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A Note on Complete Convergence for Arrays of Rowwise Independent Banach Space Valued Random Elements

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We obtain complete convergence results for arrays of rowwise independent Banach space valued random elements. Compared with similar results presented in the probabilistic literature our conditions are weaker.

Keywords Banach space valued random element; Complete convergence; Convergence in probability; Rowwise independence.

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1. Introduction

Let $(\Omega, \mathscr{F}, P)$ be a probability space and let B be a real separable Banach space with norm $\|\cdot\|$. A *random element* is defined to be an $\mathscr{F}$-measurable mapping of Ω into B with the Borel σ-algebra (i.e., the σ-algebra generated by the open sets determined by $\|\cdot\|$). The concept of independent random elements is a direct extension of the

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concept of independent random variables. A detailed account of basic properties of random elements in real separable Banach spaces can be found in Taylor [1].

Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise independent, but not necessarily identically distributed, random elements taking values in B. In general the case $k_n = \infty$ is not precluded. Rowwise independence means that the random elements within each row are independent but that no independence is assumed between rows.

A sequence of Banach space valued random elements $\{U_n, n \geq 1\}$ is said to *converge completely*, if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\{\|U_n\| > \varepsilon\} < \infty.$$

By the Borel-Cantelli lemma, this implies $\|U_n\| \to 0$ almost surely as $n \to \infty$. The converse implication is true if $\{U_n, n \geq 1\}$ are independent. This notion was given first by Hsu and Robbins [2] in the case of real-valued random variables. From then on, there are many authors who devote their study to the complete convergence for partial sums or weighted sums of independent real-valued random variables or Banach space valued random elements (see [3–5, 7]).

Hu et al.'s [5, Theorem 3.1] presents a general result establishing complete convergence for row sums of an array of rowwise independent but not necessary identically distributed Banach space valued random elements. Their result also specified the corresponding rate of convergence. Thus, Hu et al. [5] result unifies and extends previously obtained results in the literature and is given in the following theorem.

Theorem A. *Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise independent random elements and let $\{c_n, n \geq 1\}$ be a sequence of positive constants such that*

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P\big\{\|X_{nk}\| > \varepsilon\big\} < \infty \quad \text{for all } \varepsilon > 0, \tag{1.1}$$

there exist $p \geq 1/2$, $J \geq 2$, and $\delta > 0$ such that

$$\sum_{n=1}^{\infty} c_n \left(E\left[\sum_{k=1}^{k_n} \|X_{nk} I\big\{\|X_{nk}\| \leq \delta\big\}\|^2 \right]^p \right)^J < \infty, \tag{1.2}$$

and

$$\left\| \sum_{k=1}^{k_n} X_{nk} \right\| \to 0 \quad \text{in probability}. \tag{1.3}$$

Furthermore, suppose that

$$\sum_{k=1}^{k_n} P\big\{\|X_{nk}\| > \delta\big\} = o(1) \quad \text{as } n \to \infty \tag{1.4}$$

if $\liminf_{n\to\infty} c_n = 0$. *Then*

$$\sum_{n=1}^{\infty} c_n P\left\{ \left\| \sum_{k=1}^{k_n} X_{nk} \right\| > \varepsilon \right\} < \infty \quad \textit{for all } \varepsilon > 0. \tag{1.5}$$

It is implicitly assumed in Theorem A that in the case $k_n = \infty$, the series $\sum_{k=1}^{k_n} X_{nk}$ converge almost surely.

Hu et al. [7] and Ahmed et al. [8] present applications of Theorem A to obtain new complete convergence results.

Kruglov et al. [9], shows that in the case of random variables, assumptions (1.3) and (1.4) are redundant. Namely, the following result was proved [9, Theorem 1].

