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CHARACTERIZATION OF BANACH SPACE BY THE LAW OF LARGE NUMBERS FOR WEIGHTED SUMS

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ABSTRACT

We generalize the notion of the B-convex space and study the strong laws of large numbers for weighted sums of random elements in this context. The idea behind the generalization is similar to that of changing the power function in the definition of convexity for an arbitrary function φ with some additional minor conditions. B_φ-convex spaces are characterized by the laws of large numbers for weighted sums, while the convex spaces known before were characterized by the Marcinkiewicz laws of large numbers. We study conditions for generalized spaces to be of stable type.

1. INTRODUCTION

First we introduce some notation. Throughout the article E is a real separable normed space, $\mathbf{B}(E) = \{x \in E: \|x\| \leq 1\}$ is the unit ball of E, $(X_k)_1^\infty$ is a sequence of independent symmetric random elements with values in E, $S_n = \sum_{k=1}^n X_k$ and the symbol C denotes a finite positive constant which is not necessarily the same in each appearance.

A sequence of random elements (X_k) is said to be *stochastically dominated* by a positive random variable ξ (we write $(X_k) \prec \xi$), if there exists a constant $C > 0$ such that, for all $t > 0$,

$$\sup_{k\geq 1} \mathbf{P}\{\|X_k\| > t\} \leq C\mathbf{P}\{\xi > t\}.$$

Recall the notions of Rademacher and stable types of a Banach space. We say that ε is a *Bernoulli random variable*, if $\mathbf{P}\{\varepsilon = 1\} = \mathbf{P}\{\varepsilon = -1\} = 1/2$. Let (ε_k) be a sequence of independent Bernoulli random variables and $(\gamma_k^p), 1 \leq p \leq 2$, be a sequence of independent standard p-stable random variables with characteristic function $\exp\{-|t|^p\}$.

If for any sequence $(x_k) \subset E$ the convergence of the series $\sum_{k=1}^\infty \|x_k\|^p$ implies the a.s. convergence of the series $\sum_{k=1}^\infty \gamma_k^p x_k$, then the Banach space E is of *stable type* $p, 1 \leq p \leq 2$. If for any sequence $(x_k) \subset E$ the convergence of the series $\sum_{k=1}^\infty \|x_k\|^p$ implies the a.s. convergence of the series $\sum_{k=1}^\infty \varepsilon_k x_k$, then the Banach space E is of *Rademacher type* $p, 1 \leq p \leq 2$. Additional information on type p can be found in (Pisier, 1990).

Let us say some words on the history of the question. A. Beck (Beck, 1962) introduced the notion of a B-convex Banach space and proved the strong law of large numbers characterizing these spaces. B. Maurey and G. Pisier (Maurey and Pisier, 1976) studied the connection between notions of the B-convex space and a space of stable type 1. A. Shangua (Shangua, 1978) introduced the notion of a B_p-convex space which is a generalization of the notion of the B-convex space and gave their characterization in terms of the Marcinkiewicz strong law of large numbers. Note that M. Marcus and W. A. Woyczynski (Marcus and Woyczynski, 1979) obtained a characterization of spaces of stable type p by the Marcinkiewicz strong law of large numbers which is analogous to Shangua's Theorem but without introducing the notion of B_p-convexity. The same in more detail is as follows.

Let $1 \le p < 2$. A Banach space E is said to be *B_p-convex*, if there exist an $\varepsilon > 0$ and an integer $k \ge 2$ such that for all $x_1, \ldots, x_k \in \mathbf{B}(E)$ it is possible to choose the signs $\pm$ so that the inequality $\| \pm x_1 \pm \cdots \pm x_k\| \le (1-\varepsilon)k^{1/p}$ holds.

THEOREM (M. Marcus, W. A. Woyczynski, and A. Shangua). *Let $1 \le p < 2$. The following statements are equivalent:*

(1) *E is of stable type p.*

(2) *$S_n/n^{1/p} \to 0$ a. s. for any independent centered random elements (X_k) such that $(X_k) \prec \xi$ and $\mathbf{E}|\xi|^p < \infty$.*

(3) *E is a B_p convex space.*

If we replace the condition of stochastic domination of (X_k) in the statement (2) of the Marcus–Woyczynski–Shangua Theorem by the condition of identity of distributions of (X_k), then we obtain a characterization of spaces of Rademacher type p (Azlarov and Volodin, 1982; Acosta, 1981).

Note that the problem of characterization of type p spaces by the convergence of weighted sums was treated by T. Mikosch and R. Norvaisa (Mikosch and Norvaisa, 1987, Theorem 3.2.). The result of T. Mikosch and R. Norvaisa is of particular interest.

Let (w_n, a_n) be a sequence of real numbers such that $\lim_{n\to\infty} a_n = \infty$ and $\lim_{n\to\infty} w_n/a_n = 0$. We introduce the class

$$F_p = \left\{ (w_n, a_n) : \sum_{n=k}^{\infty} \frac{n^{-p-1} w_n}{a_n} = O\left(\frac{k^{-p} w_k}{a_k}\right) \text{ as } k \to \infty \right\}.$$

Consider weighted sums $Z_n = a_n^{-1} \sum_{k=1}^{n} w_k X_k$, where (X_k) are independent symmetric random elements.

THEOREM (T. Mikosch and R. Norvaisa). *Let $1 \le p < 2$ and $(w_n, a_n) \in F_q$ for all $q > p$. The following statements are equivalent:*

(1) *E is of stable type p;*

(2) *$Z_n \to 0$ a. s. as $n \to \infty$ for all independent symmetric random elements (X_k) such that $(X_k) \prec \xi$ and $\mathbf{E}\xi^p < \infty$;*

(3) *$S_n/n^{1/p} \to 0$ a. s. as $n \to \infty$ for all independent symmetric random elements (X_k) such that $(X_k) \prec \xi$ and $\mathbf{E}\xi^p < \infty$.*

Consider implication (2) $\Rightarrow$ (3). It is obvious, but it allows us to conclude that (3) $\Rightarrow$ (1) by virtue of the Marcus–Woyczynski–Shangua Theorem. As it can be seen from (2) ⇒ (3) the validity of the law of large numbers "for all at once" weights $(w_n, a_n) \in F_q$ is required. We study somewhat another problem: what can we say about the stable type, if the law of large numbers holds "even for one" weight?

2. DEFINITION AND SIMPLE PROPERTIES OF B_φ-CONVEX SPACES

Let E be a normed space and let $\varphi = (\varphi(k))_1^\infty$ be an increasing sequence of positive numbers.

Definition. Let $\varepsilon > 0$ and $k \geq 2$. A space E is said to be $(\varphi, k, \varepsilon)$-*convex* if for all $x_i \in \mathbf{B}(E), 1 \leq i \leq k$, we can choose signs $\pm$ so that the inequality $\| \pm x_1 \pm \cdots \pm x_k\| \leq (1-\varepsilon)\varphi(k)$ holds.
A space E is called B_φ-*convex*, if there exists an $\varepsilon > 0$ such that E is $(\varphi, k, \varepsilon)$ convex for all $k \in \mathbf{N}$.

