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On the weak laws for arrays of random variables ☆

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Abstract

The convergence in probability of the sequence of sums $\sum_{i=u_n}^{v_n}(X_{ni}-c_{ni})/b_n$ is obtained, where $\{u_n, n\geqslant 1\}$ and $\{v_n, n\geqslant 1\}$ are sequences of integers, $\{X_{ni}, u_n\leqslant i\leqslant v_n, n\geqslant 1\}$ are random variables, $\{c_{ni}, u_n\leqslant i\leqslant v_n, n\geqslant 1\}$ are constants or conditional expectations, and $\{b_n, n\geqslant 1\}$ are constants satisfying $b_n\to\infty$ as $n\to\infty$. The work is proved under a Cesàro-type condition which does not assume the existence of moments of X_{ni}. The current work extends that of Gut (1992, Statist. Probab. Lett. 14, 49–52), Hong and Oh (1995, Statist. Probab. Lett. 22, 52–57), Hong and Lee (1996, Bull. Inst. Math. Acad. Sinica 24, 205–209), and Sung (1998, Statist. Probab. Lett. 38, 10–105).

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1. Introduction

The classical weak law of large numbers (WLLN) says that if $\{X_n, n \geqslant 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables satisfying $nP(|X_1| > n) = o(1)$, then $\sum_{i=1}^{n}(X_i - EX_1 I(|X_1| \leqslant n))/n \to 0$ in probability as $n \to \infty$. The WLLN has been extended to the arrays of random variables or random elements (for random variables, see Hong and Oh (1995), Hong and Lee (1996), and Sung (1998), and for random elements, see Adler et al. (1997) and Hong et al. (2000)).

Let $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$, where $\{u_n \geqslant -\infty, n \geqslant 1\}$ and $\{v_n \leqslant +\infty, n \geqslant 1\}$ are two sequences of integers. Set $\mathscr{F}_{nj} = \sigma\{X_{ni}, u_n \leqslant i \leqslant j\}, u_n \leqslant j \leqslant v_n, n \geqslant 1$, and $\mathscr{F}_{n,u_n-1} = \{\emptyset, \Omega\}, n \geqslant 1$. When $u_n = 1, v_n = k_n$ ($k_n \to \infty$ as $n \to \infty$), $n \geqslant 1$, WLLNs for the array have been established by Hong and Oh (1995), Hong and Lee (1996), and Sung (1998).

In this paper, we obtain a WLLN for the more general array $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ under a Cesàro-type condition. Our work extends that of Gut (1992), Hong and Oh (1995), Hong and Lee (1996), and Sung (1998). Also, we obtain the convergence in probability of the sequence of (infinite) weighted sums $\sum_{i=1}^{\infty} a_{ni}(X_i - c_{ni})$, where $\{|a_{ni}|^r, 1 \leqslant i < \infty, n \geqslant 1\}$, $0 < r < 2$, is a Toeplitz array of constants, $\{X_n, n \geqslant 1\}$ is a sequence of random variables which is uniformly bounded by a random variable X with $aP(|X|^r > a) \to 0$ as $a \to \infty$, and $\{c_{ni}, 1 \leqslant i < \infty, n \geqslant 1\}$ is an array of constants or conditional expectations. The result generalizes and extends that of Jamison et al. (1965), Pruitt (1966), and Rohatgi (1971).

2. Main results

Throughout this section, let $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of random variables, and let $\{k_n, n \geqslant 1\}$ be a sequence of positive integers such that $k_n \to \infty$ as $n \to \infty$.

To prove our main results, we will need the following lemma.

Lemma 1. *Let $\{k_n, n \geqslant 1\}$ and $\{b_n, n \geqslant 1\}$ be sequences of positive constants such that $k_n \to \infty, b_n \to \infty$, and*

$$k_n / b_n^{\beta} \to 0 \quad \textit{for some } \beta > 0. \tag{1}$$

Suppose that there exists a positive nondecreasing function g on $[0, \infty)$ satisfying

$$\lim_{a \to 0} g(a) = 0, \quad \sum_{j=1}^{\infty} g^{\beta}(1/j) < \infty, \tag{2}$$

and

$$\frac{k_n}{b_n^{\beta}} \sum_{j=1}^{k_n - 1} \frac{g^{\beta}(j+1) - g^{\beta}(j)}{j} = O(1). \tag{3}$$