Theorem B. *Let* $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ *be an array of rowwise independent random variables and let* $\{c_n, n \ge 1\}$ *be a sequence of positive constants that satisfy condition* (1.1), *and assume that there exist* $J > 0$, $\delta > 0$ *and* $p \ge 1$ *such that*

$$\sum_{n=1}^{\infty} c_n \left(E \left| \sum_{k=1}^{k_n} [X_{nk} I[|X_{nk}| \le \delta] - EX_{nk} I(|X_{nk}| \le \delta)] \right|^p \right)^J < \infty.$$

Then

$$\sum_{n=1}^{\infty} c_n P\left\{ \max_{1\le m\le k_n} \left| \sum_{k=1}^{m} [X_{nk} - EX_{nk} I(|X_{nk}| \le \delta)] \right| > \varepsilon \right\} < \infty,$$

for all $\varepsilon > 0$.

It is possible to show that condition (1.3) cannot be omitted from Theorem A if we consider a general (without any geometrical conditions) Banach space setting. It is also shown in Hu et al. [5] in Proposition 3.1 that condition (1.1) is necessary for (1.5). It is interesting to "remove" condition (1.4). The first result in this direction in a Banach space setting was presented in Sung et al.'s [10, Theorem 3], as follows.

Theorem C. *Let* $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ *be an array of rowwise independent random elements and* $\{c_n, n \ge 1\}$ *a sequence of positive constants. Suppose that assumption* (1.1) *is fulfilled and there exist* $J > 0, \delta > 0$, *and* $p \ge 1$ *such that*

$$\sum_{n=1}^{\infty} c_n \left(E \left\| \sum_{k=1}^{k_n} (X_{nk} I(\|X_{nk}\| \le \delta) - EX_{nk} I(\|X_{nk}\| \le \delta)) \right\|^p \right)^J < \infty.$$

Then

$$\sum_{n=1}^{\infty} c_n P\left(\max_{1\le m\le k_n} \left\| \sum_{k=1}^{m} (X_{nk} - EX_{nk} I(\|X_{nk}\| \le \delta)) \right\| > \varepsilon \right) < \infty \quad \textit{for all } \varepsilon > 0.$$

Note that the conclusion of Theorems B and C are different than the conclusion of Theorem A. One of the differences is that the conclusions of Theorems B and C deal with maxima of partial sums. But the main difference appears in Sung et al. [10], where the authors required the centering by expectations of truncated random variables. In this note, we show that Theorem A holds without condition (1.4), and also establish the rate of convergence for maxima of partial sums without centering. The proofs are quite different from Sung et al. [10] and Hu et al. [5].

In the following, C always stands for a positive constant which may differ from one place to another.

2. Formulation of the Main Results

The main result of this article is the following theorem which shows that condition (1.4) in Theorem A in the case $\liminf_{n\to\infty} c_n = 0$ is redundant.

Theorem 1. *Let* $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ *be an array of rowwise independent random elements and let* $\{c_n, n \geq 1\}$ *be a sequence of positive constants such that* (1.1)–(1.3) *hold. Then* (1.5) *is true.*

The following example shows that Theorem 1 is more general than Theorem 3.1 of Hu et al. [5].

Example. Define sequences $\{c_n, n \geq 1\}$ and $\{k_n, n \geq 1\}$ by $c_n = \frac{1}{n}$ and $k_n = n$. Define an array $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ by $X_{n1} = \cdots = X_{n,n-1} = j/n$ and $X_{nn} = j/n - j$ if $n = 2^j$, and $X_{nk} = 0$ for all $1 \leq k \leq n$ if $n \neq 2^j$. Then condition (1.1) holds and there exists $p \geq 1/2$, $J \geq 2$ and $\delta > 0$ such that (1.5) holds and $\sum_{k=1}^{k_n} X_{nk} \to 0$ in probability. But for any $\delta > 0$ we have $\limsup_{n\to\infty} \sum_{k=1}^{n} P\{|X_{nk}| > \delta\} = 1$. Hence, we can apply Theorem 1 present in this article, but not Theorem 3.1 of Hu et al. [5].

Theorem 1 shows that Theorem 3.2 of Hu et al. [5] also holds without redundant condition (1.4) in the case $\liminf_{n\to\infty} c_n = 0$.

We note that Theorem 1 has already been obtained in the special case $p=1$ by Sung et al. [6]. Next, it appears that in the case of *symmetric* summands Theorems B and C are slightly stronger results than Theorem 1 because they deal with maximums of partial sums. Theorem 1 does not require symmetry of distributions and we also provide some results for maximums of partial sums in Theorems 3 and 4 below.