Remark. If $\varphi(k) = k^{1/p}$, $1 < p < 2$, then the class of B_φ-convex spaces coincides with the class of B_p-convex spaces introduced by A. Shangua, and for $\varphi(k) = k$ it coincides with the class of B-convex spaces introduced by A. Beck.
Note some simple properties of B_φ-convex spaces.
If a normed space E is $(\varphi, k, \varepsilon)$-convex, then:

1°) For any $\delta: \; 0 < \delta < \varepsilon$ the space E is (φ, k, δ)-convex.

2°) Its completion is $(\varphi, k, \varepsilon)$-convex.

3°) Any subspace of E is $(\varphi, k, \varepsilon)$-convex.

4°) Its second dual is $(\varphi, k, \varepsilon)$-convex.

5°) There exist $p < 1$ and $\delta < \varepsilon$ such that, for any increasing sequence $\varkappa$ with $\varkappa(n) \geq \varphi^p(n)$ for all $n \in \mathbf{N}$, the space E is $(\varkappa, k, \delta)$-convex.

Proof. Note that properties 1°) and 3°) are evident.

2°) Let $\overline{x}_1, \ldots, \overline{x}_n$ belong to the complement of space E, that is, to the Banach space $\overline{E}$, $\|\overline{x}_i\| \leq 1, 1 \leq i \leq k$, and $\delta > 0$. Since E is dense in $\overline{E}$, there exist $x_1, \ldots, x_k \in \mathbf{B}(E)$ such that $\|x_i - \overline{x}_i\| < \delta/k, 1 \leq i \leq k$. Since E is $(\varphi, k, \varepsilon)$-convex, for some choice of signs $\pm$ we have $\| \pm x_1 \pm \cdots \pm x_k\| \leq (1-\varepsilon)\varphi(k)$. For the same collection of signs

$$\begin{aligned} \| \pm \overline{x}_1 \pm \cdots \pm \overline{x}_k\| &= \big\|(\pm x_1 \pm \cdots \pm x_k) + (\pm(\overline{x}_1 - x_1) \pm \cdots \pm (\pm(\overline{x}_k - x_k)\big\| \\ &\leq (1-\varepsilon)\varphi(k) + \delta. \end{aligned}$$

Since $\delta > 0$ is arbitrary, $\inf_{\pm} \| \pm \overline{x}_1 \pm \cdots \pm \overline{x}_k\| \leq (1-\varepsilon)\varphi(k)$.

4°) Let $\mathbf{B} = \mathbf{B}(E), \mathbf{B}^{**} = \mathbf{B}(E^{**})$ and let $J\colon E \to E^{**}$ be the canonical imbedding of E into E^{**}. Then $J(\mathbf{B})$ is dense in $\mathbf{B}^{**}$ in $*$-weak topology (Dunford and Schwartz, 1958, Theorem V.5).
Let $x_1^{**}, \ldots, x_k^{**} \in \mathbf{B}^{**}$, $\delta > 0$, and let a vector $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_k)$ have coordinates $\alpha_i \in \{-1, +1\}, 1 \leq i \leq k$. By the definition of the norm in E^{**} for any $\overrightarrow{\alpha}$ there exists an $x^*_{\overrightarrow{\alpha}} \in E^*$ with $\|x^*_{\overrightarrow{\alpha}}\| = 1$ and

$$\left\| \sum_{i=1}^k \alpha_i x_i^{**} \right\| \leq \left| \left\langle x^*_{\overrightarrow{\alpha}}, \sum_{i=1}^k \alpha_i x_i^{**} \right\rangle \right| + \delta.$$

Consider a neighbourhood of zero of E^{**} defined as

$$U_\delta = \bigcap_{\overrightarrow{\alpha}} \left\{ x^{**} \in E^{**}\colon \ |\langle x^*_{\overrightarrow{\alpha}}, x^{**} \rangle| < \delta \right\},$$

where the intersections are taken over all possible values of $\overrightarrow{\alpha}$. Choose $x_1, \dots, x_k \in E$ so that $J(x_i) \in U_\delta + x_i^{**}, 1 \le i \le k$. We have

$$\begin{aligned}\left\| \sum_{i=1}^k \alpha_i x_i^{**} \right\| &\le \left| \left\langle x^*_{\overrightarrow{\alpha}}, \sum_{i=1}^k \alpha_i x_i^{**} \right\rangle \right| + \delta \\ &= \left| \left\langle x^*_{\overrightarrow{\alpha}}, \sum_{i=1}^k \alpha_i J(x_i) \right\rangle + \left\langle x^*_{\overrightarrow{\alpha}}, \sum_{i=1}^k \alpha_i (x_i^{**} - J(x_i)) \right\rangle \right| + \delta \\ &\le \left\| \sum_{i=1}^k \alpha_i x_i \right\| + k\delta + \delta.\end{aligned}$$

Choosing $\overrightarrow{\alpha}$ with $\|\sum_{i=1}^k \alpha_i x_i\| < (1-\varepsilon)\varphi(k)$, we see that

$$\left\| \sum_{i=1}^k \alpha_i x_i^{**} \right\| \le (1-\varepsilon)\varphi(k) + \delta(k+1),$$

or by the arbitrariness of $\delta > 0$

$$\inf_{\pm} \left\| \sum_{i=1}^k \pm x_i^{**} \right\| \le (1-\varepsilon)\varphi(k).$$

5°) Set

$$D(k) = \sup\left\{ \inf_{\pm} \left\| \sum_{i=1}^k \pm x_i \right\| : x_1, \dots, x_k \in \mathrm{B}(E) \right\}.$$

The space E is $(\varphi, k, \varepsilon)$-convex, if and only if $D(k) \le (1-\varepsilon)\varphi(k)$. Then there exist $p < 1$ and $\delta < \varepsilon$, such that $\varphi^p(k) \ge D(k)/(1-\delta)$. Hence for all $\varkappa(k) \ge \varphi^p(k)$ the space E is $(\varkappa, k, \delta)$-convex.

3. TWO IMPORTANT SEQUENCES

Consider two sequences:

$$A_\varphi(n) = \frac{\sup\{\min_{\pm} \|\sum_{k=1}^n \pm x_k\| : x_k \in \mathrm{B}(E)\}}{\varphi(n)},$$

$$B_\varphi(n) = \frac{\sup\{(\mathbf{E}\|\sum_{k=1}^n \varepsilon_k x_k\|^2)^{1/2} : \sum_{k=1}^n \|x_k\|^2 \le 1\}}{\varphi^{1/2}(n)},$$

where (ε_k) are Bernoulli random variables.
Describe some properties of these sequences.

1) $A_\varphi(n)$ and $B_\varphi(n)$ are the smallest numbers for which

$$\min_{\pm}\left\|\sum_{k=1}^{n}\pm x_k\right\| \le \varphi(n)\,A_\varphi(n)\max_{1\le k\le n}\|x_k\|,$$

$$\mathbf{E}\left\|\sum_{k=1}^{n}\varepsilon_k x_k\right\|^2 \le \varphi(n)\,B_\varphi^2(n)\sum_{k=1}^{n}\|x_k\|^2.$$

for all $x_k \in E$;

2) $(\varphi^{1/2}(n)B_\varphi(n))$ and $(\varphi(n)A_\varphi(n))$ are increasing sequences and $A_\varphi(1) = B_\varphi(1) = 1/\varphi(1)$;

3) if X_k are symmetric random elements, then

$$\mathbf{E}\left\|\sum_{k=1}^{n}X_k\right\|^2 \le \varphi(n)\,B_\varphi^2(n)\sum_{k=1}^{n}\mathbf{E}\left\|X_k\right\|^2;$$

4) $B_\varphi(n) \le (\varphi(n)\varphi(k)/\varphi(nk))^{1/2}B_\varphi(n)B_\varphi(k)$;

5) $B_\varphi(N^k) \le (\varphi^k(N)/\varphi(N^k))^{1/2}B_\varphi^k(N)$;

6) $B_\varphi(n) \le (\varphi(n)/n)^{1/2}A_\varphi(n) \le (B_\varphi^2(n)2^n/(1+(2(n-\varphi(n)B_\varphi^2(n))))^{1/2} - n(2^n - 1)/\varphi(n))^{1/2}$.