Let $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of random variables such that

$$\sup_{a > 0} \sup_{n \geqslant 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} aP(|X_{ni}| > g(a)) < \infty \tag{4}$$

and

$$\lim_{a\to\infty}\sup_{n\geqslant 1}\frac{1}{k_n}\sum_{i=u_n}^{v_n} aP(|X_{ni}|>g(a))=0. \tag{5}$$

Then

$$\sum_{i=u_n}^{v_n} E|X_{ni}|^{\beta} I(|X_{ni}|\leqslant g(k_n)) = \mathrm{o}(b_n^{\beta}).$$

Proof. Since g is a nondecreasing function, it follows that

$$\begin{aligned}
&\frac{1}{b_n^{\beta}}\sum_{i=u_n}^{v_n} E|X_{ni}|^{\beta} I(|X_{ni}|\leqslant g(k_n))\\
&= \frac{1}{b_n^{\beta}}\sum_{i=u_n}^{v_n} E|X_{ni}|^{\beta} I(|X_{ni}|\leqslant g(1)) + \frac{1}{b_n^{\beta}}\sum_{i=u_n}^{v_n}\sum_{j=2}^{k_n} E|X_{ni}|^{\beta} I(g(j-1)<|X_{ni}|\leqslant g(j)),\\
&=: A_n + B_n.
\end{aligned}$$

By (1), (2), and (4), we have

$$\begin{aligned}
A_n &= \frac{1}{b_n^{\beta}}\sum_{i=u_n}^{v_n}\sum_{j=1}^{\infty} E|X_{ni}|^{\beta} I(g(1/(j+1))<|X_{ni}|\leqslant g(1/j))\\
&\leqslant \frac{1}{b_n^{\beta}}\sum_{i=u_n}^{v_n}\sum_{j=1}^{\infty} g^{\beta}(1/j) P(g(1/(j+1))<|X_{ni}|\leqslant g(1/j))\\
&= \frac{1}{b_n^{\beta}}\sum_{i=u_n}^{v_n}\left[\sum_{j=2}^{\infty}(g^{\beta}(1/(j-1)) - g^{\beta}(1/j))P(|X_{ni}|>g(1/j)) - g^{\beta}(1)P(|X_{ni}|>g(1))\right]\\
&\leqslant \frac{1}{b_n^{\beta}}\sum_{i=u_n}^{v_n}\sum_{j=2}^{\infty}(g^{\beta}(1/(j-1)) - g^{\beta}(1/j))P(|X_{ni}|>g(1/j))\\
&\leqslant \frac{k_n}{b_n^{\beta}}\sum_{j=2}^{\infty} j(g^{\beta}(1/(j-1)) - g^{\beta}(1/j))\sup_{n\geqslant 1}\left\{\frac{1}{k_n}\sum_{i=u_n}^{v_n}\frac{1}{j}P(|X_{ni}|>g(1/j))\right\}\\
&\leqslant \frac{k_n}{b_n^{\beta}}\left(g^{\beta}(1)+\sum_{j=1}^{\infty} g^{\beta}(1/j)\right)\cdot\sup_{a>0}\sup_{n\geqslant 1}\left\{\frac{1}{k_n}\sum_{i=u_n}^{v_n} aP(|X_{ni}|>g(a))\right\}\to 0.
\end{aligned}$$

The proof that $\lim_{n\to\infty} B_n = 0$ can be obtained by the slight modification of the last part in the proof of Hong et al. (2000) (from the middle of p. 180 to the end of the proof) with changes: $||V_{ni}|| = |X_{ni}|$ and $\sum_{j=1}^{k_n} = \sum_{j=u_n}^{v_n}$. □

Remark 1. When $u_n = 1, v_n = k_n, n\geqslant 1$, condition (5) was introduced by Hong and Oh (1995). In this case, condition (5) implies condition (4), and so condition (4) can be omitted.

Now we state and prove one of our main results.

Theorem 1. *Let* $0<\beta\leqslant 2$. *Under the same conditions of Lemma* 1,

$$\sum_{i=u_n}^{v_n}(X_{ni}-c_{ni})/b_n \to 0 \textit{ in probability,}$$

where $c_{ni}=0$ *if* $0<\beta\leqslant 1$ *and* $c_{ni}=E(X_{ni}I(|X_{ni}|\leqslant g(k_n))|\mathscr{F}_{n,i-1})$ *if* $1<\beta\leqslant 2$.

