})$

$$P\{\|S_n-A_n\|>t\}\leqslant 2P\left\{\|S_n^s\|>\frac{t}{2}\right\}.$$

(iii) If V is a random element, A is a (nonrandom) member of $\mathscr{X}$, and $t>0$, then

$$P\{\|V^s\|>t\}\leqslant 2P\left\{\|V-A\|>\frac{t}{2}\right\}.$$

Proof.

(i) Since $\mathscr{V}$ is infinitesimal, $\sup_{k \geqslant 1} \operatorname{med}(\|V_{nk}\|) \leqslant T_i(\frac{1}{2})$ for all $n \geqslant N_i(\frac{1}{2})$. Then for all $t \geqslant 2T_i(\frac{1}{2})$, $n \geqslant N_i(\frac{1}{2})$, and $k \geqslant 1$, by the weak symmetrization inequality (see, e.g., Loève,[10] p. 257)

$$\begin{aligned} P\{\|V_{nk}\| > t\} &\leqslant P\left\{\left|\|V_{nk}\| - \operatorname{med}(\|V_{nk}\|)\right| > \frac{t}{2}\right\} \\ &\leqslant 2P\left\{\left|\|V_{nk}\| - \|\tilde{V}_{nk}\|\right| > \frac{t}{2}\right\} \\ &\leqslant 2P\left\{\|V^s_{nk}\| > \frac{t}{2}\right\} \end{aligned}$$

where $\tilde{V}_{nk}$ is an independent copy of V_{nk}.

(ii) This is proved in the same way as part (i).

(iii) This is evident. □

We need some additional inequalities which will be provided by the next lemma.

Lemma 5. Let $\{A_n, n \geqslant 1\}$ be a (nonrandom) sequence in $\mathscr{X}$.

(i) For every function $\varphi \in \Theta$, $t > 0$, and $n \geqslant 1$,

$$P\{\|S_n - A_n\| > t\} \leqslant \frac{\Lambda_\varphi(\|S_n^{(t)} - A_n\|)}{\varphi(t)} + \sum_{k=1}^{\infty} P\{\|V_{nk}\| > t\}.$$

(ii) If the array $\mathscr{V}$ is infinitesimal, then for all $t \geqslant 2T_i(\frac{1}{2})$ and $n \geqslant N_i(\frac{1}{2})$

$$\sum_{k=1}^{\infty} P\{\|V_{nk}\| > t\} \leqslant \frac{8P\{\|S_n - A_n\| > \frac{t}{4}\}}{1 - 4P\{\|S_n - A_n\| > \frac{t}{4}\}}$$

provided $P\{\|S_n - A_n\| > \frac{t}{4}\} \leqslant \frac{1}{4}$.

(iii) If the array $\mathscr{V}$ is $\{A_n, n \geqslant 1\}$-limitable and the function $\varphi \in \Theta$ satisfies the Δ_2-condition, then (letting $B = B(12)$ be as in (2.1)) for all $\varepsilon > 0$, $H \geqslant \varepsilon$, $n \geqslant N_\ell(\frac{1}{2}) = 1$, and

$$t \geqslant \max\left\{\frac{T_\ell(\frac{1}{2})}{6H}, \frac{T_\ell(\frac{1}{64B})}{H}\right\}$$

we have

$$\sup_{u \geqslant \varepsilon} \varphi(u)\, P\{\|S_n - A_n\| > tu\}$$

$$\leqslant 2B\left(\varphi(H)\, P\{\|S_n - A_n\| > t\varepsilon\} + 4 \sup_{u \geqslant H} \varphi(u) \sum_{k=1}^{\infty} P\{\|V_{nk}\| > tu\}\right). \tag{3.5}$$

Proof.

(i) By Lemma 3 and the definition of $\Lambda_\varphi(\cdot)$, we have

$$P\{\|S_n - A_n\| > t\} \leqslant P\{\|S_n^{(t)} - A_n\| > t\} + \sum_{k=1}^{\infty} P\{\|V_{nk}\| > t\}$$

$$\leqslant \frac{\Lambda_\varphi(\|S_n^{(t)} - A_n\|)}{\varphi(t)} + \sum_{k=1}^{\infty} P\{\|V_{nk}\| > t\}.$$

(ii) By Lemma 4(i), Lemma 1, and Lemma 4(iii), for all $t \geqslant 2T_i(\frac{1}{2})$ and $n \geqslant N_i(\frac{1}{2})$,

$$\sum_{k=1}^{\infty} P\{\|V_{nk}\| > t\} \leqslant 2 \sum_{k=1}^{\infty} P\left\{\|V_{nk}^s\| > \frac{t}{2}\right\}$$

$$\leqslant \frac{4P\{\|S_n^s\| > \frac{t}{2}\}}{1 - 2P\{\|S_n^s\| > \frac{t}{2}\}}$$

$$\leqslant \frac{8P\{\|S_n - A_n\| > \frac{t}{4}\}}{1 - 4P\{\|S_n - A_n\| > \frac{t}{4}\}}.$$