Proof. Let us note that: properties **1)**–**2)** are trivial; **3)** easily follows by integrating **2)**; **4)** follows from the definition and **3)**; **5)** is only a particular case of **4)**. Now we shall prove **6)**. The first inequality follows from the inequalities

$$\min_{\pm}\left\|\sum_{k=1}^{n}\pm x_k\right\| \le \left(\mathbf{E}\left\|\sum_{k=1}^{n}\varepsilon_k x_k\right\|^2\right)^{1/2} \le \varphi^{1/2}(n)B_\varphi(n)\left(\sum_{k=1}^{n}\|x_k\|^2\right)^{1/2}$$
$$\le (n\varphi(n))^{1/2}B_\varphi(n)\max_{1\le k\le n}\|x_k\|.$$

For the second inequality let $\gamma < B_\varphi(n)$. Then there exist $x_1,\dots,x_n \in E$ such that

$$\sum_{k=1}^{n}\|x_k\|^2 = n \quad\text{and}\quad \left(\mathbf{E}\left\|\sum_{k=1}^{n}\varepsilon_k x_k\right\|^2\right)^{1/2} \ge (n\varphi(n))^{1/2}\gamma.$$

Let $1 \le i \le n$. Then we have

$$\sum_{k=1}^{n}(\|x_k\|-\|x_i\|)^2 \le \sum_{i=1}^{n}\sum_{k=1}^{n}(\|x_k\|-\|x_i\|)^2 = 2n\sum_{k=1}^{n}\|x_k\|^2 - 2\left(\sum_{k=1}^{n}\|x_k\|\right)^2$$
$$\le 2n^2 - 2\mathbf{E}\left\|\sum_{k=1}^{n}\varepsilon_k x_k\right\|^2 \le 2n\left(n-\varphi(n)\gamma^2\right).$$

So we have

$$n^{1/2} = \left(\sum_{k=1}^{n}\|x_k\|^2\right)^{1/2} \ge \left(\sum_{k=1}^{n}\|x_k\|^2\right)^{1/2} - \sum_{k=1}^{n}(\|x_k\|-\|x_i\|)^2$$
$$\ge n^{1/2}\|x_k\| - \left(2n(n-\varphi(n)\gamma^2)\right)^{1/2}.$$

That is,

$$\max_{1\le k\le n}\|x_k\|\le 1+\left(2n(n-\varphi(n)\gamma^2)\right)^{1/2}.$$

Hence we find that

$$\max_{1\le k\le n}\frac{\|x_k\|\gamma(n\varphi(n))^{1/2}}{1+(2n(n-\varphi(n)\gamma^2))^{1/2}}\le\gamma\left(n\varphi(n)\right)^{1/2}$$
$$\le\left(\mathrm{E}\left\|\sum_{k=1}^n\varepsilon_k x_k\right\|^2\right)^{1/2}=\left(2^{-n}\sum_{\pm}\left\|\sum_{k=1}^n\pm x_k\right\|^2\right)^{1/2}$$
$$\le\left(2^{-n}\min_{\pm}\left\|\sum_{k=1}^n\pm x_k\right\|^2+(1-2^{-n})n^2\right)^{1/2}\left(\max_{1\le k\le n}\|x_k\|^2\right)^{1/2}$$
$$\le\left(2^{-n}\varphi^2(n)A_\varphi^2(n)\right)\left(\max_{1\le k\le n}\|x_k\|\right)^2+(1-2^{-n})n^2\left(\max_{1\le k\le n}\|x_k\|^2\right)^{1/2}.$$

Letting $\gamma\to B_\varphi(n)$ after some simple computations we obtain the second inequality **6**). The proof is complete.

Let $(B,\|\cdot\|_B)$ be a symmetric Banach space of real sequences. By this we mean that $\|(t_k)\|_B=\|(t_{\pi(k)})\|_B$ for all permutations π of integers and if $(t_k)\in B$, then $(|t_k|)\in B$. It is known that symmetric spaces are ideal, that is, if $t_k\le s_k$, then $\|(t_k)\|_B\le\|(s_k)\|_B$. We shall use the notation $(t_k)_{k\le n}$ for the sequence $(t_1,\ldots,t_n,0,0,\ldots)$. Let

$$\tau_B(n)=\left\|(1)_{k\le n}\right\|_B.$$

Then

$$\|(t_k)_{k\le n}\|_B\le\tau_B(n)\min_{1\le k\le n}|t_k|.$$

We shall say that B is *finitely represented* into E, if there exists a $\gamma<1$ such that for all $n\ge 2$ and all $(t_k)_{k\le n}\in B$ there exist $x_k\in E,1\le k\le n$, such that

$$\gamma\|(t_k)_{k\le n}\|_B\le\left\|\sum_{k=1}^n t_kx_k\right\|\le\|(t_k)_{k\le n}\|_B.$$

PROPOSITION 1. *If $\tau_B(n)\ge\varphi(n)A_\varphi(n)$ for all $n\ge 2$, then B is not finitely represented into E.*

Proof. If E is finitely represented into E, then there exists $\gamma<1$ such that for all $n\ge 2$ and all $(t_k)_{k\le n}\in B$ there exist $x_k\in E,1\le k\le n$, such that

$$\gamma\left\|(t_k)_{k\le n}\right\|_B\le\left\|\sum_{k=1}^n t_kx_k\right\|.$$

We can choose a collection of signs $\pm$ such that

$$\left\|\sum_{k=1}^n\pm x_k\right\|\le\varphi(n)A_\varphi(n).$$

Taking for t_k this particular collection of signs, we obtain

$$\gamma\tau_B(n) \le \varphi(n)A_\varphi(n)$$

which contradicts the hypothesis. The proof is complete.

Definition. We shall say that E is of type B if $(\mathbf{E}\|\sum_{k=1}^n \varepsilon_k x_k\|^2)^{1/2} \le \|(\|x_k\|)_{k\le n}\|_B$ for all $x_k \in E, 1 \le k \le n$.

Examples. 1) If $B = \mathcal{L}_p, 1 \le p \le 2$, then type l_p is the well-known Rademacher type p.

2) Let $B = l_\Phi$ be an Orlicz space of sequences, where Φ: $\mathbf{R}^+ \to \mathbf{R}^+$ is an increasing function. In this space the norm is defined as

$$\|(t_k)\|_B = \inf\left\{\lambda > 0\colon \sum_{k=1}^{\infty}\Phi\left(\frac{t_k}{\lambda}\right) \le 1\right\}.$$

Type l_Φ is the type Φ introduced in (Ledoux, 1981; Fazekas, 1988; Chuprunov, 1991).