We can simplify condition (1.2) of Theorem 1 when absolute moments of some order $0 < q \leq 2$ exist for the random elements comprising the arrays.

Theorem 2. *Let* $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ *be an array of rowwise independent random elements and let* $\{c_n, n \geq 1\}$ *be a sequence of positive constants. Suppose that* $E\|X_{nk}\|^q < \infty$, $1 \leq k \leq k_n$, $n \geq 1$ *for some* $0 < q \leq 2$, (1.1), *and* (1.3) *hold. If*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E\|X_{nk}\|^q \right)^J < \infty, \tag{2.1}$$

then (1.5) *holds.*

Remark 1. By Theorem 2, it is not necessary to verify the condition (1.4) in the proofs of all results obtained in Hu et al. [5, 7], and Ahmed et al. [8].

If we replace the condition (1.3) that $\sum_{k=1}^{k_n} X_{nk} \to 0$ in probability by the strictly stronger condition

$$\max_{1\le m\le k_n} P\left\{\left\|\sum_{k=1}^{m} X_{nk}\right\| > \varepsilon\right\} \to 0 \quad \text{for any } \varepsilon > 0 \tag{2.2}$$

as $n \to \infty$, we obtain the following results.

Theorem 3. *Let* $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ *be an array of rowwise independent random elements and let* $\{c_n, n \ge 1\}$ *be a sequence of positive constants such that* (1.1) *holds and there exists* $p \ge 1/2$, $J \ge 2$ *and* $\delta > 0$ *such that* (1.2) *holds. Then* (2.2) *implies that*

$$\sum_{n=1}^{\infty} c_n P\left\{\max_{1\le m\le k_n}\left\|\sum_{k=1}^{m} X_{nk}\right\| > \varepsilon\right\} < \infty \tag{2.3}$$

for all $\varepsilon > 0$.

By Theorem 3 we have the following result.

Theorem 4. *Let* $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ *be an array of rowwise independent random elements and let* $\{c_n, n \ge 1\}$ *be a sequence of positive constants. Suppose that* $E\|X_{nk}\|^q < \infty$, $1 \le k \le k_n$, $n \ge 1$ *for some* $0 < q \le 2$, (1.1), (2.1), *and* (2.2) *hold. Then* (2.3) *holds.*

Remark 2. By Theorem 4 we have that Corollary 4.2 of Hu et al. [5], Theorem 3.2 of Ahmed et al. [8], and Theorem 3.2 of Hu et al. [7] are still true if we replace weighted sums by maxima of weighted sums.

Remark 3. If the condition (4.11) of Corollary 4.8 of Hu et al. [5] is strengthened to $\max_{1\le m\le k_n} |\sum_{k=1}^{m} E\xi_{nk} I(\|\xi_{nk}\| \le \delta)| \to 0$ as $n \to \infty$, then the corollary also holds if we replace sums by maxima of sums.

3. Lemmata

In this section we present a few lemmas that play significant role in the proofs of the main results.

Lemma 1 (Lemma 2.2(ii) of Hu et al. [5]). *If* $\{Y_n, n \ge 1\}$ *is a sequence of random elements with* $Y_n \to 0$ *in probability, then for all* $t > 0$ *and sufficiently large* n

$$P\{\|Y_n\| > t\} \le 2P\{\|Y_n^s\| > t/2\}.$$

The next lemma is a simple corollary of the contraction principle presented in Lemma 6.5 of Ledoux and Talagrand [11].

Lemma 2. *Let* $\{Y_i, i \ge 1\}$ *be a sequence of symmetric random elements and* $\{a_i, i \ge 1\}$ *be a sequence of constants. Then for every* $t > 0$:

$$P\left\{\left\|\sum_i Y_i I\{\|X_i\| \le a_i\}\right\| > t\right\} \le 2P\left\{\left\|\sum_i Y_i\right\| > t\right\}.$$

The next lemma concerns the relationship between convergence in probability and mean convergence for sums of independent bounded random variables and can be found in Hu et al. [5, Lemma 2.1].

