ON $B_p$-CONVEX SPACES

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§ 1. Introduction

In the present article we generalize the notion of $B$-convex space and establish strengthened laws of large numbers for weighted sums of random elements with respect to such a generalization. The generalization's essence is concerned with the replacement of power function in the convexity definition by an arbitrary function $\varphi$ which satisfies certain conditions on growth. Introduced $B_p$-convex spaces are characterized by the laws of large numbers for weighted sums, while the known before convex spaces were characterized by the Markinewicz laws of large numbers. We also establish conditions, under which the generalized space is of a stable type $p$.

Let us introduce notation. Henceforward $(E, \| \cdot \|)$ means a real separable normed space, $B(E) = \{ x \in E : \| x \| < 1 \}$ is the unit ball in $E$, $(X^m)_{i=1}^{m}$ is a sequence of independent symmetric random elements with their values in $E$, $\sum_{n=1}^{\infty} X_n$ and we denote by $C$ certain constant quantity which can assume different values even within the same formula.

A sequence of random elements $(X^m_i)$ is said to be stochastically dominated by a positive random variable $\xi$ (notation is $(X^m_i) < \xi$) if there exists a $C > 0$ such that $\text{supp} \{ \| X^m_i \| > t \} < CP(\xi > t)$ for all $t > 0$.

We recall the notions of Rademacher's and stable types of Banach spaces. We shall say that $\xi$ is a Bernoulli random variable if $P(\xi = 1) = P(\xi = -1) = 1/2$. Let $(\xi^m_i)$ be a sequence of independent Bernoulli random variables and $(\gamma^p_i)$, $1 \leq p \leq 2$, be a sequence of independent standard $p$-stable random variables with the characteristic function $\exp(-|t|^p)$.

If for any sequence $(x^m_i) \in E$ the convergence of the series $\sum_{i=1}^{\infty} \| x^m_i \|^p$ implies the almost sure (a.s.) convergence of the series $\sum_{i=1}^{\infty} \| x^m_i \|^p$ implies the convergence of the series $\sum_{i=1}^{\infty} \| x^m_i \|^p$ implies the a.s. convergence of the series $\sum_{i=1}^{\infty} \gamma^p_i x^m_i$, then the Banach space $E$ is said to possess a stable type $p$, $1 \leq p \leq 2$. If for any sequence $(x^m_i) \in E$, the convergence of the series $\sum_{i=1}^{\infty} \| x^m_i \|^p$ implies the convergence of the series $\sum_{i=1}^{\infty} \| x^m_i \|^p$ implies the a.s. convergence of the series $\sum_{i=1}^{\infty} \gamma^p_i x^m_i$, then the Banach space $E$ is of Rademacher type $p$, $1 \leq p \leq 2$. One can find additional information on the type $p$ in [1] (ch.2, §2), [2] and [3] (ch. 3-4).

A few words about the history of the problem. A. Beck in [4] had introduced the notion of a $B$-convex Banach space and proved the strong law of large numbers, which characterizes these spaces. In [2], B. Maurey and G. Pisier studied the connection between the notion of $B$-convex space and this of the space of stable type $1$. A. Shangda (see [5] and [6]) had introduced the notion of $B_p$-convex space, which is a generalization of the notion of $B$-convex space. The same author had given its characterization in terms of the Martingale strong law of large numbers. M. Marcus and W.A. Woyczynski in [7] obtained as well a characterization of spaces of stable type $p$ (which is similar to the Shangda's theorem, but the $B_p$-convexity had not been introduced) via the Martingale strong law of large numbers. Let us dwell on this question in greater detail.
Assume that $1 < p < 2$. A Banach space $E$ is said to be $B_p$-convex if there exist $c > 0$ and integer $k > 1$ such that for all $x_1, \ldots, x_k \in B(E)$ it is possible to choose the signs $\pm$ so that the inequality $\| \pm x_1 + \ldots + \pm x_k \| \leq (1 - c) k^{1/p}$ will be valid.

**Theorem** (M. Marcus, W.A. Woyczynski and A. Shangha). Let $1 < p < 2$. The following statements are equivalent:

1. $E$ is of stable type $p$;
2. for any independent centered random elements $(X_n)$ such that $\mathbb{E} X_n^2 < \infty$ and $\mathbb{E} |X_n|^p < \infty$, the sequence $S_n/n^{1/p} \to 0$ a.s.;
3. $E$ is a $B_p$-convex space.

If we replace the condition of stochastic domination of $(X_n)$ in the statement (2) of Marcus-Woyczynski-Shangha’s Theorem by the condition of a common distribution of $(X_n)$, then we obtain the characterization of spaces of Rademacher type $p$ (see [8] and [9]).