3) Let $B = d(w,p)$ be a *Lorentz space* of sequences and let $w = (w_k)$ is a nonincreasing sequence of positive reals, $p \ge 1$. In this space the norm is defined as

$$\|(t_k)\|_B = \left(\sum_{k=1}^{\infty}(t_k^*)^p w_k\right)^{1/p},$$

where (t_k^*) is a nonincreasing rearrangement of $(|t_k|)$. In the particular case of power sequences w_k we obtain the type (p,q,r) introduced in (Norvaisa, 1984).

PROPOSITION 2. *Let the sequence φ satisfy the condition* (**A**) *(see next Section) and $\alpha = \inf_{m\ge 2}\log_m B_\varphi(m)$. If there exist $\varepsilon > 0$ and $C > 0$ such that $\tau_B(n) \ge Cn^{1+\varepsilon+\alpha}$, then E is of type B.*

Proof. Let $\log_N B_\varphi(N) < \alpha + \varepsilon/2 \equiv \gamma$ for some $N \ge 2$. Then for all n we can choose $k \ge 1$ so that $N^{k-1} \le n < N^k$ and by properties **2)** and **5)** and condition (**A**) we have

$$B_\varphi(n) \le B_\varphi(N^k), \quad \left(\frac{\varphi(N^k)}{\varphi(n)}\right)^{1/2} \le B_\varphi(N),$$

$$\frac{\varphi^k(N)}{\varphi(N^{k-1})} \le N^{(k-1)\log_N B_\varphi(N)}\frac{\varphi^{1/2}(N)}{B_\varphi(N)} \le C_1 n^\gamma.$$

Now let $x_1,\dots,x_n \in E$. Split them in groups of almost the same norm: $D_k = \{1 \le j \le n \mid \beta\tau_B(f(k-1) \le x_j < \beta\tau_B(f(k))\}, k \ge 1, \beta = \|(\|x_k\|)_{k\le n}\|_B$, and an increasing sequence $f(k)$ will be chosen further. Then (D_k) is a disjoint partition of $\{1,\dots,n\}$. If $g_k = \#D_k$, then we have:

$$\left(\mathbf{E}\left\|\sum_{k=1}^n \varepsilon_k x_k\right\|^2\right)^{1/2} \le \sum_{k=1}^{\infty}\left(\mathbf{E}\left\|\sum_{j\in D_k}\varepsilon_j x_j\right\|^2\right)^{1/2} \le \sum_{k=1}^{\infty} g_k^{1/2}B_\varphi(g_k),$$

$$\left(\sum_{j\in D_k}\|x_j\|^2\right)^{1/2} \le \sum_{k=1}^{\infty}\frac{g_k B_\varphi(g_k)\beta}{\tau_B(f(k))} \le C_1\beta\sum_{k=1}^{\infty}\frac{g_k^{1+\gamma}}{\tau_B(f(k))}.$$

Now g_k can be estimated:

$$\beta = \left\|(\|x_k\|)_{k\le n}\right\|_B \ge \left\|(\|x_j\|)_{j\in D_k}\right\|_B \ge \tau_B(g_k)\min_{j\in D_k}\|x_j\| \ge \frac{\tau_B(g_k)\beta}{\tau_B(g_{k-1})}.$$

So, $\tau_B(g_k) \le \tau_B(f(k-1))$ and $g_k \le f(k-1)$. Hence

$$\left(\mathbf{E}\left\|\sum_{k=1}^{n}\varepsilon_k x_k\right\|^2\right)^{1/2} \le \left\|(\|x_k\|)_{k\le n}\right\|_B C_1\sum_{k=1}^{\infty}\frac{f(k)^{1+\gamma}}{\tau_B(f(k))}.$$

Since $n^{1+\gamma}/\tau_B(n) \to 0$ as $n \to \infty$ we can choose a sequence $f(k)$ in such a way that the latter series converges. The proof is complete.

4. CHARACTERIZATION OF B_φ-CONVEX SPACES BY LAWS OF LARGE NUMBERS FOR WEIGHTED SUMS

Let E be a Banach space and let (X_n) be a sequence of independent centered random elements taking values in E. Put

$$T_n = \sum_{k=1}^{n} a_k(n)X_k,$$

where $a = (a_k(n), 1 \le k \le n, n \in \mathbf{N})$ is a triangular array of constants which will be called a *weight*. A weight a determines the sequences

$$\varphi(n) = \frac{1}{\max_{1\le k\le n}|a_k(n)|} \quad \text{and} \quad \varkappa(n) = \frac{1}{\min_{1\le k\le n}|a_k(n)|}, \qquad n \in \mathbf{N}$$

and we require that $(\varphi(n))_1^\infty$ and $(\varkappa(n))_1^\infty$ are increasing sequences.
Now we put the same conditions on the sequences φ and $\varkappa$ corresponding to the weight a.

Condition (**A**). $\varphi(N^m) \ge \varphi^m(N)$ *for all* $N, m \in \mathbf{N}$.

Condition (B). *There exist* $M > 1$ *and* $D > 1$ *such that for all* $n \in \mathbf{N}$ *the inequality* $\varkappa(Mn) \le D\varkappa(n)$ *holds.*
For a sequence φ introduce the following characteristics:

$$p(\varphi) = \inf\left\{p\colon \sum_{n=1}^{\infty}\varphi^{-p}(n) < \infty\right\},$$

$$q(\varphi) = \sup\left\{q\colon \sup_{n\ge 2}\frac{\varphi^q(n)}{n} \le 1\right\}.$$

The following lemma exposes the connection between these characteristics.

LEMMA 1. *If* φ *satisfies the condition* (**A**), *then* $p(\varphi) = q(\varphi) = \lim_{n\to\infty}\ln n/\ln\varphi(n)$. *The supremum in the definition of* $q(\varphi)$ *is attained, that is,* $\sup_{n\ge 2}\varphi^{q(\varphi)}(n)/n \le 1$.

Proof 1. We first prove that $\lim_{n\to\infty}\ln n/\ln\varphi(n)$ exists and is equal to $q(\varphi)$. Take any $q < q(\varphi)$. Then for all $n \ge 2$ one has $\varphi^q(n) \le n$, hence $\underline{\lim}\ln n/\ln\varphi(n) \ge$

$\varliminf \ln n/(q^{-1}\ln n) = q$, that is, $\varliminf \ln n/\ln\varphi(n) \geq q(\varphi)$. Now take any $q > q(\varphi)$. There exists an $N \geq 2$ such that $\varphi(N) > N^{1/q}$. For all $n \geq N$ we have

$$\varphi(n) = \varphi(N^{\log_N n}) \geq \varphi(N^{[\log_N n]}),$$

where $[t]$ is the integer part of t. Applying the condition $(\mathbf{A})\,[\log_N n]$ times, we see that

$$\varphi(n) \geq \varphi^{[\log_N n]}(N) \geq \varphi^{\log_N n-1}(N) \geq (N^{1/q})^{\log_N n-1} = (nN)^{1/q}.$$

Hence $\varlimsup \ln n/\ln\varphi(n) \leq \varlimsup \ln n/(q^{-1}\ln n + \ln N^{-1/q}) = q$. That is, $\varlimsup \ln n/\ln\varphi(n) \leq q(\varphi)$. So $\lim_{n\to\infty} \ln n/\ln\varphi(n) = q(\varphi)$.