Lemma 3. *Let the array* $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ *of rowwise independent random elements be symmetric and suppose there exists* $\delta > 0$ *such that* $\|X_{nk}\| \leq \delta$ *a.s. for all* $1 \leq k \leq k_n, n \geq 1$. *If* $\sum_{k=1}^{k_n} X_{nk} \to 0$ *in probability, then* $E\|\sum_{k=1}^{k_n} X_{nk}\| \to 0$ *as* $n \to \infty$.

Lemma 4 presented now is a combination of well-known inequalities by Von Bahr and Esseen (part (i)) and Rosenthal (part (ii)). The proofs can be found in Gut [12, Theorems 6.1(iii) and 9.1].

Lemma 4. *Let* $X_1, X_2, \ldots, X_n$ *be independent mean zero random variables,* $p \geq 1$, *and suppose that* $E|X_k|^p < \infty$ *for all* k.

(i) *If* $1 \leq p \leq 2$, *then* $E|\sum_{k=1}^n X_k|^p \leq C\sum_{k=1}^n E|X_k|^p$.
(ii) *If* $p > 2$, *then* $E|\sum_{k=1}^n X_k|^p \leq \max\{\sum_{k=1}^n E|X_k|^p, (\sum_{k=1}^n EX_k^2)^{p/2}\}$.

The next lemma is a famous c_r-inequality (cf., e.g., [12, Theorem 2.2]).

Lemma 5. *Let* $r > 0$. *Suppose that* $E|X|^r < \infty$ *and* $E|Y|^r < \infty$. *Then*

$$E|X+Y|^r \leq c_r(E|X|^r + E|Y|^r),$$

where $c_r = 1$ *when* $r \leq 1$ *and* $c_r = 2^{r-1}$ *when* $r \geq 1$.

We also need the Ottaviani inequality (cf., e.g., [12, Theorem 7.7], or [11, Lemma 6.2]).

Lemma 6. *Suppose* $X_1, X_2, \ldots, X_n$ *are independent random variables, and let* x *and* y *be positive reals. If*

$$\beta = \max_{1\leq k\leq n} P\{\|S_n - S_k\| > y) < 1,$$

then

$$P\left\{\max_{1\leq m\leq n}\left\|\sum_{k=1}^m X_k\right\| > x+y\right\} \leq \frac{1}{1-\beta}P\left\{\left\|\sum_{k=1}^n X_k\right\| > x\right\}.$$

The the last lemma is an iterated form of the Kahane–Hoffmann–Jorgensen inequality (cf., e.g., [12, Theorem 7.5(iii)]).

Lemma 7. *If* $X_1, X_2, \ldots, X_n$ *are independent symmetric random elements, then for all* $t \geq 0$ *and* $j \geq 1$

$$P\left\{\left\|\sum_{k=1}^n X_k\right\| > 3^j t\right\} \leq C_j P\left\{\sup_{1\leq k\leq n}\|X_k\| > t\right\} + D_j\left(P\left\{\left\|\sum_{k=1}^n X_k\right\| > t\right\}\right)^{2^j}$$

where C_j *and* D_j *are positive constants depending only on* j.

4. Proofs

Proof of Theorem 1. Let $N_1 = \{n : \sum_{k=1}^{k_n} P\{\|X_{nk}\| > \delta\} > 1\}$ and $N_2 = \{n : \sum_{k=1}^{k_n} P\{\|X_{nk}\| > \delta\} \le 1\}$. Then by condition (1.1)

$$\sum_{n \in N_1} c_n < \sum_{n \in N_1} c_n \sum_{k=1}^{k_n} P\big\{\|X_{nk}\| > \delta\big\} \le \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P\big\{\|X_{nk}\| > \delta\big\} < \infty.$$

Note that for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} c_n P\left\{\left\| \sum_{k=1}^{k_n} X_{nk} \right\| > \varepsilon\right\} \le \sum_{n \in N_1} c_n + \sum_{n \in N_2} c_n P\left\{\left\| \sum_{k=1}^{k_n} X_{nk} \right\| > \varepsilon\right\}.$$

Hence, to prove the theorem, it is enough to show that

$$\sum_{n \in N_2} c_n P\left\{\left\| \sum_{k=1}^{k_n} X_{nk} \right\| > \varepsilon\right\} < \infty, \quad \text{for all } \varepsilon > 0.$$

Without loss of generality, we assume that $N_2 = \{n, n \ge 1\}$.