Proof. Let $X'_{ni}=X_{ni}I(|X_{ni}|\leqslant g(k_n))$ for $u_n\leqslant i\leqslant v_n, n\geqslant 1$. By the same arguments as the proof of Sung (1998), it suffices to show that

$$\sum_{i=u_n}^{v_n}(X'_{ni}-c_{ni})/b_n \to 0 \text{ in probability.}$$

First, we consider the case of $0<\beta\leqslant 1$. Since $c_{ni}=0$, it follows by the Markov's inequality and Lemma 1 that

$$P\left(\left|\sum_{i=u_n}^{v_n}(X'_{ni}-c_{ni})\right|\bigg/ b_n>\varepsilon\right)\leqslant E\left|\sum_{i=u_n}^{v_n}X'_{ni}\right|^{\beta}/(\varepsilon^{\beta}b_n^{\beta})\leqslant \sum_{i=u_n}^{v_n}E|X'_{ni}|^{\beta}/(\varepsilon^{\beta}b_n^{\beta})\to 0.$$

Now we consider the case of $1<\beta\leqslant 2$. In this case, $X'_{ni}-c_{ni}, u_n\leqslant i\leqslant v_n$, form a martingale difference sequence. Using Burkholder's (1966) inequality, Jensen's inequality, and Lemma 1, we get

$$\begin{aligned}
P\left(\frac{|\sum_{i=u_n}^{v_n}(X'_{ni}-c_{ni})|}{b_n}>\varepsilon\right) &\leqslant \frac{1}{\varepsilon^{\beta}b_n^{\beta}}E\left|\sum_{i=u_n}^{v_n}(X'_{ni}-c_{ni})\right|^{\beta}\\
&\leqslant \frac{C_\beta}{\varepsilon^{\beta}b_n^{\beta}}\sum_{i=u_n}^{v_n}E|X'_{ni}-c_{ni}|^{\beta}\\
&\leqslant \frac{C_\beta 2^{\beta-1}}{\varepsilon^{\beta}b_n^{\beta}}\sum_{i=u_n}^{v_n}E|X'_{ni}|^{\beta}+E|c_{ni}|^{\beta}\\
&\leqslant \frac{C_\beta 2^{\beta}}{\varepsilon^{\beta}b_n^{\beta}}\sum_{i=u_n}^{v_n}E|X'_{ni}|^{\beta}\to 0,
\end{aligned}$$

where C_β is a constant depending only on β. □

Note that condition (3) can be difficult to check. A sufficient condition for (3) was given by Hong et al. (2000) as follows:

$$\frac{g(k_n)}{b_n}=\mathrm{O}(1) \quad \text{and} \quad \sum_{j=1}^{k_n}\frac{g^{\beta}(j)}{j^2}=\mathrm{O}\left(\frac{b_n^{\beta}}{k_n}\right). \tag{6}$$

Corollary 1. *Let* $0<r<2$ *and* $k_n\to\infty$ *as* $n\to\infty$. *Let* $\{X_{ni}, u_n\leqslant i\leqslant v_n, n\geqslant 1\}$ *be an array of random variables such that*

$$\sup_{a>0}\sup_{n\geqslant 1}\frac{1}{k_n}\sum_{i=u_n}^{v_n}aP(|X_{ni}|^r>a)<\infty \tag{7}$$

and

$$\lim_{a\to\infty}\sup_{n\geqslant 1}\frac{1}{k_n}\sum_{i=u_n}^{v_n} aP(|X_{ni}|^r>a)=0. \tag{8}$$

Then

$$\sum_{i=u_n}^{v_n}(X_{ni}-c_{ni})/k_n^{1/r}\to 0 \text{ in probability},$$

where $c_{ni}=0$ *if* $0<r<1$ *and* $c_{ni}=E(X_{ni}I(|X_{ni}|^r\leqslant k_n)|\mathscr{F}_{n,i-1})$ *if* $1\leqslant r<2$.

Proof. Let $g(t)=t^{1/r}$ and $b_n=k_n^{1/r}$. Take $\beta=1$ if $0<r<1$ and $\beta=2$ if $1\leqslant r<2$. Then conditions (1), (2), (4), and (5) are clearly satisfied. By Theorem 1, it remains to show that condition (3) holds. To prove (3), it is enough to show that condition (6) holds. Clearly $g(k_n)/b_n=\mathrm{O}(1)$. Since $\beta/r>1$, it follows that

$$\sum_{j=1}^{k_n}\frac{g^{\beta}(j)}{j^2}=\sum_{j=1}^{k_n} j^{(\beta/r)-2}=\mathrm{O}(k_n^{(\beta/r)-1}).$$

Thus condition (6) is satisfied. □

Remark 2. When $u_n=1, v_n=k_n, n\geqslant 1$, Hong and Oh (1995) proved Corollary 1 for $1\leqslant r<2$. For the case $0<r<1$, they used a centering sequence whereas we have no centering.