II. Now let us turn to the proof of $p(\varphi) = q(\varphi)$. It is evident that $p(\varphi) \geq q(\varphi)$, since for any $p > p(\varphi)$ the series $\sum_{n=1}^{\infty}\varphi^{-p}(n)$ is convergent, hence $\varphi^{p}(n)/n \to \infty$. Prove that $p(\varphi) \leq q(\varphi)$. To do this, we show that for all $p > q(\varphi)$ the inequality $p \geq p(\varphi)$ holds. Let $p > q(\varphi)$. For any $\varepsilon > 0$ there exists an N such that $\ln\varphi(n)/\ln n \geq 1/q(\varphi) - \varepsilon/p$ for all $n \geq N$. Hence

$$\sum_{n=N}^{\infty}\varphi^{-p}(n) = \sum_{n=N}^{\infty} n^{-p\log_n\varphi(n)} = \sum_{n=N}^{\infty} n^{-p\ln\varphi(n)/\ln n}$$
$$\leq \sum_{n=N}^{\infty} n^{-p(1/q(\varphi)-\varepsilon/p)} = \sum_{n=N}^{\infty} n^{-p/q(\varphi)+\varepsilon}.$$

Let $\varepsilon > 0$ be so small that $p/q(\varphi) - \varepsilon > 1$. Then the series $\sum_{n=1}^{\infty}\varphi^{-p}(n)$ is convergent, that is, $p \geq p(\varphi)$.

III. Finally we show that the supremum in the definition of $q(\varphi)$ is attained. Assume the contrary: let there exist an $N \geq 2$ such that $\varphi(N) > N^{1/q(\varphi)}$, that is, $\varphi(N) = sN^{1/q(\varphi)}$ for some $s > 1$. By condition $(\mathbf{A})$: $\varphi(N^k) \geq \varphi^k(N) = s^k N^{k/q(\varphi)}$. This is a contradiction:

$$q(\varphi) = \lim_{n\to\infty}\frac{\ln n}{\ln\varphi(n)} = \lim_{k\to\infty}\frac{\ln N^k}{\ln\varphi(N^k)} \leq \frac{\ln N}{\ln s + (\ln N)/q(\varphi)} < q(\varphi).$$

The Theorem is proved.

The main result of this paragraph is the following assertion.

THEOREM. *Consider the statements.*

(1) *E is of stable type p, $1 \leq p < 2$ (that is, B_p-convex).*

(2) *For all weights a the condition $\sum_{n=1}^{\infty}\mathbf{P}\{\xi > \varphi(n)\} < \infty$ implies $T_n \to 0$ a. s. for all independent symmetric random elements X_k with $(X_k) \prec \xi$.*

(2′) *Statement (2) holds for at least one weight a.*

(3) *For all the weights a*

$$\sum_{k=1}^{n} a_k(n)\varepsilon_k x_k \to 0 \quad \textit{in probability as} \quad n \to \infty$$

for all $x_k \in \mathrm{B}(E)$ (where ε_k are independent Bernoulli random variables).

(3′) *Statement* (3) *holds for at least one weight* a.

(4) E *is* $B_\varkappa$*-convex.*

If φ *satisfies condition* (**A**) and $p(\varphi) \le p$, *then* (**1**) $\Rightarrow$ (**2**); *if* $\varkappa$ *satisfies the condition* (**B**), *then* (**3′**) $\Rightarrow$ (4); *if* $\varkappa$ *satisfies the condition* (**A**) *and* $p \le q(\varkappa)$, *then* (4) $\Rightarrow$ (**1**); *finally, it is evident that*

$$\begin{array}{ccc} (2) & \Rightarrow & (2') \\ \Downarrow & & \Downarrow \\ (3) & \Rightarrow & (3'). \end{array}$$

For the proof of Theorem we need some lemmas.

LEMMA 2. *If* φ *satisfies condition* (**A**), *then*

$$\sum_{k=n}^{\infty} \varphi^{-p}(k) = O\left(n\varphi^{-p}(n)\right)$$

for all $p > p(\varphi)$.

Proof. In fact, by the condition imposed on φ we have

$$\sum_{k=n}^{\infty} \varphi^{-p}(k) = \sum_{l=1}^{\infty} \sum_{k=n^l}^{n^{l+1}} \varphi^{-p}(k) \le \sum_{l=1}^{\infty} n^l(n-1)\varphi^{-p}(n^l)$$
$$\le n\varphi^{-p}(n) \sum_{l=1}^{\infty} n^l \varphi^{-p(l-1)}(n).$$

Let us prove that the last series converges. Choose an r such that $p(\varphi) < r < p$, then (see the proof of Lemma 1 (II)) there exists an N such that $\varphi(N) \ge N^{1/r}$. Then

$$\varphi(n) > \left(\frac{n}{N}\right)^{1/r} \quad \text{and} \quad \sum_{l=1}^{\infty} n^l \varphi^{-p(l-1)}(n) \le \sum_{l=1}^{\infty} n^{l(1-p/r)} N^{p(l-1)/r}$$

for all $n > N$. Let $n \ge N^{2p/(p-r)}$, then $\sum_{l=1}^{\infty} n^l \varphi^{-p(l-1)}(n) \le N^{-p/r} \sum_{l=1}^{\infty} N^{-ql}$ where $q = p/(p-r) > 0$. Note that the last series is convergent. The proof is completed.

The next lemma is a generalization of the classical result (Stout, 1974, p.127–128).

LEMMA 3. *Let* $(X_k) \prec \xi$ *and* $\sum_{n=1}^{\infty} \mathbf{P}\{\xi > \varphi(n)\} < \infty$. *If*

$$\sum_{k=n}^{\infty} \varphi^{-p}(k) = O\left(n\varphi^{-p}(n)\right),$$

then

$$\sum_{n=1}^{\infty} \varphi^{-p}(n)\, \mathbf{E}\, \|X_n\|^p I\{\|X_n\| < \varphi(n)\} < \infty.$$

Proof. Note that if $(X_k) \prec \xi$, then

$$\mathbf{E}\left\|X_k I\{\|X_k\| < t\}\right\|^p \le \mathbf{E}\,\xi^p I\{\xi < t\}.$$

We have

$$\begin{aligned}\sum\nolimits_1 &= \sum_{n=1}^{\infty} \varphi^{-p}(n)\,\mathbf{E}\,\|X_n\|^p I\{\|X_n\| < \varphi(n)\}\\
&\le \sum_{n=1}^{\infty} \varphi^{-p}(n)\,\mathbf{E}\,\xi^p I\{\xi < \varphi(n)\}\\
&= \sum_{n=1}^{\infty} \varphi^{-p}(n) \sum_{k=1}^{n} \mathbf{E}\,\xi^p I\{\varphi(k-1) \le \xi < \varphi(k)\}\\
&= \sum_{k=1}^{\infty} \mathbf{E}\,\xi^p I\{\varphi(k-1) \le \xi < \varphi(k)\} \sum_{n=k}^{\infty} \varphi^{-p}(n).\end{aligned}$$

Since the sequence φ is increasing, we can define an increasing function $\psi(t)$ such that $\psi(\varphi(n)) = n$. It is well known that $\mathbf{E}\,\psi(\xi) \le 1 + \sum_{n=1}^{\infty} \mathbf{P}\{\xi > \varphi(n)\}$. Then by the condition $\mathbf{E}\,\psi(\xi) < \infty$. By the condition on φ there exists a $C > 0$ such that

$$\begin{aligned}\sum\nolimits_1 &\le \sum_{k=1}^{\infty} \mathbf{E}\,\psi(\xi)\,\frac{\xi^p}{\psi(\xi)}\, I\{\varphi(k-1) \le \xi < \varphi(k)\} C k \varphi^{-p}(k)\\
&\le C \sum_{k=1}^{\infty} \mathbf{E}\,\psi(\xi)(\varphi^p(k)/(k-1)) I\{\varphi(k-1) \le \xi < \varphi(k)\} k \varphi^{-p}(k)\\
&\le C \sum_{k=1}^{\infty} \mathbf{E}\,\psi(\xi) I\{\varphi(k-1) \le \xi < \varphi(k)\} \le C\,\mathbf{E}\,\psi(\xi) < \infty.\end{aligned}$$

The proof is completed.