Let $\{X_{nk}^s, 1 \le k \le k_n, n \ge 1\}$ be the symmetrized version of X_{nk}, i.e., $X_{nk}^s = X_{nk} - X_{nk}^*$ where X_{nk} and X_{nk}^* are independent and have the same distribution. By Lemma 1, to prove the theorem, it is enough to show that

$$\sum_{n=1}^{\infty} c_n P\left\{\left\| \sum_{k=1}^{k_n} X_{nk}^s \right\| > \varepsilon\right\} < \infty, \quad \text{for all } \varepsilon > 0.$$

Note that for any $\varepsilon > 0$ and some $\delta > 0$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} c_n P\left\{\left\| \sum_{k=1}^{k_n} X_{nk}^s \right\| > \varepsilon\right\} \\
&\quad \le \sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P\big\{\|X_{nk}^s\| > \delta\big\} + \sum_{n=1}^{\infty} c_n P\left\{\left\| \sum_{k=1}^{k_n} X_{nk}^s I(\|X_{nk}^s\| \le \delta) \right\| > \varepsilon\right\} \\
&\quad \le 2\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P\big\{\|X_{nk}\| > \delta/2\big\} + \sum_{n=1}^{\infty} c_n P\left\{\left\| \sum_{k=1}^{k_n} X_{nk}^s I(\|X_{nk}^s\| \le \delta) \right\| > \varepsilon\right\}.
\end{aligned}$$

In view of (1.1), it suffices to show that

$$\sum_{n=1}^{\infty} c_n P\left\{\left\| \sum_{k=1}^{k_n} X_{nk}^s I(\|X_{nk}^s\| \le \delta) \right\| > \varepsilon\right\} < \infty \quad \text{for all } \varepsilon > 0.$$

Since $\sum_{k=1}^{k_n} X_{nk} \to 0$ in probability implies $\sum_{k=1}^{k_n} X_{nk}^s \to 0$ in probability, hence using the contraction principle formulated above in Lemma 2, we have

$$\sum_{k=1}^{k_n} X_{nk}^s I(\|X_{nk}^s\| \le \delta) \to 0 \quad \text{in probability}.$$

By Lemma 3, we have

$$E\left\| \sum_{k=1}^{k_n} X_{nk}^s I(\|X_{nk}^s\| \le \delta) \right\| \to 0 \quad \text{as } n \to \infty.$$

Now by Lemmas 1 and 7 with $j = [\mathrm{Log}\, J] + 1$ where Log denotes the logarithm to the base 2 and $[\cdot]$ is the integer part function, and the weak symmetrization inequality (cf. [12, Proposition 6.2]), we have for all large n

$$\begin{aligned}
&P\left\{ \left\| \sum_{k=1}^{k_n} X_{nk} I(\|X_{nk}\| \le \delta) \right\| \ge \epsilon \right\} \\
&\quad \le 2P\left\{ \left\| \sum_{k=1}^{k_n} X_{nk}^s I(\|X_{nk}^s\| \le \delta) \right\| \ge \epsilon/2 \right\} \le 2C_j \sum_{k=1}^{k_n} P\left\{ \|X_{nk}^s\| \ge \frac{\epsilon}{2\cdot 3^j} \right\} \\
&\qquad + 2D_j\left(P\left\{ \left\| \sum_{k=1}^{k_n} X_{nk}^s I(\|X_{nk}^s\| \le \delta) \right\| \ge \frac{\epsilon}{2\cdot 3^j} \right\} \right)^J \\
&\quad \le 4C_j \sum_{k=1}^{k_n} P\left\{ \|X_{nk}\| \ge \frac{\epsilon}{4\cdot 3^j} \right\} + 2D_j\left(P\left\{ \left\| \sum_{k=1}^{k_n} X_{nk}^s I(\|X_{nk}^s\| \le \delta) \right\| \ge \frac{\epsilon}{2\cdot 3^j} \right\} \right)^J.
\end{aligned}$$