Finally, let us note that T. Mikosch and R. Norvaisa ([10], Theorem 3.2) had considered the problem of characterization of spaces of type $p$ via the convergence of weighted sums. These authors investigated the law of large numbers in the case of multidimensional index of summing. So, we have to reformulate slightly their result.

Let $(w_n, a_n)$ be a sequences of real numbers such that $\lim_{n \to \infty} w_n = 0$ and $\lim_{n \to \infty} a_n / a_n = 0$. We introduce the class

$$F_p = \{ (w_n, a_n) : \sum_{n=1}^{\infty} n^{p-1} w_n / a_n = O(k^{-p} w_n / a_n) \text{ with } k \to \infty \}.$$  

We consider weighted sums of the form $Z_n = \frac{1}{a_n} \sum_{k=1}^{n} w_k X_k$, where $(X_k)$ are independent symmetric random elements.

**Theorem** (T. Mikosch and R. Norvaisa). Let $1 < p < 2$ and $(w_n, a_n) \in F_q$ for all $q > p$. The following statements are equivalent:

1. $E$ is of stable type $p$;
2. for all independent symmetric random elements $(X_n)$ such that $\mathbb{E} X_n^2 < \infty$ and $\mathbb{E} |X_n|^p < \infty$, the sequence $Z_n \to 0$ a.s. with $n \to \infty$;
3. for all independent symmetric random elements $(X_n)$ such that $\mathbb{E} X_n^2 < \infty$ and $\mathbb{E} |X_n|^p < \infty$, the sequence $S_n/n^{1/p} \to 0$ a.s.

Let us dwell, in particular, on the implication (2) $\Rightarrow$ (3). This is precisely the implication which, being absolutely evident, allows to conclude that (3) $\Rightarrow$ (1) with the help of, e.g., (3) $\Rightarrow$ (1) in Marcus-Woyczynski-Shangha Theorem. The implication (2) $\Rightarrow$ (3) shows that in the item (2) the validity of law of large numbers is meant to be satisfied "for all at once" weights $(w_n, a_n) \in F_q$. The problem which we study is somewhat different: what one can say on a presence of the stable type, when the law of large numbers holds true at least for one weight? We shall give, by these means, an answer to our question posed at the end of [11].

§ 2. Definition of $B_p$-convex spaces and their simplest properties

Let $E$ be a normed space and $\{ \varphi(h) \} \_1^\infty$ be an increasing sequence of positive numbers.

**Definition.** Let $\varepsilon$ be positive $0$ and $k \geq 2$ be an integer. The space $E$ is said to be $\varphi$-convex if for all $x \in E$, one can choose the signs $\pm$ so that the inequalities $\| \pm x_1 + \ldots + \pm x_k \| \leq (1 - \varepsilon) \varphi(k)$ are valid.

The space $E$ is called $B_p$-convex if there exist $\varepsilon > 0$ and $k \geq 2$ such that $E$ is $\varphi$-convex.
Remark. If \( \varphi(h)=h^{1/p}, \ 1<p<2, \) then the class of \( B_p \)-convex spaces coincides with that of \( B_p \)-convex spaces introduced by A. Shanggu in [6], and for \( \varphi(h)=h \) — with the class of \( B \)-convex spaces introduced by A. Beck (see [4]).

Let us note some simple properties of \( B_p \)-convex spaces. Their proofs differ slightly from these of \( B_1 \)-convex spaces.

1°. If a normed space \( E \) is \( \varphi(k)=1 \)-convex, then:
   a. for any \( 0<\varepsilon<\delta \), \( E \) is \( \varphi(k,\varepsilon) \)-convex;
   b. its completion is \( \varphi(k,\varepsilon) \)-convex;
   c. any subspace of \( E \) is \( \varphi(k,\varepsilon) \)-convex;
   d. its second dual is \( \varphi(k,\varepsilon) \)-convex;
   e. there exist \( p<1 \) and \( \varepsilon<\delta \) such that for any increasing sequence \( x \) with \( x(n)\to\varphi^p(n) \)

   the space \( E \) is \( \varphi(k,\varepsilon,\delta) \)-convex for all \( n \).

Proof. Notice that the properties 1° and 3° are straightforward.

2°. Let \( x_1,\ldots,x_n \) belong to the complement of the space \( E \), i.e., to the Banach space \( \mathcal{F} \), \( \|x_i\|<1, \ 1\leq i<k, \ k>0 \). Since \( E \) is dense in \( \mathcal{F} \), there exist \( x_1,\ldots,x_k \in E \) such that \( \|x_i-x_i\|<\delta/k \), \( 1\leq i\leq k \). Since \( E \) is \( \varphi(k,\varepsilon) \)-convex, for certain collection of signs \( \pm \) we have that \( \|\pm x_1,\ldots,\pm x_k\|<(1-\varepsilon)\varphi(k) \delta \).

