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ON COMPLETE CONVERGENCE FOR ARRAYS OF ROWWISE NEGATIVELY ASSOCIATED RANDOM VARIABLES[1)]

С помощью методов, развитых в [4], мы устанавливаем результат о полной сходимости в схеме серий построчно отрицательно ассоциированных случайных величин, который обобщает результаты работ [2], [1] и [8].

Ключевые слова и фразы: отрицательно ассоциированные случайные величины, схема серий построчно отрицательно ассоциированных случайных величин, полная сходимость.

The paper [4] provides sufficient conditions for complete convergence for sums of arrays of rowwise independent random variables. The wide majority of results on complete convergence known in literature could be derived from the main result of [4]. In that paper one can find also a short history of the complete convergence notion and the corresponding references. The purpose of the present investigation is to prove by the methods developed in [4] that the main results of [2], [1], and [8] remain true for negatively associated random variables.

In this paper we assume that $\{X_{nk},\ 1 \leqslant k \leqslant k_n,\ n \geqslant 1\}$ is an array of random variables defined on a probability space $(\Omega, \mathscr{F}, \mathbf{P})$, $\{k_n,\ n \geqslant 1\}$ is a sequence of positive integers such that $\lim_{n\to\infty} k_n = \infty$, and $\{c_n,\ n \geqslant 1\}$ is a sequence of positive constants. For an event $A \in \mathscr{F}$ we denote by $I\{A\}$ the indicator function. We note that all the results of this paper remain true in the case $k_n = \infty$ for some/all $n \geqslant 1$, provided the series $\sum_{k=1}^{\infty} X_{nk}$ converges almost surely. Certainly, we should consider sup instead of max in the case of infinite sums. In what follows, C always stands for a positive constant which may differ from one place to another.

Recall that a finite family of random variables $\{X_i,\ 1 \leqslant i \leqslant n\}$ is said to be *negatively associated* (abbreviated to NA in what follows) if for any disjoint subsets A and B of $\{1, 2, \ldots, n\}$ and any real coordinate-wise nondecreasing scalar functions f on $\mathbf{R}^A$ and g on $\mathbf{R}^B$,

$$\operatorname{cov}\ (f(X_i, i \in A),\ g(X_j, j \in B)) \leqslant 0$$

whenever the covariance exists. An infinite family of random variables $\{X_i, i \geqslant 1\}$ is NA if every finite subfamily is NA.

This concept was introduced in [3]. The authors of [3] also pointed out and proved in their paper that a number of well-known multivariate distributions possess the NA property. The following exponential inequality of Kolmogorov's type is proved in [7].

Lemma 1. *Let $\{X_i,\ 1 \leqslant i \leqslant n\}$ be a sequence of NA mean zero random variables with $\mathbf{E}|X_i|^2 < \infty$ for every $1 \leqslant i \leqslant n$ and let*

$$B_n = \sum_{i=1}^{n} \mathbf{E} X_i^2.$$

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Then for all $\varepsilon > 0$ *and* $a > 0$

$$\mathbf{P}\Big\{\max_{1\leqslant k\leqslant n}\Big|\sum_{i=1}^{k}X_i\Big|\geqslant\varepsilon\Big\}\leqslant 2\mathbf{P}\Big\{\max_{1\leqslant i\leqslant n}|X_i|>a\Big\}$$
$$+4\exp\Big\{-\frac{\varepsilon^2}{4(a\varepsilon+B_n)}\Big[1+\frac{2}{3}\ln\Big(1+\frac{a\varepsilon}{B_n}\Big)\Big]\Big\}.$$

Our main result, which could be considered as a partial generalization of Theorem 1 in [4], is the following.

Theorem 1. *Let* $\{X_{nk},\ 1\leqslant k\leqslant k_n,\ n\geqslant 1\}$ *be an array of rowwise NA random variables. Suppose that for every* $\varepsilon > 0$ *and some* $\delta > 0$

(i) $\sum_{n=1}^{\infty}c_n\sum_{i=1}^{k_n}\mathbf{P}\{|X_{ni}|>\varepsilon\}<\infty$,

(ii) *there exists* $j\geqslant 1$ *such that* $\sum_{n=1}^{\infty}c_n(\sum_{i=1}^{k_n}\mathbf{D}X_{ni}\,I\{|X_{ni}|\leqslant\delta\})^j<\infty$. *Then*

$$\sum_{n=1}^{\infty}c_n\mathbf{P}\Big\{\max_{1\leqslant m\leqslant k_n}\Big|\sum_{k=1}^{m}[X_{nk}-\mathbf{E}X_{nk}I\{|X_{nk}|\leqslant\delta\}]\Big|>\varepsilon\Big\}<\infty$$

for any $\varepsilon > 0$.