Corollary 2. *Let* $\{X_{ni}, 1\leqslant i\leqslant k_n, n\geqslant 1\}$, $k_n\to\infty$ *as* $n\to\infty$, *be an array of random variables. Let* $\{d_n, n\geqslant 1\}$ *be a nondecreasing sequence of positive constants such that*

$$\frac{n}{d_n^{\beta}}\to 0 \quad\text{and}\quad \frac{k_n}{d_{k_n}^{\beta}}\sum_{j=1}^{k_n-1}\frac{d_{j+1}^{\beta}-d_j^{\beta}}{j}=\mathrm{O}(1) \tag{9}$$

for some $0<\beta\leqslant 2$. *Suppose that*

$$\lim_{j\to\infty}\sup_{n\geqslant 1}\frac{1}{k_n}\sum_{i=1}^{k_n} jP(|X_{ni}|>d_j)=0.$$

Then

$$\sum_{i=1}^{k_n}(X_{ni}-c_{ni})/d_{k_n}\to 0 \text{ in probability},$$

where $c_{ni}=0$ *if* $0<\beta\leqslant 1$ *and* $c_{ni}=E(X_{ni}I(|X_{ni}|\leqslant d_{k_n})|\mathscr{F}_{n,i-1})$ *if* $1<\beta\leqslant 2$.

Proof. Let $g(t)$ be a nondecreasing function such that $g(t)=0$ for $0\leqslant t<1$ and $g(n)=d_n$ for $n\geqslant 1$. Let $u_n=1, v_n=k_n, b_n=d_{k_n}, n\geqslant 1$. Then conditions (1)–(5) are clearly satisfied, and so the proof follows from Theorem 1. □

Remark 3. When $d_n/n^{\alpha}\uparrow$ for some $\alpha>\frac{1}{2}$, condition (9) is satisfied for $\beta=2$ by Lemma 2 of Sung (1998). In this case, Sung (1998) proved Corollary 2.

If $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ is a bounded array of constants, then Theorem 1 can be extended to weighted sums.

Theorem 2. *Let* $0 < \beta \leqslant 2$. *Let* $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ *be a bounded array of constants, i.e.,* $\sup_{n\geqslant 1} \sup_{u_n \leqslant i \leqslant v_n} |a_{ni}| < \infty$. *Under the same conditions of Lemma* 1,

$$\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - c_{ni})/b_n \to 0 \text{ in probability},$$

where $c_{ni} = 0$ *if* $0 < \beta \leqslant 1$ *and* $c_{ni} = E(X_{ni} I(|X_{ni}| \leqslant g(k_n))|\mathscr{F}_{n,i-1})$ *if* $1 < \beta \leqslant 2$.

The proof is similar to that of Theorem 1 and is omitted.

The following theorem gives a result of convergence in probability for (infinite) weighted sums of random variables.

Theorem 3. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of random variables which is uniformly bounded by a random variable* X *such that* $aP(|X|^r > a) \to 0$ *as* $a \to \infty$ *for some* $0 < r < 2$. *Let* $\{|a_{ni}|^r, 1 \leqslant i < \infty, n \geqslant 1\}$ *be a Toeplitz array of constants, i.e.,*

$$\lim_{n\to\infty} a_{ni} = 0 \text{ for every } i \tag{10}$$

and

$$\sup_{n\geqslant 1} \sum_{i=1}^{\infty} |a_{ni}|^r < C \text{ for some constant } C > 0. \tag{11}$$

If $\sup_{i\geqslant 1} |a_{ni}| \to 0$ *as* $n \to \infty$, *then*

$$\sum_{i=1}^{\infty} a_{ni}(X_i - c_{ni}) \to 0 \text{ in probability},$$

where $c_{ni} = 0$ *if* $0 < r < 1$ *and* $c_{ni} = E(X_i I(|a_{ni} X_i|^r \leqslant 1)|\mathscr{F}_{i-1})$ *if* $1 \leqslant r < 2(\mathscr{F}_n = \sigma\{X_i, 1 \leqslant i \leqslant n\}$ *and* $\mathscr{F}_0 = \{\emptyset, \Omega\})$.