By virtue of arbitrariness of \( \delta>0 \), we have \( \inf \{\|\pm x_1,\ldots,\pm x_k\|<(1-\varepsilon)\varphi(k) \delta \} \).

4°. Let \( \mathcal{B}=\mathcal{B}(E), \mathcal{B}^{**}=\mathcal{B}(E^{**}) \) and \( J:E\to E^{**} \) be the canonical embedding of \( E \) into \( E^{**} \). Then \( J(B) \) is dense in \( B^{**} \) in the sense of \( \bar{w} \)-weak topology (see [12], Theorem V.3).

Let \( \delta>0, \ \varepsilon>0, \ \alpha = (\alpha_1,\ldots,\alpha_k) \) be the coordinates \( \alpha_i \in \{-1,+1\}, \ 1\leq i\leq k \). By the norm definition in \( E^{**} \), for any \( \alpha \) there exists \( x_\alpha \in E^* \) with \( \|x_\alpha\|=1 \) and

\[
\left\| \sum_{i=1}^{k} \alpha_i x_i^{**} \right\| < \left< x_\alpha^{**}, \sum_{i=1}^{k} \alpha_i x_i^{**} \right> + \delta.
\]

Let us consider a neighborhood of zero of \( E^{**} \) defined as

\[
U_\delta = \left\{ x^{**} \in E^{**} : \left< x_\alpha^{**}, x^{**} \right> < \delta \right\},
\]

where the intersection is taken by all possible values of \( x^{**} \). Now choose \( x_1,\ldots,x_k \in E \) such that \( J(x_i) \in U_\delta \), \( 1\leq i\leq k \):

\[
\left\| \sum_{i=1}^{k} \alpha_i x_i^{**} \right\| < \left< x_\alpha^{**}, \sum_{i=1}^{k} \alpha_i x_i^{**} \right> + \delta = \left< x_\alpha^{**}, \sum_{i=1}^{k} \alpha_i J(x_i) + \left< x_\alpha^{**}, \sum_{i=1}^{k} \alpha_i (x_i^{**} - J(x_i)) \right> + \delta \right\| \sum_{i=1}^{k} \alpha_i x_i \| + R \delta + \delta.
\]

Choosing \( \alpha \) such that

\[
\left\| \sum_{i=1}^{k} \alpha_i x_i \right\| < (1-\varepsilon)\varphi(k) + \delta(1-\varepsilon)\varphi(k) + \delta,
\]

or, due to arbitrariness of \( \delta>0 \), \( \inf \{\|\sum_{i=1}^{k} \alpha_i x_i\|<(1-\varepsilon)\varphi(k) \} \).

5°. Set \( D(k)=\sup \{ \inf \{\sum_{i=1}^{k} \alpha_i x_i \} : \alpha \in \{-1,+1\}^k \} \). The space \( E \) is \( \varphi(k,\varepsilon,\delta) \)-convex if and only if

\[
D(k) < (1-\varepsilon)\varphi(k), \ \text{or} \ \varphi(k) > D(k)/(1-\varepsilon).
\]

Theorem 3. Characterization of \( \varphi \)-convex spaces via the laws of large numbers for weighted sums.

Let \( E \) be a Banach space and \( (X_n) \) be a sequence of independent centered random elements which are taking values in \( E \). Set...
\[ T_n = \sum_{k=1}^{n} q_k(n) X_k, \]

where \( q_k(n), 1 \leq k \leq n, n \in \mathbb{N} \) is a triangular array of constants, which will be called a weight. Any weight defines the sequences

\[ \varphi(n) = \frac{1}{n} \max_{1 \leq k \leq n} |q_k(n)| \quad \text{and} \quad \chi(n) = \frac{1}{n} \min_{1 \leq k \leq n} |q_k(n)|, \quad n \in \mathbb{N}. \]

In addition, we choose the \( \alpha \) so that the sequences \( (\varphi(n))_n^\alpha \) and \( (\chi(n))_n^\alpha \) be increasing.

We impose the following conditions upon the sequences \( \varphi \) and \( \chi \) (which are in correspondence with the weight \( \alpha \)). Assume that \( \chi(x(n)) \) is an increasing sequence of positive numbers.