(iii) For $H \geqslant \varepsilon$,

$$\sup_{u \geqslant \varepsilon} \varphi(u)\, P\{\|S_n - A_n\| > tu\}$$

$$\leqslant \sup_{\varepsilon \leqslant u \leqslant 12H} \varphi(u)\, P\{\|S_n - A_n\| > tu\} + \sup_{u \geqslant 12H} \varphi(u)\, P\{\|S_n - A_n\| > tu\}$$

$$\leqslant \varphi(12H)\, P\{\|S_n - A_n\| > t\varepsilon\} + 2 \sup_{u \geqslant 12H} \varphi(u)\, P\left\{\|S_n^s\| > \frac{tu}{2}\right\}$$

(by Lemma 4(ii))

$$\leqslant \varphi(12H)\, P\{\|S_n - A_n\| > t\varepsilon\} + 2 \sup_{u \geqslant 12H} \varphi(u) \sum_{k=1}^{\infty} P\left\{\|V^s_{nk}\| > \frac{tu}{6}\right\}$$

$$+ 8 \sup_{u \geqslant 12H} \varphi(u) \left(P\left\{\|S^s_n\| > \frac{tu}{6}\right\}\right)^2$$

(by Lemma 2)

$$\leqslant \varphi(12H)\, P\{\|S_n - A_n\| > t\varepsilon\} + 4 \sup_{u \geqslant 12H} \varphi(u) \sum_{k=1}^{\infty} P\left\{\|V_{nk}\| > \frac{tu}{12}\right\}$$

$$+ 32 P\{\|S_n - A_n\| > tH\} \sup_{u \geqslant 12H} \varphi(u)\, P\left\{\|S_n - A_n\| > \frac{tu}{12}\right\}$$

(by Lemma 4(iii))

$$= \varphi(12H)\, P\{\|S_n - A_n\| > t\varepsilon\} + 4 \sup_{v \geqslant H} \varphi(12v) \sum_{k=1}^{\infty} P\{\|V_{nk}\| > tv\}$$

$$+ 32 P\{\|S_n - A_n\| > tH\} \sup_{v \geqslant H} \varphi(12v)\, P\{\|S_n - A_n\| > tv\}$$

$$\leqslant B\left(\varphi(H)\, P\{\|S_n - A_n\| > t\varepsilon\} + 4 \sup_{v \geqslant H} \varphi(v) \sum_{k=1}^{\infty} P\{\|V_{nk}\| > tv\}\right)$$

$$+ 32 B P\{\|S_n - A_n\| > tH\} \sup_{v \geqslant H} \varphi(v)\, P\{\|S_n - A_n\| > tv\}$$

(by (2.1) where $B = B(12)$)

$$\leqslant B\left(\varphi(H)\, P\{\|S_n - A_n\| > t\varepsilon\} + 4 \sup_{u \geqslant H} \varphi(u) \sum_{k=1}^{\infty} P\{\|V_{nk}\| > tu\}\right)$$

$$+ 32 B P\{\|S_n - A_n\| > tH\} \sup_{u \geqslant \varepsilon} \varphi(u)\, P\{\|S_n - A_n\| > tu\}$$

(since $H \geqslant \varepsilon$)

$$\leqslant B\left(\varphi(H)\, P\{\|S_n - A_n\| > t\varepsilon\} + 4 \sup_{u \geqslant H} \varphi(u) \sum_{k=1}^{\infty} P\{\|V_{nk}\| > tu\}\right)$$

$$+ \frac{32B}{64B} \sup_{u \geqslant \varepsilon} \varphi(u)\, P\{\|S_n - A_n\| > tu\}$$

$\left(\text{since } tH \geqslant T_\ell\left(\frac{1}{64B}\right)\right)$

$$= B\left(\varphi(H)\, P\{\|S_n - A_n\| > t\varepsilon\} + 4 \sup_{u \geqslant H} \varphi(u) \sum_{k=1}^{\infty} P\{\|V_{nk}\| > tu\}\right)$$

$$+ \frac{1}{2} \sup_{u \geqslant \varepsilon} \varphi(u)\, P\{\|S_n - A_n\| > tu\}.$$

Thus,

$$\frac{1}{2} \sup_{u \geqslant \varepsilon} \varphi(u) \, P\{\|S_n - A_n\| > tu\}$$

$$\leqslant B \left(\varphi(H) \, P\{\|S_n - A_n\| > t\varepsilon\} + 4 \sup_{u \geqslant H} \varphi(u) \sum_{k=1}^{\infty} P\{\|V_{nk}\| > tu\} \right)$$

proving (3.5). □

The last lemma will be used in providing a characterization of arrays of degenerate random elements with convergent row sums in terms of a particular quasi-norm on a class of arrays.