P r o o f. Let $\{Y_{nk},\ 1\leqslant k\leqslant k_n,\ n\geqslant 1\}$ be a *monotone truncation* of $\{X_{nk},\ 1\leqslant k\leqslant k_n,\ n\geqslant 1\}$, namely, $Y_{nk}=X_{nk}I\{|X_{nk}|\leqslant\delta\}+\delta\,I\{X_{nk}>\delta\}-\delta\,I\{X_{nk}<-\delta\}$, where $\delta>0$ and $1\leqslant k\leqslant k_n,\ n\geqslant 1$. Note that by Property 6 in [3] (applied twice) we can conclude that $\{Y_{nk}-\mathbf{E}Y_{nk},\ 1\leqslant k\leqslant k_n,\ n\geqslant 1\}$ is an array of rowwise NA random variables.

For $n\geqslant 1$ and $1\leqslant m\leqslant k_n$ let

$$T_m=\sum_{k=1}^{m}Y_{nk},\quad S_m=\sum_{k=1}^{m}X_{nk},$$
$$S'_m=\sum_{k=1}^{m}X_{nk}\,I\{|X_{nk}|\leqslant\delta\},\quad Z_m=\sum_{k=1}^{m}(\delta\,I\{X_{nk}>\delta\}-\delta\,I\{X_{nk}<-\delta\}),$$
$$A=\bigcap_{k=1}^{k_n}\{X_{nk}=X_{nk}\,I\{|X_{nk}|\leqslant\delta\}\},\quad A^c=\bigcup_{k=1}^{k_n}\{X_{nk}\neq X_{nk}\,I\{|X_{nk}|\leqslant\delta\}\}.$$

Note that for any $n\geqslant 1$

$$\mathbf{P}\Big\{\max_{1\leqslant m\leqslant k_n}|S_m-\mathbf{E}S'_m|>\varepsilon\Big\}=\mathbf{P}\Big\{\Big\{\max_{1\leqslant m\leqslant k_n}|S_m-\mathbf{E}S'_m|>\varepsilon\Big\}\bigcap A^c\Big\}$$
$$+\mathbf{P}\Big\{\Big\{\max_{1\leqslant m\leqslant k_n}|S'_m-\mathbf{E}S'_m|>\varepsilon\Big\}\bigcap A\Big\}$$
$$\leqslant\sum_{k=1}^{k_n}\mathbf{P}\{|X_{nk}|>\delta\}+\mathbf{P}\Big\{\max_{1\leqslant m\leqslant k_n}|S'_m-\mathbf{E}S'_m|>\varepsilon\Big\}.$$

The last term can be estimated in the following way:

$$\mathbf{P}\Big\{\max_{1\leqslant m\leqslant k_n}|S'_m-\mathbf{E}S'_m|>\varepsilon\Big\}=\mathbf{P}\Big\{\max_{1\leqslant m\leqslant k_n}|(T_m-\mathbf{E}T_m)+(Z_m-\mathbf{E}Z_m)|>\varepsilon\Big\}$$
$$\leqslant\mathbf{P}\Big\{\max_{1\leqslant m\leqslant k_n}|T_m-\mathbf{E}T_m|>\frac{\varepsilon}{2}\Big\}+\mathbf{P}\Big\{\max_{1\leqslant m\leqslant k_n}|Z_m-\mathbf{E}Z_m|>\frac{\varepsilon}{2}\Big\}.$$

By the Markov inequality $\mathbf{P}\{\max_{1\leqslant m\leqslant k_n}|Z_m-\mathbf{E}Z_m|>\varepsilon/2\}\leqslant C\sum_{k=1}^{k_n}\mathbf{P}\{|X_{nk}|>\delta\}$. By (i) it is enough to prove that

$$\sum_{n=1}^{\infty}c_n\mathbf{P}\Big\{\max_{1\leqslant m\leqslant k_n}|T_m-\mathbf{E}T_m|>\varepsilon\Big\}<\infty.$$