**Condition (A).** \( \chi(n) = x(n) \chi(n) \) for all \( n \in \mathbb{N} \).

**Condition (B).** There exist \( M > 1 \) and \( D > 1 \) both such that the inequality \( \chi(Mn) \leq D \chi(n) \) holds true for all \( n \in \mathbb{N} \). For a sequence \( \chi \) we introduce the characteristics:

\[ p(\chi) = \inf \left\{ p: \sum_{n=1}^{\infty} \chi^p(n) < \alpha \right\}, \]

\[ q(\chi) = \sup \{ q: \sup_{n \geq 1} \chi^q(n/n^{1/q}) < \alpha \} \]

The next Lemma establishes the relation between these characteristics.

**Lemma 1.** If \( \chi \) satisfies the condition (A), then \( p(\chi) = q(\chi) = \lim_{n \to \infty} \ln n/\ln \chi(n) \). The supremum in the definition of \( q(\chi) \) is to be attained, i.e., \( \sup_{n \geq 1} \chi^q(n/n^{1/q}) < \alpha \).

**Proof.** I. First let us show that the limit \( \lim_{n \to \infty} \ln n/\ln \chi(n) \) exists and equals \( q(\chi) \).

Take an arbitrary \( q < q(\chi) \). Then for all \( n \geq 2 \) we have that \( \chi(n) < n^{1/q} \). Hence, \( \lim_{n \to \infty} \ln n/\ln \chi(n) \geq \lim_{n \to \infty} \ln n/\ln n^{1/q} = q(\chi) \).

Now let us take an arbitrary \( q > q(\chi) \). Then there exists \( N > 2 \) such that \( \chi(n) > n^{1/q} \). For all \( n > N \) we have

\[ \chi(n) = \chi(n^{1/q}) > \chi(n^{1/q}) \]

where \( \lfloor \cdot \rfloor \) is the integer part of a number. From this, applying the condition (A) \( \lfloor \log_{1/q} \rfloor \) times, we obtain

\[ \chi(n) > \chi(n^{1/q}) > \chi(n^{1/q}) > \cdots > \chi(n^{1/q}) > \chi(n^{1/q}) \]

Consequently,

\[ \lim_{n \to \infty} \ln n/\ln \chi(n) \leq \lim_{n \to \infty} \ln (\ln n^{1/q})/\ln n^{1/q} = q, \]

i.e., \( \lim_{n \to \infty} \ln n/\ln \chi(n) = q \). Finally, \( \lim_{n \to \infty} \ln n/\ln \chi(n) = q(\chi) \).

II. Now let us turn to the proof of the equality \( p(\chi) = q(\chi) \). It is clear that \( p(\chi) \geq q(\chi) \), because for any \( p > p(\chi) \) the series \( \sum_{n=1}^{\infty} \chi^p(n) \) converges. Hence \( \chi^p(n)/n \to 0 \).

Let us show that \( p(\chi) \leq q(\chi) \). To this end we have to show that for all \( p > q(\chi) \) the inequality \( p > p(\chi) \) is valid. Assume \( p > q(\chi) \). For any \( \epsilon > 0 \) there exists \( N \) such that for all \( n > N \) we have \( \ln \chi(n)/\ln n^{1/q} < q(\chi) + \epsilon \). This implies

\[ \sum_{n=N}^{\infty} \chi^p(n) = \sum_{n=N}^{\infty} \chi^{p/q}(n) \cdot n^p/n^{p/q} = \sum_{n=N}^{\infty} n^p/\chi^{p/q}(n) \cdot n^{1/q} \]

Having chosen \( \epsilon > 0 \) small enough for \( p/q(\chi) + \epsilon > 1 \), we obtain the convergence of the series \( \sum_{n=1}^{\infty} \chi^p(n) \), i.e., \( p > p(\chi) \).

III. Finally, let us show that the supremum in definition of \( q(\chi) \) is attainable.
In order to prove Theorem 1 we need the following result:

**Lemma 2.** If $\mathcal{C}$ satisfies the condition (A), then (for all $p \in \mathcal{C}$) we have

$$\sum_{x \sim} E(\mathcal{C}(x) \mid p(x) = p_{i}(x)) = \mathcal{C}(x) \mid p(x) = p_{i}(x).$$

**Proof**. First note that for $(x, y)$ we have

$$E(\mathcal{C}(x) \mid p(y) = p_{i}(y)) = \sum_{x \sim} E(\mathcal{C}(x) \mid p(y) = p_{i}(y)) = \mathcal{C}(x) \mid p(y) = p_{i}(y).$$

Thus, the lemma holds for the case $p(y) = p_{i}(y)$. It is clear that $E(\mathcal{C}(x) \mid p(y)) = \mathcal{C}(x) \mid p(y)$ for all $y$. Therefore, the lemma holds for all $p$. 