Lemma 6. Let S be a random element in a real separable Banach space $\mathscr{X}$ and let $\varphi \in \Theta$ be such that $\varphi(t) > 0$ for all $t > 0$. Then $S = A_0$ a.s. for some $A_0 \in \mathscr{X}$ if and only if

$$\inf_{A \in \mathscr{X}} \sup_{t \geqslant 0} \varphi(t) \, P\{\|S - A\| > t\} = 0. \tag{3.6}$$

Proof. The necessity half is obvious. To prove the sufficiency half, let $\varepsilon > 0$ be arbitrary. By (3.6), for all $n \geqslant 1$, there exists $A_n \in \mathscr{X}$ such that

$$\sup_{t \geqslant 0} \varphi(t) \, P\{\|S - A_n\| > t\} \leqslant \frac{1}{n^2}.$$

Hence

$$\sum_{n=1}^{\infty} P\{\|S - A_n\| > \varepsilon\} \leqslant \sum_{n=1}^{\infty} \frac{1}{\varphi(\varepsilon) \, n^2} < \infty$$

and then by the Borel–Cantelli lemma we have $S - A_n \to 0$ a.s. Thus, $A_n \to S$ a.s. and the conclusion follows since the $\{A_n, n \geqslant 1\}$ are nonrandom elements in $\mathscr{X}$. □

4. TWO CRITERIA FOR AN ARRAY BEING $\mathscr{A}$-LIMITABLE

With the preliminaries accounted for, the first main result may be established.

Theorem 1. An array $\mathscr{V}$ is infinitesimal and $\{A_n, n \geqslant 1\}$-limitable if and only if

$$\lim_{\substack{t \to \infty \\ n \to \infty}} \sum_{k=1}^{\infty} P\{\|V_{nk}\| > t\} = 0 \tag{4.1}$$

and

$$\lim_{\substack{t \to \infty \\ n \to \infty}} \frac{\Lambda_{\varphi}(\|S_n^{(t)} - A_n\|)}{\varphi(t)} = 0 \tag{4.2}$$

for every function $\varphi \in \Theta$ satisfying the Δ_2-condition.

Proof. Sufficiency. It follows from

$$\sup_{k \geqslant 1} P\{\|V_{nk}\| > t\} \leqslant \sum_{k=1}^{\infty} P\{\|V_{nk}\| > t\}$$

and (4.1) that the array $\mathscr{V}$ is infinitesimal. Moreover, it follows from Lemma 5(i), (4.1), and (4.2) that $\mathscr{V}$ is $\{A_n, n \geqslant 1\}$-limitable.

Necessity. Since $\mathscr{V}$ is infinitesimal and $\{A_n, n \geqslant 1\}$-limitable, (4.1) follows directly from Lemma 5(ii). Next, let $\varphi \in \Theta$ satisfy the Δ_2-condition. It follows from Lemma 3 that for all $t_0 > 0$, $t > 0$, and $n \geqslant 1$,

$$P\{\|S_n^{(t_0)} - A_n\| > t\} \leqslant \sum_{k=1}^{\infty} P\{\|V_{nk}\| > t_0\} + P\{\|S_n - A_n\| > t\}.$$

For arbitrary $\varepsilon > 0$, it then follows from (4.1) and from $\mathscr{V}$ being $\{A_n, n \geqslant 1\}$-limitable that for t_0, t, and n sufficiently large (say $t_0 \geqslant T_0(\varepsilon)$, $t \geqslant T^*(\varepsilon)$, and $n \geqslant N^*(\varepsilon)$) that

$$P\{\|S_n^{(t_0)} - A_n\| > t\} \leqslant \varepsilon.$$

Then, arguing exactly as in the proof of Lemma 5(iii), for all $t_0 \geqslant T_0(\frac{1}{2})$, $n \geqslant N^*(\frac{1}{2})$, $\varepsilon > 0$, $H \geqslant \varepsilon$, and $t \geqslant \max\{T^*(\frac{1}{2})/6H, T^*(\frac{1}{64B})/H\}$ we have

$$\begin{aligned} &\sup_{u \geqslant \varepsilon} \varphi(u)\, P\{\|S_n^{(t_0)} - A_n\| > tu\} \\ &\quad \leqslant 2B\Bigg(\varphi(H)\, P\{\|S_n^{(t_0)} - A_n\| > t\varepsilon\} \\ &\qquad + 4 \sup_{u \geqslant H} \varphi(u) \sum_{k=1}^{\infty} P\{\|V_{nk} I(\|V_{nk}\| \leqslant t_0)\| > tu\}\Bigg) \end{aligned}$$