Let $B_n = \sum_{k=1}^{k_n} \mathbf{D}(Y_{nk})$. For any $\varepsilon > 0$ and $a > 0$ set $d = \min\{1, a\varepsilon/(18\delta^2),\ a/(12\delta)\}$,

$$\mathbf{N}_1 = \{n\colon B_n > a\varepsilon\}, \quad \mathbf{N}_2 = \left\{n\colon \sum_{k=1}^{k_n} \mathbf{P}\left\{|X_{nk}| > \min\left\{\delta,\ \frac{a}{3}\right\}\right\} > d\right\},$$

$$\mathbf{N}_3 = \left\{n\colon \sum_{k=1}^{k_n} \mathbf{D}X_{nk}\, I\{|X_{nk}| \leqslant \delta\} > \frac{a\varepsilon}{2}\right\}, \quad \mathbf{N}_4 = \mathbf{N} - (\mathbf{N}_2 \cup \mathbf{N}_3).$$

We can state that $\mathbf{N}_1 \subseteq \mathbf{N}_2 \cup \mathbf{N}_3$, because

$$\begin{aligned} B_n &= \sum_{k=1}^{k_n} \Big[\mathbf{D}X_{nk}\, I\{|X_{nk}| \leqslant \delta\} + \mathbf{D}(\delta\, I\{X_{nk} > \delta\} - \delta\, I\{X_{nk} < -\delta\}) \\ &\qquad + 2\operatorname{cov}(X_{nk}\, I\{|X_{nk}| \leqslant \delta\},\ \delta\, I\{X_{nk} > \delta\} - \delta\, I\{X_{nk} < -\delta\})\Big] \\ &\leqslant \sum_{k=1}^{k_n} \mathbf{D}X_{nk}\, I\{|X_{nk}| \leqslant \delta\} + 9\delta^2 \sum_{k=1}^{k_n} \mathbf{P}\{|X_{nk}| > \delta\}. \end{aligned}$$

Note that

$$\begin{aligned} &\sum_{n\in\mathbf{N}_2\cup\mathbf{N}_3} c_n \mathbf{P}\Big\{\max_{1\leqslant m\leqslant k_n} |T_m - \mathbf{E}T_m| > \varepsilon\Big\} \leqslant \sum_{n\in\mathbf{N}_2\cup\mathbf{N}_3} c_n \\ &\quad \leqslant \frac{1}{d} \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \mathbf{P}\{|X_{nk}| > \delta\} + \frac{2^j}{(a\varepsilon)^j} \sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} \mathbf{D}X_{nk}\, I\{|X_{nk}| \leqslant \delta\}\right)^j < \infty. \end{aligned}$$

Hence it suffices to prove that $\sum_{n\in\mathbf{N}_4} c_n \mathbf{P}\{\max_{1\leqslant m\leqslant k_n} |\sum_{k=1}^{m}(Y_{nk} - \mathbf{E}Y_{nk})| > \varepsilon\} < \infty$. By Lemma 1 we have

$$\begin{aligned} &\sum_{n\in\mathbf{N}_4} c_n \mathbf{P}\left\{\max_{1\leqslant m\leqslant k_n} \left|\sum_{k=1}^{m}(Y_{nk} - \mathbf{E}Y_{nk})\right| > \varepsilon\right\} \\ &\quad \leqslant \sum_{n\in\mathbf{N}_4} c_n \Big[2\mathbf{P}\Big\{\max_{1\leqslant k\leqslant k_n} |Y_{nk} - \mathbf{E}Y_{nk}| > a\Big\} \\ &\qquad\qquad + 4\exp\left\{-\frac{\varepsilon^2}{4(a\varepsilon + B_n)}\left[1 + \frac{2}{3}\ln\left(1 + \frac{a\varepsilon}{B_n}\right)\right]\right\}\Big]. \end{aligned}$$