Proof. Since $aP(|X|^r > a) \to 0$ as $a \to \infty$, there exists a_0 such that $aP(|X|^r > a) < \varepsilon$ if $a > a_0$. It follows that

$$\sup_{a>0} aP(|X|^r > a) \leqslant \sup_{0<a\leqslant a_0} aP(|X|^r > a) + \sup_{a>a_0} aP(|X|^r > a) \leqslant a_0 + \varepsilon. \tag{12}$$

Let $k_n = 1/\sup_{i\geqslant 1} |a_{ni}|^r, u_n = 1, v_n = \infty, X_{ni} = k_n^{1/r} a_{ni} X_i$, and $b_n = k_n^{1/r}$. Then we have by (11) and (12) that

$$\begin{aligned}
\sup_{n\geqslant 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} aP(|X_{ni}|^r > a) &\leqslant \sup_{n\geqslant 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} aP\left(|X|^r > \frac{a}{k_n |a_{ni}|^r}\right), \\
&= \sup_{n\geqslant 1} \sum_{i=u_n}^{v_n} |a_{ni}|^r \frac{a}{k_n |a_{ni}|^r} P\left(|X|^r > \frac{a}{k_n |a_{ni}|^r}\right), \\
&\leqslant (a_0 + \varepsilon) \sup_{n\geqslant 1} \sum_{i=1}^{\infty} |a_{ni}|^r \leqslant (a_0 + \varepsilon)C.
\end{aligned}$$

So (7) is satisfied. If $a > a_0$, then $a/k_n|a_{ni}|^r \geqslant a > a_0$ for all n and i. For $a > a_0$, we get

$$\sup_{n\geqslant 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} aP(|X_{ni}|^r > a) \leqslant \sup_{n\geqslant 1} \sum_{i=u_n}^{v_n} |a_{ni}|^r \frac{a}{k_n|a_{ni}|^r} P\left(|X|^r > \frac{a}{k_n|a_{ni}|^r}\right),$$
$$\leqslant \varepsilon \sup_{n\geqslant 1} \sum_{i=1}^{\infty} |a_{ni}|^r \leqslant \varepsilon C.$$

So (8) is satisfied. Hence the proof follows from Corollary 1. □

Pruitt (1966) proved Theorem 3 for $r = 1$ under the stronger condition that $\{X_n, n\geqslant 1\}$ is a sequence of i.i.d. random variables with $E|X_1| < \infty$. Rohatgi (1971) proved Theorem 3 for $0 < r \leqslant 1$ under the stronger condition that $\{X_n, n\geqslant 1\}$ is a sequence of independent random variables which is uniformly bounded by a random variable X with $E|X|^r < \infty$. Note that the condition $aP(|X|^r > a) \to 0$ as $a \to \infty$ is weaker than $E|X|^r < \infty$. Hence Theorem 3 extends the results of Pruitt (1966) and Rohatgi (1971).

The following corollary was proved by Jamison et al. (1965).

Corollary 3. *Let $\{X_n, n\geqslant 1\}$ be i.i.d. random variables such that*

$$\lim_{a\to\infty} aP(|X_1| > a) = 0 \quad \text{and} \quad \lim_{a\to\infty} EX_1 I(|X_1| \leqslant a) = \mu. \tag{13}$$

Let $\{a_n, n\geqslant 1\}$ be a sequence of positive constants satisfying

$$\max_{1\leqslant i\leqslant n} a_i \Big/ \sum_{i=1}^{n} a_i \to 0. \tag{14}$$

Then $\sum_{i=1}^n a_i X_i / \sum_{i=1}^n a_i \to \mu$ in probability.

Proof. Let $a_{ni} = a_i / \sum_{i=1}^n a_i$ for $1 \leqslant i \leqslant n, n\geqslant 1$. Then $\{a_{ni}, 1\leqslant i\leqslant n, n\geqslant 1\}$ is a Toeplitz array. By Theorem 3 with $r = 1$,

$$\frac{\sum_{i=1}^n a_i (X_i - EX_1 I(|X_1| \leqslant 1/a_{ni}))}{\sum_{i=1}^n a_i} \to 0 \text{ in probability.}$$

Since $1/a_{ni} \geqslant \sum_{i=1}^n a_i / \max_{1\leqslant i\leqslant n} a_i \to \infty$ as $n \to \infty$, we have by (13) that $EX_1 I(|X_1| \leqslant 1/a_{ni}) \to \mu$ as $n \to \infty$ uniformly in i, which implies $\sum_{i=1}^n a_i EX_1 I(|X_1| \leqslant 1/a_{ni}) / \sum_{i=1}^n a_i \to \mu$. Thus the proof is complete. □

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