LEMMA 4. *If (X_k) are independent symmetric random elements, then* $\mathbf{E}\,(\sup_{n\ge k} \|T_n\|) \le 4\,\mathbf{E}\,(\sup_{n \ge k} \|S_n\|/\varphi(n))$ *for any $k \ge 1$.*

Proof. By Levy's inequality (Vakhania et al., 1987, Proposition V.2.3) for all $k \le m$: $\mathbf{Q} = \mathbf{P}\{\sup_{k \le n \le m} \|T_n\| > t\} \le 2\mathbf{P}\{\|T_m\| > t\}$. Then, by Kwapien's inequality (Vakhania et al., 1987, Lemma V.4.1 (a)):

$$\mathbf{Q} \le 4\mathbf{P}\left\{\frac{\|S_m\|}{\varphi(m)} > t\right\} \le 4\mathbf{P}\left\{\sup_{k \le n \le m} \frac{\|S_n\|}{\varphi(n)} > t\right\}.$$

Hence, $\mathbf{P}\{\sup_{n\ge k} \|T_n\| > t\} \le 4\mathbf{P}\{\sup_{n \ge k} \|S_n\|/\varphi(n) > t\}$. So

$$\begin{aligned}\mathbf{E}\left(\sup_{n\ge k} \|T_n\|\right) &= \int_0^{\infty} \mathbf{P}\left\{\sup_{n \ge 1} \|T_n\| > t\right\} dt\\
&\le 4 \int_0^{\infty} \mathbf{P}\left\{\sup_{n\ge 1} \frac{\|S_n\|}{\varphi(n)} > t\right\} dt = 4\,\mathbf{E}\left(\sup_{n\ge k} \frac{\|S_n\|}{\varphi(n)}\right).\end{aligned}$$

The proof is completed.

LEMMA 5. *Let E be a space of Rademacher type p with $1 < p < 2$. If (X_k) are independent symmetric random elements satisfying $\sum_{k=1}^{\infty} \mathbf{E}\|X_k\|^p/\varphi^p(k) < \infty$, then $\mathbf{E}(\sup_{n\geq 1}\|S_n\|/\varphi(n)) < \infty$.*

Proof. For any $k \geq 0$ set $m_k = \min\{n\colon \varphi(n) \geq 2^k\}$ and $N_k = \{n\colon m_k \leq n < m_{k+1}\}$. Then by the choice of N_k and by Levy's inequality we have

$$\begin{aligned}\mathbf{Q} = \mathbf{P}\left\{\sup_{n\geq 1}\frac{\|S_n\|}{\varphi(n)} > t\right\} &= \mathbf{P}\left\{\sup_{k\geq 0}\max_{n\in N_k}\frac{\|S_n\|}{\varphi(n)} > t\right\}\\ &\leq \sum_{k=0}^{\infty}\mathbf{P}\left\{\max_{n\in N_k}\frac{\|S_n\|}{\varphi(n)} > t\right\} \leq \sum_{k=0}^{\infty}\mathbf{P}\left\{\max_{n\in N_k}\|S_n\| > 2^k t\right\}\\ &\leq 2\sum_{k=0}^{\infty}\mathbf{P}\{\|S_{m_{k+1}-1}\| > 2^k t\}.\end{aligned}$$

Now applying Chebyshev's inequality and taking into account the type p of E (Pisier, 1990), we observe that

$$\mathbf{Q} \leq 2t^{-p}\sum_{k=0}^{\infty}2^{-kp}\mathbf{E}\|S_{m_{k+1}-1}\|^p \leq Ct^{-p}\sum_{k=0}^{\infty}2^{-kp}\sum_{i=1}^{m_{k+1}-1}\mathbf{E}\|X_i\|^p.$$

Changing the order of summation, we obtain that

$$\mathbf{Q} \leq Ct^{-p}\sum_{i=1}^{\infty}\mathbf{E}\|X_i\|^p\sum_{k\geq k_0}2^{-kp},$$

where $k_0 = \min\{k\colon m_{k+1} - 1 \geq i\}$. Note that for k_0 the inequalities $m_{k_0+1} - 1 \geq i$ and $\varphi(m_{k_0+1} - 1) > 2^{k_0+1}$ are valid.
Next we estimate the following sum:

$$\sum_{k\geq k_0}2^{-kp} = \frac{2^p}{1-2^{-p}}2^{-(k_0+1)p} \leq C(\varphi(m_{k_0+1}-1))^{-p} < C\varphi^{-p}(i).$$

So $\mathbf{Q} \leq Ct^{-p}\sum_{i=1}^{\infty}\mathbf{E}\|X_i\|^p/\varphi^p(i)$. Then

$$\begin{aligned}\mathbf{E}\left(\sup_{n\geq 1}\frac{\|S_n\|}{\varphi(n)}\right) &\leq 1 + \int_1^{\infty}\mathbf{P}\left\{\sup_{n\geq 1}\frac{\|S_n\|}{\varphi(n)} > t\right\}dt\\ &\leq 1 + C\sum_{i=1}^{\infty}\frac{\mathbf{E}\|X_i\|^p}{\varphi^p(i)}\int_1^{\infty}t^{-p}dt < \infty.\end{aligned}$$

The proof is completed.

The next lemma can be considered as an analogue of Chung's law of large numbers for the weighted sums.

LEMMA 6. *If E is of Rademacher type $p, 1 < p < 2$, then for all independent symmetric random elements (X_k) the condition $\sum_{k=1}^{\infty} \mathbf{E}\,\|X_k\|^p/\varphi^p(k) < \infty$ implies $T_n \to 0$ a. s.*

Proof. Since E is of type $p, \mathbf{E}\,\|\sum_{k=1}^{\infty} X_k/\varphi(k)\|^p \le C\sum_{k=1}^{\infty}\mathbf{E}\,\|X_k\|^p/\varphi^p(k)$. Hence the series $\sum_{k=1}^{\infty} X_k/\varphi(k)$ converges in $L^p(E)$. Then by (Vakhania et al., 1987; Theorem V.2.3) it converges a. s. By Kronecker's lemma $S_n/\varphi(n) \to 0$ a. s. as $n \to \infty$. Thus $\sup_{k \ge n} S_k/\varphi(k) \to 0$ a.s. as $n \to \infty$. By Lemma 5 one has $\mathbf{E}\,\sup_{n\ge 1}\|S_n\|/\varphi(n) < \infty$. Hence, by Lebesgue dominated convergence theorem $\mathbf{E}\,\sup_{k\ge n}\|S_k\|/\varphi(k) \to 0$ as $n \to \infty$. By Lemma 4 we have $\mathbf{E}\,\sup_{k \ge n}\|T_k\| \to 0$ as $n \to \infty$. Finally, by Chebyshev's inequality $\mathbf{P}\{\sup_{k\ge n}\|T_k\| > \varepsilon\} \to 0$ as $n \to \infty$ for all $\varepsilon > 0$. Thus $T_n \to 0$ a. s. which completes the proof.