Note that for any $n \in \mathbf{N}_4$

$$\begin{aligned} &\max_{1\leqslant k\leqslant k_n} |\mathbf{E}Y_{nk}| \leqslant \max_{1\leqslant k\leqslant k_n} \mathbf{E}|Y_{nk}| \\ &\quad \leqslant \max_{1\leqslant k\leqslant k_n} \left(\delta\mathbf{P}\{|X_{nk}| > \delta\} + \mathbf{E}|X_{nk}|\, I\left\{|X_{nk}| \leqslant \frac{a}{3}\right\} + \mathbf{E}|X_{nk}|\, I\left\{\frac{a}{3} < |X_{nk}| \leqslant \delta\right\}\right) \\ &\quad \leqslant 2\delta \sum_{k=1}^{k_n} \mathbf{P}\left\{|X_{nk}| > \min\left\{\frac{a}{3},\ \delta\right\}\right\} + \frac{a}{3} \leqslant 2\delta d + \frac{a}{3} \leqslant \frac{a}{2} \end{aligned}$$

(the last but one inequality holds, because $n \notin \mathbf{N}_2$). This implies that, for any $n \in \mathbf{N}_4$, $\max_{1\leqslant k\leqslant k_n} |\mathbf{E}Y_{nk}| \leqslant a/2$ and

$$\begin{aligned} &\sum_{n\in\mathbf{N}_4} c_n \mathbf{P}\Big\{\max_{1\leqslant k\leqslant k_n} |Y_{nk} - \mathbf{E}Y_{nk}| > a\Big\} \leqslant \sum_{n=1}^{\infty} c_n \mathbf{P}\left\{\max_{1\leqslant k\leqslant k_n} |Y_{nk}| > \frac{a}{2}\right\} \\ &\quad \leqslant \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} \mathbf{P}\left\{|X_{nk}| > \min\left\{\delta,\ \frac{a}{2}\right\}\right\} < \infty \quad \text{(by (i))}. \end{aligned}$$

Therefore it is sufficient to prove that

$$A = \sum_{n\in \mathbf{N}_4} c_n \exp\left\{ -\frac{\varepsilon^2}{4(a\varepsilon + B_n)} \left[1+\frac{2}{3}\ \ln\left(1+\frac{a\varepsilon}{B_n}\right)\right]\right\} < \infty.$$

When $n \in \mathbf{N}_4$, $B_n \leqslant a\varepsilon$, and $\sum_{k=1}^{k_n} \mathbf{P}\{|X_{nk}| > \delta\} \leqslant 1$. Let $a = \varepsilon/(12j)$. We have

$$\begin{aligned} A &\leqslant \exp\left\{-\frac{3}{2}j\right\} \sum_{n\in\mathbf{N}_4} c_n \exp\left\{-j\ln\frac{B_n + a\varepsilon}{B_n}\right\} \\ &= C\sum_{n\in\mathbf{N}_4} c_n\left(\frac{B_n}{B_n+a\varepsilon}\right)^j \leqslant C\sum_{n\in\mathbf{N}_4} c_n\left(\frac{B_n}{a\varepsilon}\right)^j = C\sum_{n\in\mathbf{N}_4} c_n (B_n)^j \\ &\leqslant C\sum_{n\in\mathbf{N}_4} c_n\left[9\delta^2\sum_{k=1}^{k_n}\mathbf{P}\{|X_{nk}|>\delta\} + \sum_{k=1}^{k_n}\mathbf{D}X_{nk}\, I\{|X_{nk}|\leqslant\delta\}\right]^j \\ &\leqslant C\sum_{n\in\mathbf{N}_4} c_n\left\{(9\delta^2)^j\left[\sum_{k=1}^{k_n}\mathbf{P}\{|X_{nk}|>\delta\}\right]^j + \left[\sum_{k=1}^{k_n}\mathbf{D}X_{nk}\, I\{|X_{nk}|\leqslant\delta\}\right]^j\right\} \\ &\leqslant C\sum_{n\in\mathbf{N}_4} c_n\left\{(9\delta^2)^j\sum_{k=1}^{k_n}\mathbf{P}\{|X_{nk}|>\delta\} + \left[\sum_{k=1}^{k_n}\mathbf{D}X_{nk}\, I\{|X_{nk}|\leqslant\delta\}\right]^j\right\} < \infty \end{aligned}$$

by the assumptions. The proof is completed.