Note that the latter series converges, because $p_{i}(x)$ is bounded above by $1$.

The following Lemma is a generalization of well-known result see (131), pp. 127-128.

**Theorem.** Consider the following statement:

1. For all $x \in X$, the class $F_{x}$ is $\mathcal{B}$-measurable.
2. For all $x \in X$, the statement (A) is fulfilled for at least one weight $q_{x}$.
3. The space $\mathcal{B}$ is $\mathcal{B}$-measurable.

Then the space $\mathcal{B}$ is $\mathcal{B}$-measurable. Consequently, $\mathcal{B}$ is measurable.
Lemma 4. If \((X_k)\) are independent symmetric random elements, then 
\[ E(\sup_{n \in \mathbb{N}} |T_n|) < \infty. \]

Proof. By the Levy's inequality (see [14], Proposition V.2.3), for all \(k \in \mathbb{N}\) we have that 
\[ G = P(\sup_{n \in \mathbb{N}} |T_n| > t) < 2P(\sum_{n \in \mathbb{N}} |X_n| > t), \]
but then, by the Kwapien's inequality ([14], Lemma V.4.1(a)), it follows that 
\[ G < 4P(\sup_{n \in \mathbb{N}} |X_n| > t) = 4P(\sup_{n \in \mathbb{N}} |S_n| > t). \]
Consequently, 
\[ P(\sup_{n \in \mathbb{N}} |T_n| > t) < 4P(\sup_{n \in \mathbb{N}} |S_n| > t). \]
Hence,
\[ E(\sup_{n \in \mathbb{N}} |T_n|) = \int_0^\infty P(\sup_{n \in \mathbb{N}} |T_n| > t) dt < 4\int_0^\infty P(\sup_{n \in \mathbb{N}} |S_n| > t) dt = 4E(\sup_{n \in \mathbb{N}} |S_n| > t). \]

Lemma 5. Let \(E\) possess the Rademacher type \(p, 1 < p < 2\). If \((X_k)\) are independent symmetric random elements satisfying 
\[ \sum_{n \in \mathbb{N}} E|X_n|^p/\varphi^p(k) < \infty, \]
then 
\[ E(\sup_{n \in \mathbb{N}} |S_n| > t) < \infty. \]

Proof. Let us set 
\[ m_k = \min(n: \varphi(n) > 2^k) \]
and 
\[ N_k = \{n: m_k < n < m_{k+1}\} \]
for any \(k \geq 0\). Then, by the choice of \(N_k\) and by Levy's inequality, we have that 
\[ G = P(\sup_{n \in \mathbb{N}} |S_n| > t) = P(\sup_{n \in \mathbb{N}} \max_{k \geq 0} |S_n| > t) < \sum_{k = 0}^\infty \sum_{n \in N_k} P(\max_{k \geq 0} |S_n| > 2^k t). \]
Now, using Chebyshev's inequality and taking into account that \(E\) is of the Rademacher type \(p\) (see [11], ch.2, §2; [2], [3], ch.3-4), we obtain that
\[ G < 8t^p \sum_{k = 0}^\infty \sum_{n \in N_k} \frac{\varphi(n)}{\varphi^p(n)} \sum_{i = 1}^{m_{k+1} - 1} |X_i|^p \leq C t^p \sum_{k = 0}^\infty \sum_{n \in N_k} \varphi(n) \sum_{i = 1}^{m_{k+1} - 1} |X_i|^p. \]
Altering the order of summing, we obtain 
\[ G < C t^p \sum_{i = 1}^\infty \sum_{k \geq 0} |X_i|^p/\varphi^p(k) \]
where \(k_0 = \min(k: m_{k+1} - 1 > i)\). Note that the for \(k_0\) the inequalities 
\[ m_{k_0} - 1 > i, \quad \varphi(m_{k_0} - 1) > 2^{k_0 - 1} \]
are valid. Let us estimate the sum:
\[ \sum_{k \geq k_0} 2^{-k p} - \frac{2^{p - (k_0 - 1)p}}{1 - 2^{-p}} = C \varphi(m_{k_0} - 1)^p \leq C \varphi^p(i). \]
Hence, 
\[ G < C t^p \sum_{i = 1}^\infty \sum_{k \geq 0} |X_i|^p/\varphi^p(k). \]
Then
\[ E(\sup_{n \in \mathbb{N}} |S_n| > t) = 1 + \sum_{i = 1}^\infty P(\sup_{n \in \mathbb{N}} |S_n| > t) \frac{\varphi(n)}{\varphi^p(t)} \frac{\varphi^p(n)}{\varphi^p(t)} dt < 1 + C \sum_{i = 1}^\infty \sum_{k \geq 0} |X_k|^p/\varphi^p(k) < \infty. \]
The following Lemma can be considered as an analog of Chung's law of large numbers (cf. [15]) for weighted sums.