LEMMA 7. *Let $\varkappa$ satisfy condition* (**B**) *and E is not $B_\varkappa$-convex. Then there exist sequences $(x_n) \in \mathbf{B}(E)$ and $(n_k)_{k=1}^{\infty}$, such that*

$$\inf_{\pm}\left\|\varkappa(n_k)^{-1}\sum_{i=1}^{n_k} \pm x_i\right\| > \frac{1}{2D}.$$

Proof will be carried out by induction. Put $n_1 = 1$ and take for x_1 an element of E with $1/(2D) < \|x_1\| \le 1$.
Assume that $n_1 < n_2 < \cdots < n_m$ and $(x_i)_{i=1}^{n_m} \subset \mathbf{B}(E)$ satisfy condition of Lemma. Choose an n_{m+1} so that $n_{m+1} > Mn_m/(M-1)$ (that is, $n_{m+1} < M(n_{m+1} - n_m)$) and $\varkappa(n_{m+1}) > 4Dn_m$.
Since E is not $B_\varkappa$-convex, there exist $x_{n_m+1}, \dots, x_{n_{m+1}} \in \mathbf{B}(E)$ such that for any choice of signs $\pm$ we have

$$\| \pm x_{n_m+1} \pm \cdots \pm x_{n_{m+1}}\| > \left(1 - \tfrac{1}{4}\right)\varkappa(n_{m+1} - n_m).$$

Hence, by condition (**B**)

$$\begin{aligned}
&\frac{1}{\varkappa(n_{m+1})}\| \pm x_1 \pm \cdots \pm x_{n_{m+1}}\| \\
&\ge \frac{1}{\varkappa(n_{m+1})}\| \pm x_{n_m+1} \pm \cdots \pm x_{n_{m+1}}\| - \frac{1}{\varkappa(n_{m+1})}\| \pm x_1 \pm \cdots \pm x_{n_m}\| \\
&\ge \frac{1}{\varkappa(M(n_{m+1} - n_m))}\| \pm x_{n_m+1} \pm \cdots \pm x_{n_{m+1}}\| - \frac{n_m}{\varkappa(n_{m+1})} \\
&\ge \left(1 - \frac{1}{4}\right)\frac{\varkappa(n_{m+1} - n_m)}{\varkappa(M(n_{m+1} - n_m))} - \frac{1}{4}D \ge \frac{1}{2D}.
\end{aligned}$$

Now we have everything we need to prove Theorem.

Proof of Theorem. First we show that (**1**) $\Rightarrow$ (**2**). Since E is of stable type $p < 2$ and $p(\varphi) \le p$, there exists a $q > p(\varphi)$ such that E is of Rademacher type q (Pisier, 1990). Let $X_n' = X_n I\{\|X_n\| \le \varphi(n)\}$. Then, by virtue of Lemmas 2 and 3, $\sum_{n=1}^{\infty}\varphi^{-q}(n)\,\mathbf{E}\,\|X_n'\|^q < \infty$. Hence by Lemma 6 we have $\sum_{k=1}^{n} a_k(n)X_k' \to 0$ a. s. Moreover, $\sum_{n=1}^{\infty}\mathbf{P}\{X_n \ne X_n'\} = \sum_{n=1}^{\infty}\mathbf{P}\{X_n > \varphi(n)\} \le \sum_{n=1}^{\infty}\mathbf{P}\{\xi > \varphi(n)\} < \infty$. By the Borel–Cantelli lemma $\mathbf{P}\{X_n \ne X_n' \text{ infinitely often}\} = 0$. Hence $T_n \to 0$ a. s. as $n \to \infty$.

Now show that $(3') \Rightarrow (4)$. Suppose E is $B_\varkappa$-convex. Then the conclusion of Lemma 7 holds for some sequence $(x_n) \subset \mathbf{B}(E)$. By the condition, $Y_n = \sum_{k=1}^n a_k(n)\varepsilon_k x_k \to 0$ in probability as $n \to \infty$. Fix an $\varepsilon > 0$ and let n be so large that $\mathbf{P}\{\|Y_n\| > \varepsilon\} < 1/8$. Put $Z_k = a_k(n)\varepsilon_k x_k$, then $\|Z_k\| \le a_k(n) \le 1/\varphi(n)$ and $Y_n = \sum_{k=1}^n Z_k$. By Levy's and Kolmogorov's inequalities (Vakhania et al., 1987, Propositions V.2.3 and V.3.1)

$$\frac{1}{8} \ge \mathbf{P}\left\{\left\|\sum_{k=1}^n Z_k\right\| \ge \varepsilon\right\} \ge \frac{1}{4}\left(1 - \frac{(\varepsilon + 1/\varphi(n))^2 + \varepsilon^2/2}{\mathbf{E}\|Y_n\|^2}\right).$$

Hence $\mathbf{E}\|Y_n\|^2 \le 2((\varepsilon + 1/\varphi(n))^2 + \varepsilon^2/2)$. By the arbitrariness of ε, $\mathbf{E}\|Y_n\|^2 \to 0$ as $n \to \infty$. Note that by Lemma V.4.1(b) (Vakhania et al., 1987)

$$\mathbf{E}\left\|\frac{1}{\varkappa(n)}\sum_{k=1}^n \varepsilon_k x_k\right\|^2 \le 2\mathbf{E}\left\|\sum_{k=1}^n a_k(n)\varepsilon_k x_k\right\|^2 = \mathbf{E}\|Y_n\|^2 \to 0,$$

that is, $(\varkappa(n))^{-1}\sum_{k=1}^n \varepsilon_k x_k \to 0$ in $L^2(E)$ and hence in probability. But this contradicts to the conclusion of Lemma 6.

Finally we prove that $(4) \Rightarrow (1)$. Suppose E is not of stable type $p = q(\varkappa)$. Then l_p is finitely represented in E (Pisier, 1990), that is, for all $n \in \mathbf{N}$ and $\varepsilon > 0$ there exist $x_1, \dots, x_n \in E$ such that for all $s_1, \dots, s_n \in \mathbf{R}$ the inequality

$$(1-\varepsilon)\left(\sum_{k=1}^n |s_k|^p\right)^{1/p} \le \left\|\sum_{k=1}^n s_k x_k\right\| \le \left(\sum_{k=1}^n |s_k|^p\right)^{1/p}$$

is fulfilled.