where $B = B(12)$ is as in (2.1). Let $n \geqslant N^*(\frac{1}{2})$, $\varepsilon > 0$, $H \geqslant \max\{\varepsilon, T^*(\frac{1}{2})/6, T^*(\frac{1}{64B})\}$, $t_0 \geqslant \max\{T_0(\frac{1}{2}), H\}$, and $t = 1$. Then

$$\begin{aligned}
&\sup_{u \geqslant \varepsilon} \varphi(u)\, P\{\|S_n^{(t_0)} - A_n\| > u\} \\
&\quad \leqslant 2B\left(\varphi(H) + 4 \sup_{u \geqslant H} \varphi(u) \sum_{k=1}^{\infty} P\{\|V_{nk} I(\|V_{nk}\| \leqslant t_0)\| > u\}\right) \\
&\quad = 2B\left(\varphi(H) + 4 \sup_{H \leqslant u \leqslant t_0} \varphi(u) \sum_{k=1}^{\infty} P\{\|V_{nk} I(\|V_{nk}\| \leqslant t_0)\| > u\}\right) \\
&\quad \leqslant 2B\left(\varphi(H) + 4\varphi(t_0) \sum_{k=1}^{\infty} P\{\|V_{nk}\| > H\}\right).
\end{aligned}$$

Then for arbitrary $\delta > 0$, we have recalling (4.1) that for all n and H sufficiently large and all $t_0 \geqslant H$ that

$$\begin{aligned}
\frac{\Lambda_\varphi(\|S_n^{(t_0)} - A_n\|)}{\varphi(t_0)} &= \frac{1}{\varphi(t_0)} \lim_{\varepsilon \to 0} \sup_{u \geqslant \varepsilon} \varphi(u)\, P\{\|S_n^{(t_0)} - A_n\| > u\} \\
&\leqslant 2B \frac{\varphi(H)}{\varphi(t_0)} + 8B \sum_{k=1}^{\infty} P\{\|V_{nk}\| \geqslant H\} \\
&\leqslant 2B \frac{\varphi(H)}{\varphi(t_0)} + \delta.
\end{aligned}$$

Then since $\varphi(t_0) \uparrow \infty$ as $t_0 \uparrow \infty$,

$$\lim_{\substack{t_0 \to \infty \\ n \to \infty}} \frac{\Lambda_\varphi(\|S_n^{(t_0)} - A_n\|)}{\varphi(t_0)} \leqslant \delta$$

and (4.2) follows since $\delta > 0$ is arbitrary. □

Remark 3. Note that the Δ_2-condition was not used in the sufficiency half of Theorem 1.

Let us now introduce some notation. For an array $\mathscr{V}$, a (nonrandom) sequence $\mathscr{A} = \{A_n, n \geqslant 1\}$ in $\mathscr{X}$, a function $\varphi \in \Theta$, and $t \geqslant 0$, let

$$\rho_{n,t}(\mathscr{V}, \mathscr{A}, \varphi) = \sup_{u \geqslant 0} \varphi(u)\, P\{\|S_n - A_n\| > tu\}, \qquad n \geqslant 1.$$

Theorem 2. Let $\mathscr{V}$ be an array which is φ-suitable for some function $\varphi \in \Theta$ satisfying the Δ_2-condition. Then $\mathscr{V}$ is $\mathscr{A} = \{A_n, n \geqslant 1\}$-limitable if and only if

$$\lim_{\substack{t \to \infty \\ n \to \infty}} \rho_{n,t}(\mathscr{V}, \mathscr{A}, \varphi) = 0. \tag{4.3}$$

Proof. Sufficiency. The implication follows immediately from

$$P\{\|S_n - A_n\| > t\} \leqslant \frac{\rho_{n,t}(\mathscr{V}, \mathscr{A}, \varphi)}{\varphi(1)}, \qquad t \geqslant 0, \quad n \geqslant 1.$$

Necessity. Let $\varepsilon > 0$. Since φ is continuous at 0 with $\varphi(0) = 0$, there exists $\delta > 0$ such that $\varphi(u) \leqslant \varepsilon$ whenever $0 \leqslant u \leqslant \delta$. It may be assumed that $\delta \leqslant 1$. Then

$$\begin{aligned} &\rho_{n,t}(\mathscr{V}, \mathscr{A}, \varphi) \\ &\quad \leqslant \sup_{0 \leqslant u \leqslant \delta} \varphi(u)\, P\{\|S_n - A_n\| > tu\} + \sup_{\delta \leqslant u \leqslant 1} \varphi(u)\, P\{\|S_n - A_n\| > tu\} \\ &\qquad + \sup_{u \geqslant 1} \varphi(u)\, P\{\|S_n - A_n\| > tu\}. \end{aligned} \tag{4.4}$$