Lemma 6. If the space \(E\) is of Rademacher type \(p, 1 < p < 2\), then for any independent symmetric random elements \((X_k)\) the condition 
\[ \sum_{i = 1}^\infty |X_k|^p/\varphi^p(k) < \infty \]
implies \(T_n \to 0\) a.s.
\[ \mathbb{E} \left\| \sum_{k=1}^{\infty} X_k/\varphi(k) \right\|_p^p \leq C \sum_{k=1}^{\infty} \mathbb{E} \left| X_k \right|^p/\varphi_p(k). \]

Therefore, the series \( \sum_{k=1}^{\infty} X_k/\varphi(k) \) converges in \( L^p(E) \). But then ([14], Theorem V.2.3) it converges a.s. By the Kronecker's Lemma, we have \( S_n/\varphi(n) \to 0 \) a.s. It means that \( \sup_{k \geq n} S_k/\varphi(k) \to 0 \) a.s. with \( n \to \infty \). By Lemma 5 we have that \( \mathbb{E} \sup_{k \geq n} S_k/\varphi(k) \to 0 \). By the Lebesgue Theorem, \( \mathbb{E} \sup_{k \geq n} S_k/\varphi(k) \to 0 \) with \( n \to \infty \). By Lemma 4, \( \mathbb{E} \sup_{k \geq n} T_k \to 0 \) with \( n \to \infty \). Finally, by the Chebyshev's inequality, we have that \( \mathbb{P}(\sup_{k \geq n} T_k \geq \varepsilon) \to 0 \) with \( n \to \infty \) for all \( \varepsilon > 0 \). Hence \( T_n \to 0 \) a.s.

**Lemma 7.** Let \( \chi \) satisfy the condition (B) and the space \( E \) be non-\( B_{1,\infty} \)-convex. Then there exist the sequences \( (x_n) \in \mathcal{B}(E) \) and \( (n_k) \) such that

\[ \inf \left\{ \frac{1}{X(n_k)} \sum_{i=1}^{n_k} x_i \right\} > 1/(2D). \]

We will prove Lemma by the induction. Put \( n_1 = 1 \) and take in the capacity of \( x_1 \) an element \( E \) with \( 1/(2D) \leq x_1 \).

Assume that there are \( n_1 < n_2 < \cdots < n_m \) and \( (x_{n_1}, x_{n_2}, \ldots, x_{n_m}) \in \mathcal{B}(E) \) which satisfy the conditions of Lemma. Choose \( n_{m+1} \) so that the following inequalities hold true: \( n_{m+1} > Mn_m/(M-1) \) (i.e., \( n_{m+1} > M(n_{m+1} - n_m) \)) and \( \chi(n_{m+1}) > 4Dn_m \).

As the space \( E \) is not \( B_{1,\infty} \)-convex, so there exist \( x_{n_{m+1}}, \ldots, x_{n_{m+1}} \in \mathcal{B}(E) \) such that for arbitrary chosen signs "\( \pm \)" one has

\[ \|
\begin{pmatrix}
\pm x_{n_{m+1}} & \ldots & \pm x_{n_{m+1}} \\
\end{pmatrix}
\| > (1-1/4) \chi(n_{m+1}, n_m). \]

But then, by virtue of the condition (B), it follows

\[ \frac{1}{X(n_{m+1})} \| x_{n_{m+1}}, \ldots, x_{n_{m+1}} \| > \frac{1}{X(n_{m+1})} \| x_{n_{m+1}}, \ldots, x_{n_{m+1}} \| > \frac{1}{X(n_{m+1})} \| x_{n_{m+1}}, \ldots, x_{n_{m+1}} \| > (1-1/4) \chi(n_{m+1}, n_m). \]

**Proof of Theorem.** First let us show that (1) \( \Rightarrow \) (2). As \( E \) is of stable type \( p < 2 \) and \( \varphi(n) = p \) exists \( q > p \) such that \( E \) is of the Rademacher type \( q \) (see [11], ch.2, §2; [21], ch.3-4). Let \( X_n = X_{n,l} \mathbb{I}\{X \leq \omega(n)\} \). Then, by virtue of Lemmas 2 and 3, the series \( \sum_{n=1}^{\infty} X_n^2 \mathbb{I}\{X \leq \omega(n)\} < \infty \). From this, by Lemma 6, we have \( \sum_{n=1}^{\infty} a_n(n)X_n = 0 \) a.s. Moreover,

\[ \sum_{n=1}^{\infty} P(X_n \neq X_n') = \sum_{n=1}^{\infty} P(X_n > \omega(n)) = \sum_{n=1}^{\infty} P(X \geq \omega(n)) < \infty. \]

By the Borel-Cantelli's Lemma,

\[ P(X_n \neq X_n' \text{ infinite times}) = 0. \]

Therefore, \( T_n \to 0 \) a.s. with \( n \to \infty \).