Hence, by (1.1) it is enough to prove that

$$\sum_{n=1}^{\infty} c_n \left(E\left[\sum_{k=1}^{k_n} \left\| X_{nk}^s I(\|X_{nk}^s\| \le \delta) \right\|^2 \right]^p \right)^J < \infty. \tag{4.1}$$

Note that

$$\begin{aligned}
\|X_{nk}^s I(\|X_{nk}^s\| \le \delta)\| &= \|(X_{nk} - X_{nk}^*) I(\|X_{nk} - X_{nk}^*\| \le \delta)\| \\
&\le \|(X_{nk} - X_{nk}^*) I(\|X_{nk} - X_{nk}^*\| \le \delta, \|X_{nk}\| \le \delta)\| \\
&\quad + \|(X_{nk} - X_{nk}^*) I(\|X_{nk} - X_{nk}^*\| \le \delta, \|X_{nk}\| > \delta)\| \\
&\le \|(X_{nk} - X_{nk}^*) I(\|X_{nk} - X_{nk}^*\| \le \delta, \|X_{nk}\| \le \delta)\| + \delta I(\|X_{nk}\| > \delta) \\
&= \|(X_{nk} I(\|X_{nk}\| \le \delta) - X_{nk}^* I(\|X_{nk}\| \le \delta)) I(\|X_{nk} - X_{nk}^*\| \le \delta)\| \\
&\quad + \delta I(\|X_{nk}\| > \delta)
\end{aligned}$$

and in the same way

$$\begin{aligned}
&\|(X_{nk} I(\|X_{nk}\| \le \delta) - X_{nk}^* I(\|X_{nk}\| \le \delta)) I(\|X_{nk} - X_{nk}^*\| \le \delta)\| \\
&\quad \le \|X_{nk} I(\|X_{nk}\| \le \delta, \|X_{nk}^*\| \le \delta) - X_{nk}^* I(\|X_{nk}\| \le \delta, \|X_{nk}^*\| \le \delta)\| \\
&\qquad \times I(\|X_{nk} - X_{nk}^*\| \le \delta) + \delta I(\|X_{nk}^*\| > \delta) \\
&\quad \le \|X_{nk}\| I(\|X_{nk}\| \le \delta) + \|X_{nk}^*\| I(\|X_{nk}^*\| \le \delta) + \delta I(\|X_{nk}^*\| > \delta).
\end{aligned}$$

Hence,

$$\begin{aligned}
\|X_{nk}^s I(\|X_{nk}^s\| \le \delta)\| &\le \|X_{nk}\| I(\|X_{nk}\| \le \delta) + \|X_{nk}^*\| I(\|X_{nk}^*\| \le \delta) \\
&\quad + \delta I(\|X_{nk}\| > \delta) + \delta I(\|X_{nk}^*\| > \delta).
\end{aligned}$$

By the c_r-inequality presented in Lemma 5 we have

$$E\left[\sum_{k=1}^{k_n}\left\|X_{nk}^s I(\|X_{nk}^s\|\le\delta)\right\|^2\right]^p \le CE\left[\sum_{k=1}^{k_n}\left\|X_{nk} I(\|X_{nk}\|\le\delta)\right\|^2\right]^p + CE\left(\sum_{k=1}^{k_n} I(\|X_{nk}\|>\delta)\right)^p.$$

Now we estimate the last expression in different ways for different values of $p \ge 1/2$.