Clearly

$$\sup_{0 \leqslant u \leqslant \delta} \varphi(u)\, P\{\|S_n - A_n\| > tu\} \leqslant \varepsilon. \tag{4.5}$$

Moreover, since $\mathscr{V}$ is $\{A_n, n \geqslant 1\}$-limitable

$$\lim_{\substack{t \to \infty \\ n \to \infty}} \sup_{\delta \leqslant u \leqslant 1} \varphi(u)\, P\{\|S_n - A_n\| > tu\} \leqslant \lim_{\substack{t \to \infty \\ n \to \infty}} \varphi(1)\, P\{\|S_n - A_n\| > t\delta\} = 0. \tag{4.6}$$

Finally, by Lemma 5(iii) with $\varepsilon = H = 1$

$$\begin{aligned} &\lim_{\substack{t \to \infty \\ n \to \infty}} \sup_{u \geqslant 1} \varphi(u)\, P\{\|S_n - A_n\| > tu\} \\ &\quad \leqslant \lim_{\substack{t \to \infty \\ n \to \infty}} 2B\left(\varphi(1)\, P\{\|S_n - A_n\| > t\} + 4 \sup_{u \geqslant 1} \varphi(u) \sum_{k=1}^{\infty} P\{\|V_{nk}\| > tu\}\right) \\ &\qquad \text{(for some } B < \infty) \\ &\quad = 0 \end{aligned} \tag{4.7}$$

since $\mathscr{V}$ is $\{A_n, n \geqslant 1\}$-limitable and φ-suitable. Combining (4.4), (4.5), (4.6), and (4.7) yields (4.3) since $\varepsilon > 0$ is arbitrary. □

Remark 4. Note that the hypothesis that $\mathscr{V}$ is φ-suitable was not used in the sufficiency half of Theorem 2.

5. SPACES OF $\mathscr{A}$-LIMITABLE ARRAYS

Let us now introduce the following spaces:

$$SB = \{\mathscr{V} : \mathscr{V} \text{ is } \{A_n, n \geqslant 1\}\text{-limitable for some sequence } \{A_n, n \geqslant 1\} \subseteq \mathscr{X}\}$$

and

$$WLLN = \{\mathscr{V} : \mathscr{V} \text{ is } \{A_n, n \geqslant 1\}\text{-limitable for some sequence } \{A_n, n \geqslant 1\} \subseteq \mathscr{X} \text{ with } T_\ell(\varepsilon) \leqslant \varepsilon \text{ for all } \varepsilon > 0\}.$$

Remarks 5.

(i) Trivially, SB and $WLLN$ are linear spaces with the natural operations.

(ii) In view of Proposition 1 and Remark 2, if $\mathscr{V} \in SB$, then $S_n - A_n$ is stochastically bounded and $N_\ell(\varepsilon)$ can be taken to be 1 for all $\varepsilon > 0$ (provided $T_\ell(\varepsilon)$ is taken to be sufficiently large).

(iii) If $\mathscr{V} \in WLLN$, then S_n obeys the weak law of large numbers with some centering sequence $\{A_n, n \geqslant 1\}$; that is, $S_n - A_n \xrightarrow{P} 0$.

Theorem 3. Let $\mathscr{V}$ be an array which is φ-suitable for some function $\varphi \in \Theta$ satisfying the Δ_2-condition.

(i) $\mathscr{V} \in SB$ if and only if $\lim_{\substack{t \to \infty \\ n \to \infty}} \rho_{n,t}(\mathscr{V}, \mathscr{A}, \varphi) = 0$ for some sequence $\mathscr{A} = \{A_n, n \geqslant 1\} \subseteq \mathscr{X}$.

(ii) Suppose that $N_s(\varepsilon)$ can be taken to be 1 for all $\varepsilon > 0$. Then $\mathscr{V} \in SB$ if and only if $\lim_{t \to \infty} \sup_{n \geqslant 1} \rho_{n,t}(\mathscr{V}, \mathscr{A}, \varphi) = 0$ for some sequence $\mathscr{A} = \{A_n, n \geqslant 1\} \subseteq \mathscr{X}$.

Proof.

(i) The assertion is a restatement of Theorem 2.

(ii) Sufficiency. This follows immediately from the sufficiency half of part (i).