Now let us demonstrate that (3') \( \Rightarrow \) (4). Let \( E \) be non-\( B_{1,\infty} \)-convex. Then for a sequence \( (x_n) \in \mathcal{B}(E) \) the assertion of Lemma 7 takes place. By the condition,

\[ Y_n = \sum_{k=1}^{n} a_k(n)x_k \]

in probability with \( n \to \infty \). We fix \( \varepsilon > 0 \) and choose \( n \) large enough for \( P(\|Y_n\| > \varepsilon) < 1/8 \). Put \( \eta_n = \varepsilon a_n(n)^{1/\varphi(n)} \). Then \( \|Y_n\| < \varepsilon a_n(n)^{1/\varphi(n)} \) and \( Y_n \to 0 \) a.s. By virtue of Levy's and Kolmogorov's
inequalities (see [14], pp. 211, 223), we have
\[ \frac{1}{2} \geq P \left\{ \left\| \sum_{k=1}^{n} \eta_k \| > \varepsilon \right\} \geq 1 - \frac{(\varepsilon + 1/p(n))^2 + \varepsilon^2/2}{E \| Y_n \|^2} \].

Therefore, \( E \| Y_n \|^2 \geq 2((\varepsilon + 1/p(n))^2 + \varepsilon^2/2) \). As \( \varepsilon \) is arbitrary, so \( E \| Y_n \|^2 \rightarrow 0 \) with \( n \rightarrow \infty \).

Note that due to Lemma V.4.1 (b) (see [14]) it follows that
\[ E \left[ \left( \frac{1}{\gamma(n)} \sum_{k=1}^{n} \xi_k x_k \right)^2 \right] \leq 2E \left[ \sum_{k=1}^{n} \alpha_k(n) \xi_k x_k \right]^2 - E \| Y_n \|^2 \rightarrow 0. \]

i.e., \( E \left[ \frac{1}{\gamma(n)} \sum_{k=1}^{n} \xi_k x_k \right] \rightarrow 0 \) in \( L^2(\Omega) \) and, consequently, "in probability". But this contradicts the statement of Lemma.

Finally, let us show that (4) \( \Rightarrow \) (1). Suppose that \( E \) is being not of the stable type \( p=q(x) \). Then \( 1/p \) is finitely representable in \( E \) (see [1], ch. 2; [2], ch. 3.; [3], ch. 3-4); i.e., for all \( n \in \mathbb{N} \) and \( \varepsilon > 0 \) there exist \( x_1, \ldots, x_n \in E \) such that for all \( s_1, \ldots, s_n \in E \) the inequality
\[ (1-\varepsilon) \left( \sum_{k=1}^{n} |s_k|^p \right)^{1/p} \left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \leq \left( \sum_{k=1}^{n} |\xi_k|^p \right)^{1/p} \]

is valid.

As \( E \) is \( B_p \)-convex, so there exist \( n \geq 2 \) and \( \varepsilon > 0 \) such that for all \( \xi \in B(E) \) we have that \( \left\| \sum_{k=1}^{n} \xi x_k \right\| \geq (1-\varepsilon) \gamma(n) \). Taking namely this collection of signs in the capacity of \( s_1 \), we obtain that
\[ n^{1/p}(1-\varepsilon/2) \leq \left| \sum_{k=1}^{n} \xi x_k \right| \leq (1-\varepsilon) \gamma(n), \]

or
\[ \gamma(n)/n \geq ((1-\varepsilon/2)/(1-\varepsilon))^p > 1. \]

But this contradicts the choice of \( p=q(x) \), because due to Lemma 1 we have that
\[ \sup_{n \geq 1} \gamma(n)/n \leq 1. \]

§ 4. Subsequent properties of \( B_p \)-convex spaces

With the help of proved Theorem we can obtain a number of properties of \( B_p \)-convex spaces, which are not so trivial as \( 1^p-5^p \) were.