Since E is $B_\varkappa$-convex, there exist $n \ge 2, \varepsilon > 0$ such that $\|\pm x_1 \pm \cdots \pm x_n\| \le (1-\varepsilon)\varkappa(n)$ for all $x_k \in \mathbf{B}(E)$. Taking for s_i this particular collection of signs, we see that

$$n^{1/p}\left(1 - \frac{\varepsilon}{2}\right) \le \left\|\sum_{k=1}^n \pm x_k\right\| \le (1-\varepsilon)\,\varkappa(n)$$

or

$$\frac{\varkappa^p(n)}{n} \ge \left(\frac{1-\varepsilon/2}{1-\varepsilon}\right)^p > 1.$$

But this contradicts to the choice of $p = q(\varkappa)$, since from our first lemma one has $\sup_{n\ge 2} \varkappa^{q(\varkappa)}(n)/n \le 1$.

5. SUBSEQUENT PROPERTIES OF B_φ-CONVEX SPACES

With the help of the last theorem we can obtain a number of other properties of B_φ convex spaces.

6°. If φ satisfies condition (**A**) and E is B_φ-convex, then E is of stable type $p \le p(\varphi)$.

Remark. It is evident that the power function satisfies conditions (**A**) and (**B**). In this case we obtain the corollary to the result of Marcus–Woyczynski–Shangua. Hence a more interesting example is $\varphi(n) = n^{1/p}(\ln n + 1)$, where $1 < p < 2$.
Note that the implication (**4**) $\Rightarrow$ (**1**) of Theorem implies the next statement: B_φ-convexity (φ satisfies the condition (**A**)) makes sense only, if $p(\varphi) < 2$, because the stable type p makes sense only for $p \le 2$. Moreover, if φ satisfies the condition (**A**) and $p(\varphi) < 1$, then any space is B_φ-convex. This is the elementary corollary to the following property **7°**.

7°. Let φ satisfy condition (**A**). E is B_φ-convex if and only if $A_\varphi(k) \to 0$ as $k \to \infty$.

Proof. It is evident that if $A_\varphi(k) \to 0$, then E is B_φ-convex.
On the other hand, if E is B_φ-convex, then by implication (**4**) $\Rightarrow$ (**1**) of Theorem, E is of stable type $p > q(\varphi)$. For any $x_1, \dots, x_k \in \mathbf{B}(E)$ we have $\sum_{k=1}^{\infty} \|x_k\|/\varphi^p(k) \le \sum_{k=1}^{\infty} \varphi^{-p}(k)$. Hence, the former series converges ($p > p(\varphi)$). By Kronecker's lemma $\varphi^{-p}(k)\sum_{i=1}^{k} \|x_i\|^p \to 0$ as $k \to \infty$. Since E is of Rademacher type p

$$\left(\mathbf{E}\left\|\frac{1}{\varphi(k)}\right\|\sum_{i=1}^{k}\|\varepsilon_i x_i\|^p\right)^{1/p} \le C\left(\frac{1}{\varphi^p(k)}\sum_{i=1}^{k}\|x_i\|^p\right)^{1/p} \to 0 \quad \text{as} \quad k \to \infty.$$

The proof is completed.

With the help of the last property it is easy to prove the following property.

8°. Let E_1 and E_2 be normed spaces and let $R\colon E_1 \to E_2$ be a linear continuous and open operator. Let E_1 be B_φ-convex and φ satisfy condition (**A**). Then $R(E_1) \subseteq E_2$ is B_φ-convex.

Proof. Since R is continuous, there exists an $L > 0$, such that $\|R(x)\| \le L$ for all $x \in \mathbf{B} = \mathbf{B}(E_1)$. Since the mapping is open, it follows that $R(\mathbf{B})$ is the neighborhood of zero in $R(E_1)$, that is, there exists a $\delta > 0$ for which $\{y \in R(E_1)\colon \|y\| < \delta\} \subset R(\mathbf{B})$.
From 7° it follows that we can choose k so large that $A_\varphi(k) \le \delta/2L$. Let $y_1, \dots, y_k$ belong to the unit ball of $R(E_1)$. Then $\delta y_1, \dots, \delta y_k \in R(\mathbf{B})$, that is, there exist $x_1, \dots, x_k \in \mathbf{B}$ such that $Rx_1 = \delta y_1, 1 \le i \le k$. Select signs $\pm$ so that the inequality $\|\sum_{i=1}^{k} \pm x_i\|/\varphi(k) \le \delta/(2L)$ holds. Then

$$\begin{aligned}\frac{1}{\varphi(k)}\left\|\sum_{i=1}^{k} \pm y_i\right\| &= \frac{1}{\delta\varphi(k)}\left\|\sum_{i=1}^{k} \pm \delta y_i\right\| = \frac{1}{\delta\varphi(k)}\left\|\sum_{i=1}^{k} \pm Rx_i\right\| \\ &\le \frac{L}{\delta\varphi(k)}\left\|\sum_{i=1}^{k} \pm x_i\right\| \le \frac{1}{2}.\end{aligned}$$

Hence $R(E_1)$ is $(\varphi, k, 1/2)$-convex. The proof is completed.

COROLLARY. *If the normed spaces E_1 and E_2 are isomorphic and φ satisfies condition* (**A**), *then E_1 is B_φ-convex if and only if E_2 is B_φ-convex.*

REFERENCES

Acosta, A. de (1981). Inequalities for B-valued random vectors with applications to the strong law of large numbers. *Ann. Probab.* **9**, 157–161.

Azlarov, T., and Volodin, N. (1981). Law of large numbers for identically distributed Banach space valued random variables. *Theory Probab. Appl.* **26**, 573–580.

Beck, A. (1962). Convexity condition in Banach spaces and the law of large numbers. *Proc. Amer. Math. Soc.* **13**, 329–334.

Dunford, N., and Schwartz, J. (1958). *Linear Operators. General theory.* Interscience Publ., London.

Chuprunov, A. (1991). On a generalization of spaces of stable and Rademacher types p. *Theory Probab. Appl.* **36**, 521–534.

Fazekas, I. (1988). On Banach spaces of type Φ. In: *Function Spaces. Proceedings of the International Conference held in Poznan, 1986.* Ed. by J. Musielak, *Ser. Teubner Texte zur Math.* **103**. Teubner, Leipzig. pp. 20–25.

Ledoux, M. (1981). Loi du logarithme itere et types Φ d'espaces de Banach. *Ann. Sci. Univ. Clermont-Ferrand II, Math.* **19**, 207–220.

Marcus, M., and Woyczynski, W. (1979). Stable measures and central limit theorem in Banach spaces. *Trans. Amer. Math. Soc.* **271**, 71–102.

Maurey, B., and Pisier, G. (1976). Series de variables aleatoires vectorielles indepedantes et proprietes geometrique des espaces de Banach. *Studia Math.* **58**, 45–90.

Mikosch, T., and Norvaisa, R. (1987). Strong law of large numbers for fields of Banach space valued random variables. *Probab. Theory Rel. Fields* **74**, 241–253.

Norvaisa, R. (1984). The law of large numbers for identically distributed random variables taking values in Banach spaces. *Lietuvos Math. Rinkings* **24**, 133–151.

Pisier, G. (1990). Probabilistic methods in the geometry of Banach spaces. *Lect. Notes Math.* **1206**, 167–241.

Shangua, A. (1978). Laws of large numbers in Banach spaces. *Proceedings of Georgia Academy of Sciences* **91**, 16–20.

Stout, W. (1974). *Almost Sure Convergence.* Academic Press, New York.

Vakhania, N., Tarieladze, V., and Chobanian, S. (1987). *Probability Distributions on Banach Spaces.* D. Reidel, Dordrecht.