Necessity. Proceeding as in the proof of the necessity half of Theorem 2, we have that for arbitrary $\varepsilon > 0$, there exist $\delta > 0$ and $B < \infty$ such that

$$\begin{aligned}\sup_{n \geqslant 1} \rho_{n,t}(\mathscr{V}, \mathscr{A}, \varphi) &\leqslant \varepsilon + \sup_{n \geqslant 1} \varphi(1)\, P\{\|S_n - A_n\| > t\delta\} \\ &\quad + 2B\left(\varphi(1) \sup_{n \geqslant 1} P\{\|S_n - A_n\| > t\} + 4 \sup_{n \geqslant 1} \sup_{u \geqslant 1} \varphi(u) \sum_{k=1}^{\infty} P\{\|V_{nk}\| > tu\}\right)\end{aligned}$$

and the result follows by letting $t \to \infty$ since $N_\ell(\varepsilon)$ and $N_s(\varepsilon)$ can be taken to be 1 and since $\varepsilon > 0$ is arbitrary. □

Theorem 4. Let $\mathscr{V}$ be an array which is φ-suitable for some function $\varphi \in \Theta$ satisfying the Δ_2-condition.

(i) If $\lim_{n \to \infty} \rho_{n,t}(\mathscr{V}, \mathscr{A}, \varphi) = 0$ for some sequence $\mathscr{A} = \{A_n, n \geqslant 1\} \subseteq \mathscr{X}$ and all $t > 0$, then $\mathscr{V} \in WLLN$.

(ii) If $\mathscr{V} \in WLLN$ and $T_s(\varepsilon) \leqslant \varepsilon$ for all $\varepsilon > 0$, then $\lim_{n \to \infty} \rho_{n,t}(\mathscr{V}, \mathscr{A}, \varphi) = 0$ for some sequence $\mathscr{A} = \{A_n, n \geqslant 1\} \subseteq \mathscr{X}$ and all $t > 0$.

Proof. The proof is a slight modification of that of Theorem 2 and the details are left to the reader. □

For an array $\mathscr{V}$ and a function of $\varphi \in \Theta$, let us introduce the notation

$$\lambda_\varphi(\mathscr{V}) = \inf_{\{A_n, n \geqslant 1\} \subseteq \mathscr{X}} \sup_{\substack{t \geqslant 0 \\ n \geqslant 1}} \varphi(t)\, P\{\|S_n - A_n\| > t\}.$$

The next theorem provides a characterization of φ-suitable arrays $\mathscr{V}$ (where $\varphi \in \Theta$ satisfies the Δ_2-condition) being $\{A_n, n \geqslant 1\}$-limitable for some sequence $\{A_n, n \geqslant 1\} \subseteq \mathscr{X}$ by the finiteness of $\lambda_\varphi(\mathscr{V})$.

Theorem 5. Let $\mathscr{V}$ be an array which is φ-suitable for some function $\varphi \in \Theta$ satisfying the Δ_2-condition and suppose that $N_s(\varepsilon)$ can be taken to be 1 for all $\varepsilon > 0$. Then $\mathscr{V} \in SB$ if and only if $\lambda_\varphi(\mathscr{V}) < \infty$.

Proof. Sufficiency. If $\lambda_\varphi(\mathscr{V}) < \infty$, then there exists a sequence $\{A_n, n \geqslant 1\} \subseteq \mathscr{X}$ such that

$$C \equiv \sup_{\substack{t \geqslant 0 \\ n \geqslant 1}} \varphi(t)\, P\{\|S_n - A_n\| > t\} < \infty.$$

Then

$$\lim_{\substack{t\to\infty\\ n\to\infty}} P\{\|S_n - A_n\| > t\} = \lim_{\substack{t\to\infty\\ n\to\infty}} \frac{\varphi(t)\, P\{\|S_n - A_n\| > t\}}{\varphi(t)} \leqslant \lim_{t\to\infty} \frac{C}{\varphi(t)} = 0$$

and so $\mathscr{V} \in SB$.

Necessity. If $\mathscr{V} \in SB$, then by Theorem 3(ii)

$$\lim_{t\to\infty} \sup_{\substack{u\geqslant 0\\ n\geqslant 1}} \varphi(u)\, P\{\|S_n - A_n\| > tu\} = 0$$

for some sequence $\{A_n, n \geqslant 1\} \subseteq \mathscr{X}$. Let $t_0 \geqslant 1$ be such that

$$\sup_{\substack{u\geqslant 0\\ n\geqslant 1}} \varphi(u)\, P\{\|S_n - A_n\| > t_0 u\} \leqslant 1. \tag{5.1}$$

Then

$$\begin{aligned}
\lambda_\varphi(\mathscr{V}) &\leqslant \sup_{\substack{u\geqslant 0\\ n\geqslant 1}} \varphi(u)\, P\{\|S_n - A_n\| > u\}\\
&= \sup_{\substack{u\geqslant 0\\ n\geqslant 1}} \varphi(t_0 u)\, P\{\|S_n - A_n\| > t_0 u\}\\
&\leqslant B \sup_{\substack{u\geqslant 0\\ n\geqslant 1}} \varphi(u)\, P\{\|S_n - A_n\| > t_0 u\}\\
&\qquad\qquad (\text{for some } B < \infty \text{ by } (2.1))\\
&\leqslant B \qquad (\text{by } (5.1))\\
&< \infty.
\end{aligned}$$