If $1/2 \le p \le 1$, by the c_r-inequality presented in Lemma 5 we have

$$E\left(\sum_{k=1}^{k_n} I(\|X_{nk}\|>\delta)\right)^p \le E\left(\sum_{k=1}^{k_n} I(\|X_{nk}\|>\delta)\right) = \sum_{k=1}^{k_n} P\{\|X_{nk}\|>\delta\}.$$

If $1 < p \le 2$, by the c_r-inequality presented in Lemmas 5 and 4(i) and noticing that $\sum_{k=1}^{k_n} P\{\|X_{nk}\|>\delta\} \le 1$ for all $n \ge 1$ we have

$$\begin{aligned}
&E\left(\sum_{k=1}^{k_n} I(\|X_{nk}\|>\delta)\right)^p \\
&\le E\left(\left|\sum_{k=1}^{k_n} I(\|X_{nk}\|>\delta) - \sum_{k=1}^{k_n} P\{\|X_{nk}\|>\delta\}\right| + \sum_{k=1}^{k_n} P\{\|X_{nk}\|>\delta\}\right)^p \\
&\le CE\left|\sum_{k=1}^{k_n} I(\|X_{nk}\|>\delta) - \sum_{k=1}^{k_n} P\{\|X_{nk}\|>\delta\}\right|^p + C\sum_{k=1}^{k_n} P\{\|X_{nk}\|>\delta\} \\
&\le C\sum_{k=1}^{k_n} E\left|I(\|X_{nk}\|>\delta) - P\{\|X_{nk}\|>\delta\}\right|^p + C\sum_{k=1}^{k_n} P\{\|X_{nk}\|>\delta\} \\
&\le C\sum_{k=1}^{k_n} P\{\|X_{nk}\|>\delta\}.
\end{aligned}$$

If $p > 2$, by the c_r-inequality presented in Lemmas 5 and 4(ii)

$$\begin{aligned}
&E\left(\sum_{k=1}^{k_n} I(\|X_{nk}\|>\delta)\right)^p \\
&\le CE\left|\sum_{k=1}^{k_n} I(\|X_{nk}\|>\delta) - \sum_{k=1}^{k_n} P\{\|X_{nk}\|>\delta\}\right|^p + C\sum_{k=1}^{k_n} P\{\|X_{nk}\|>\delta\} \\
&\le C\sum_{k=1}^{k_n} E\left|I(\|X_{nk}\|>\delta) - P\{\|X_{nk}\|>\delta\}\right|^p \\
&\quad + C\left(\sum_{k=1}^{k_n} E\left|I(\|X_{nk}\|>\delta) - P\{\|X_{nk}\|>\delta\}\right|^2\right)^{p/2} + C\sum_{k=1}^{k_n} P\{\|X_{nk}\|>\delta\} \\
&\le C\sum_{k=1}^{k_n} P\{\|X_{nk}\|>\delta\}.
\end{aligned}$$

Therefore,

$$E\left[\sum_{k=1}^{k_n} \left\| X_{nk}^s I(\|X_{nk}^s\| \le \delta)\right\|^2\right]^p \le CE\left[\sum_{k=1}^{k_n} \left\| X_{nk} I(\|X_{nk}\| \le \delta)\right\|^2\right]^p + C\sum_{k=1}^{k_n} P\{\|X_{nk}\| > \delta\}.$$

Hence, by the c_r-inequality (Lemma 5) and assumptions (1.1) and (1.2), (4.1) holds. □

Proof of Theorem 2. In view of Theorem 1, it suffices to verify that (1.2) holds with $p = 1$. For $n \ge 1$ and any $\delta > 0$

$$E\left(\sum_{k=1}^{k_n} \left\| X_{nk} I(\|X_{nk}\| \le \delta)\right\|^2\right) = \sum_{k=1}^{k_n} E\left\| X_{nk} I(\|X_{nk}\| \le \delta)\right\|^2 \le \delta^{2-q} \sum_{k=1}^{k_n} E\|X_{nk}\|^q$$

and (1.2) with $p = 1$ follows from (2.1). □

Proof of Theorem 3. By the Ottaviani inequality (Lemma 6), for any $\varepsilon > 0$, for n large enough we have

$$P\left\{\max_{1\le m\le k_n} \left\|\sum_{k=1}^{m} X_{nk}\right\| > \varepsilon\right\} \le 2P\left\{\left\|\sum_{k=1}^{k_n} X_{nk}\right\| > \varepsilon/2\right\}.$$

Hence, by Theorem 1 we have Theorem 3 at once. □

Proof of Theorem 4. The proof is the same as that of Theorem 2, only instead of Theorem 1 we use Theorem 3. □

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