In the following two corollaries of Theorem 1 we assume that $\{X_{nk},\ 1 \leqslant k \leqslant k_n,\ n \geqslant 1\}$ is an array of rowwise NA random variables such that conditions (i) and (ii) are satisfied. Corollary 1 shows that the main result of [2] and [8] remains true for negative associated random variables.

Corollary 1. *If* $\sum_{i=1}^{k_n} \mathbf{E}X_{ni}\, I\{|X_{ni}| \leqslant \delta\} \longrightarrow 0$ *as* $n \to \infty$, *then*

$$\sum_{n=1}^{\infty} c_n \mathbf{P}\left\{\left|\sum_{k=1}^{k_n} X_{nk}\right| > \varepsilon\right\} < \infty \qquad \text{for any} \quad \varepsilon > 0.$$

Corollary 2 shows that the main result of [1] remains true for negatively associated random variables.

Corollary 2. *If* $\max_{1\leqslant m\leqslant k_n} |\sum_{i=1}^{m} \mathbf{E}X_{ni} I\{|X_{ni}| \leqslant \delta\}| \longrightarrow 0$ *as* $n \to \infty$, *then* $\sum_{n=1}^{\infty} c_n \mathbf{P}\{\max_{1\leqslant m\leqslant k_n} |\sum_{k=1}^{m} X_{nk}| > \varepsilon\} < \infty$ *for any* $\varepsilon > 0$.

Similar argument as in [2, Remark 2] gives the following result.

Corollary 3. *Let* $\{X_{ni},\ 1 \leqslant i \leqslant k_n,\ n \geqslant 1\}$ *be an array of rowwise NA random variables with* $\mathbf{E}X_{nk} = 0$ *for all* $n \geqslant 1$ *and* $1 \leqslant k \leqslant k_n$. *Let* $\phi(x)$ *be a real function such that for some* $\delta > 0$

$$\sup_{x>\delta} \frac{x}{\phi(x)} < \infty \quad \text{and} \quad \sup_{0\leqslant x\leqslant\delta} \frac{x^2}{\phi(x)} < \infty.$$

Suppose that for all $\varepsilon > 0$, *conditions* (i) *and*

(ii)$'$ *there exists* $j \geqslant 1$ *such that* $\sum_{n=1}^{\infty} c_n (\sum_{k=1}^{k_n} \mathbf{E}\phi(|X_{nk}|))^j < \infty$ *and* $\sum_{k=1}^{k_n} \mathbf{E}\phi(|X_{nk}|) \to 0$ *as* $n \to \infty$ *are satisfied. Then*

$$\sum_{n=1}^{\infty} c_n \mathbf{P}\left\{\max_{1\leqslant m\leqslant k_n}\left|\sum_{k=1}^{m} X_{nk}\right| > \varepsilon\right\} < \infty$$

for any $\varepsilon > 0$.

P r o o f. We show that the conditions of Corollary 2 are satisfied.

For (ii) note that

$$\sum_{k=1}^{k_n} \mathbf{E}X_{nk}^2\, I\{|X_{nk}| \leqslant \delta\} \leqslant \sup_{0\leqslant x\leqslant\delta} \frac{x^2}{\phi(x)} \sum_{k=1}^{k_n} \mathbf{E}\phi(|X_{nk}|).$$

Since $\mathbf{E}X_{nk} = 0$ it follows that

$$\max_{1\leqslant m\leqslant k_n}\left|\sum_{k=1}^{m}\mathbf{E}X_{nk}\, I\{|X_{nk}|\leqslant\delta\}\right| = \max_{1\leqslant m\leqslant k_n}\left|\sum_{k=1}^{m}\mathbf{E}X_{nk}\, I\{|X_{nk}|>\delta\}\right|$$

$$\leqslant \sup_{x>\delta}\frac{x}{\phi(x)}\sum_{k=1}^{k_n}\mathbf{E}\phi(|X_{nk}|)\longrightarrow 0 \quad \text{as} \quad n\to\infty.$$

R e m a r k. By Theorem 1 and Corollary 3 we have that Corollaries 1 and 2 in [2] are also true if we assume negative association instead of independence and replace weighted sums by the maxima of weighted sums. By Theorem 1, it is easy to obtain the sufficiency parts of Theorems 2.1 and 2.3 in [6] and Theorems 2.1 and 2.2 in [5].

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