□

For a function $\varphi \in \Theta$ satisfying the Δ_2-condition, the space

$$SB_\varphi \equiv \{\mathscr{V} \in SB : \mathscr{V} \text{ is } \varphi\text{-suitable with } N_s(\varepsilon) = 1 \text{ for all } \varepsilon > 0\}$$

is also a linear space with the natural operations. According to Theorem 5,

$$SB_\varphi = \{\mathscr{V} : \lambda_\varphi(\mathscr{V}) < \infty \text{ and } \mathscr{V} \text{ is } \varphi\text{-suitable with } N_s(\varepsilon) = 1 \text{ for all } \varepsilon > 0\}.$$

It is easy to see that $\lambda_\varphi(\cdot)$ is a *quasi-norm* on SB_φ; that is, for $\mathscr{V}, \mathscr{W} \in SB_\varphi$:

(i) $0 \leqslant \lambda_\varphi(\mathscr{V}) < \infty$ and $\lambda_\varphi(\{0, k \geqslant 1, n \geqslant 1\}) = 0$,

(ii) $\lambda_\varphi(-\mathscr{V}) = \lambda_\varphi(\mathscr{V})$,

(iii) $\lambda_\varphi(\mathscr{V} + \mathscr{W}) \leqslant B(\lambda_\varphi(\mathscr{V}) + \lambda_\varphi(\mathscr{W}))$ for some constant $1 \leqslant B < \infty$.

Inequality (iii) is called the *B-triangle inequality* and the constant B can be taken to be that of (2.1) with $A = 2$.

If the array $\mathscr{V} = \{V_{nk}, k \geqslant 1, n \geqslant 1\}$ is such that $\lambda_\varphi(\mathscr{V}) = 0$, then for all $n \geqslant 1$

$$\inf_{A \in \mathscr{X}} \sup_{t \geqslant 0} \varphi(t)\, P\{\|S_n - A\| > t\} = 0$$

and it follows from Lemma 6 that $S_n = A_n$ a.s. for some $A_n \in \mathscr{X}$. We will now argue that this implies that $\mathscr{V}$ is an array of degenerate random elements. For suppose that V_{nK} is nondegenerate for some $n \geqslant 1$ and some $K \geqslant 1$. Since the sum of a finite number of independent random elements (at least one of which is nondegenerate) is nondegenerate, we have that $\sum_{k=1}^{K} V_{nk}$ is nondegenerate. But then for the same reason

$$A_n = S_n = \sum_{k=1}^{K} V_{nk} + \sum_{k=K+1}^{\infty} V_{nk} \quad \text{a.s.}$$

is nondegenerate, a contradiction. We have thus shown that

$$\begin{aligned} &\mathscr{V} \text{ is an array of degenerate random elements} \\ &\qquad \text{with convergent row sums if and only if } \lambda_\varphi(\mathscr{V}) = 0. \end{aligned} \tag{5.2}$$

Next, for $\mathscr{V}$ and $\mathscr{W}$ in SB_φ, let us declare $\mathscr{V} \sim \mathscr{W}$ if $\lambda_\varphi(\mathscr{V} - \mathscr{W}) = 0$. (In particular, $\lambda_\varphi(\mathscr{V}) = 0$ if and only if $\mathscr{V} \sim \{0, k \geqslant 1, n \geqslant 1\}$.) Then $\sim$ is an equivalence relation on SB_φ in view of (5.2). Moreover, if $\mathscr{V}, \mathscr{W} \in SB_\varphi$ with $\mathscr{V} \sim \mathscr{W}$, then (5.2) ensures that $\lambda_\varphi(\mathscr{V}) = \lambda_\varphi(\mathscr{W})$ whence the value of $\lambda_\varphi(\mathscr{V})$ does not depend on the particular representative $\mathscr{V}$ selected from an equivalence class of arrays determined by $\sim$.