6. If \( \varphi \) satisfies (A) and (B), then the following statements are equivalent:
(1) the space \( E \) is of stable type \( p=\varphi(\varphi) \);
(2) for any independent centered random elements \( (X_k) \) such that \( (X_k) \xi \) and \( \sum_{k=1}^{n} \varphi(\varphi) > \varphi(n) \)
\( \rightarrow \varphi(n) \rightarrow 0 \) a.s. with \( n \rightarrow \infty \);
(3) the space \( E \) is \( B_p \)-convex.

Remark. It is obvious that power functions satisfy both the conditions (A) and (B). Hence, we obtain the statements of Marcus-Woyczyński-Shangai's Theorem. Therefore, the example \( \varphi(n)=n^{1/p}(\ln n+1), 1<p<2 \), must be more interesting one.

Note that the implication (4) \( \Rightarrow \) (1) of Theorem implies the next statement: the notion of \( B_p \)-convexity \( (\varphi \) satisfies (A)) makes sense only for \( p(\varphi)<2 \), because the notion of
Let \( \varphi \) satisfy the condition (A). The space \( E \) then is \( B_\varphi \)-convex if and only if \( A_\varphi(k) \to 0 \) with \( k \to \infty \).

Proof. If \( A_\varphi(k) \to 0 \), then, obviously, the space \( E \) is \( B_\varphi \)-convex.

Vice versa, if \( E \) is \( B_\varphi \)-convex, then, by implication (4)\( \Rightarrow \) (1) of Theorem, \( E \) is of the stable type \( p > q(\varphi) \). For any \( x_1, \ldots, x_n \in B(E) \) the series is such that

\[
\sum_{k=1}^n \left\| x_k \right\|^p < \sum_{k=1}^n \left\| x_k \right\|^p.
\]

Consequently, it converges, because \( p > q(\varphi) \). By the Kronecker's Lemma,

\[
\frac{1}{\varphi^p(k)} \sum_{i=1}^k |x_i|^p \to 0 \text{ with } k \to \infty.
\]

Since \( E \) is of Rademacher type \( \rho \), we have that

\[
\left( E \left\| \frac{1}{\varphi(k)} \sum_{i=1}^k \varepsilon_i x_i \right\|^{1/p} \right)^{1/p} \leq \left( E \left\| \frac{1}{\varphi^p(k)} \sum_{i=1}^k |x_i|^p \right\|^{1/p} \right)^{1/p} \to 0
\]

with \( k \to \infty \).

With the help of the property 7\( ^o \) it is easy to establish

8\( ^o \). Let \( E_1 \) and \( E_2 \) be normed spaces, \( R: E_1 \to E_2 \) be linear continuous and open operator. Moreover, assume that \( E_1 \) is \( B_\varphi \)-convex and \( \varphi \) satisfies (A). Then \( R(E_1) \subset \subset E_2 \) is \( B_\varphi \)-convex.

Proof. Since \( R \) is continuous, there exists \( L > 0 \) such that \( |R(x)| < L \) for all \( x \in B(E_1) \). The openness of the mapping implies that \( R(B) \) is the neighborhood of zero in \( R(E_1) \), i.e., there exists \( \delta > 0 \), for which we have \( \|y\in R(E_1) : \|y\| < \delta \| \in R(B) \).

Let us choose \( k \) so large that \( A_\varphi(k) < \delta / 2L \). This can be done due to the property 7\( ^o \).

Let \( y_1, \ldots, y_k \) belong to the unit ball of the space \( R(E_1) \). Then we have that \( \delta y_1, \ldots, \delta y_k \in R(B) \), i.e., there exist \( x_1, \ldots, x_n \in B(E_1) \) such that \( Rx_i - \delta y_i, \ 1 \leq i \leq k \). Let us arrange the signs in such an order that the inequality

\[
\frac{1}{\varphi(k)} \left\| \sum_{i=1}^k \pm x_i \right\| < \frac{\delta}{2L}
\]

holds true. Then

\[
\frac{1}{\varphi(k)} \left\| \sum_{i=1}^k \pm y_i \right\| = \frac{1}{\varphi(k)} \left\| \sum_{i=1}^k \pm \delta y_i \right\| = \frac{1}{\varphi(k)} \left\| \sum_{i=1}^k \pm Rx_i \right\| < \frac{1}{\varphi(k)} \left\| \sum_{i=1}^k \pm x_i \right\| < 1/2.
\]

Hence, \( R(E_1) \) is \( (\varphi, k, \delta / 2L) \)-convex.

Corollary. If the normed spaces \( E_1 \) and \( E_2 \) are isomorphic and \( \varphi \) satisfies (A), then \( E_1 \) is \( B_\varphi \)-convex if and only if \( E_2 \) is \( B_\varphi \)-convex.

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