A topology on SB_φ can be defined in the natural way: a basis for the topology is the collection

$$\big\{\{\mathscr{W} \in SB_\varphi : \lambda_\varphi(\mathscr{V} - \mathscr{W}) < \varepsilon\} : \mathscr{V} \in SB_\varphi, \varepsilon > 0\big\}.$$

By Lemma 3.10.1 of Bergh and Löfström,[1] p. 59, the space SB_φ is *metrizable*; that is, there is a nonnegative function $\lambda_\varphi^*(\cdot)$ on SB_φ with

$$\lambda_\varphi^*(\cdot) \leqslant \lambda_\varphi^\rho(\cdot) \leqslant 2\lambda_\varphi^*(\cdot)$$

where $\rho = (1 + \log B)^{-1}$ (Log denotes the logarithm to the base 2) such that

$$d_\varphi(\mathscr{V}, \mathscr{W}) \equiv \lambda_\varphi^*(\mathscr{V} - \mathscr{W})$$

is a metric which defines the topology on SB_φ. The function $\lambda_\varphi^*(\cdot)$ is defined by

$$\lambda_\varphi^*(\mathscr{V}) = \inf\left\{\sum_{j=1}^{n} \lambda_\varphi^\rho(\mathscr{V}_j) \colon \sum_{j=1}^{n} \mathscr{V}_j = \mathscr{V}, n \geqslant 1\right\}, \qquad \mathscr{V} \in SB_\varphi.$$

Since $\lambda_\varphi(\cdot)$ is constant on each equivalence class determined by $\sim$, it is easy to see that $\lambda_\varphi^*(\cdot)$ likewise enjoys this property.

Theorem 6. Let $\varphi \in \Theta$ satisfy the Δ_2-condition and let

$$WLLN_\varphi = \{\mathscr{V} \in WLLN : \mathscr{V} \text{ is } \varphi\text{-suitable with } N_s(\varepsilon) = 1 \text{ for all } \varepsilon > 0\}.$$

Then $WLLN_\varphi$ is a closed subspace of the metric space (SB_φ, d_φ) or, equivalently,

$$\{\mathscr{V} \in SB_\varphi : \forall \varepsilon > 0\ \exists \mathscr{W} \in WLLN_\varphi \text{ such that } d_\varphi(\mathscr{V}, \mathscr{W}) < \varepsilon\} \subseteq WLLN_\varphi.$$

Proof. Let $\mathscr{V} \in SB_\varphi$ be such that for all $\varepsilon > 0$, there exists $\mathscr{W} \in WLLN_\varphi$ such that $d_\varphi(\mathscr{V}, \mathscr{W}) < \varepsilon$. Then $\mathscr{V}$ is φ-suitable with $N_s(\varepsilon) = 1$ for all $\varepsilon > 0$ and it remains to show that $\mathscr{V} \in WLLN$. Let $\varepsilon > 0$ be arbitrary. Let $\mathscr{W} = \{W_{nk}, k \geqslant 1, n \geqslant 1\} \in WLLN_\varphi$ be such that

$$d_\varphi(\mathscr{V}, \mathscr{W}) < \left(\frac{\varepsilon}{2}\varphi\left(\frac{\varepsilon}{2}\right)\right)^\rho \Big/ 2. \tag{5.3}$$

Since $\mathscr{W} \in WLLN$,

$$\lim_{n\to\infty} P\left\{\left\|\sum_{k=1}^{\infty} W_{nk} - A_n'\right\| > \frac{\varepsilon}{2}\right\} = 0 \tag{5.4}$$

for some sequence $\{A_n', n \geqslant 1\} \subseteq \mathscr{X}$. Since

$$\begin{aligned}\lambda_\varphi(\mathscr{V} - \mathscr{W}) &\leqslant (2\lambda_\varphi^*(\mathscr{V} - \mathscr{W}))^{1/\rho}\\ &= (2d_\varphi(\mathscr{V}, \mathscr{W}))^{1/\rho}\\ &< \frac{\varepsilon}{2}\varphi\left(\frac{\varepsilon}{2}\right) \qquad \text{(by (5.3))},\end{aligned}$$

there exists a sequence $\{A''_n, n \geqslant 1\} \subseteq \mathscr{X}$ such that for all $t \geqslant 0$ and $n \geqslant 1$

$$\varphi(t)\, P\left\{\left\| S_n - \sum_{k=1}^{\infty} W_{nk} - A''_n \right\| > t\right\} < \frac{\varepsilon}{2}\,\varphi\left(\frac{\varepsilon}{2}\right). \tag{5.5}$$

Then setting $A_n = A'_n + A''_n$, $n \geqslant 1$, we have for all large n

$$\begin{aligned} P\{\|S_n - A_n\| > \varepsilon\} &\leqslant P\left\{\left\| S_n - \sum_{k=1}^{\infty} W_{nk} - A''_n \right\| > \frac{\varepsilon}{2}\right\} + P\left\{\left\| \sum_{k=1}^{\infty} W_{nk} - A'_n \right\| > \frac{\varepsilon}{2}\right\} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad \text{(by (5.4) and (5.5))} \\ &= \varepsilon. \end{aligned}$$

Thus $\mathscr{V} \in WLLN$. □

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