ON CONVERGENCE OF SERIES OF INDEPENDENT RANDOM ELEMENTS IN BANACH SPACES

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ABSTRACT

The rate of convergence for an almost certainly convergent series $S_n = \sum_{i=1}^{n} V_i$ of independent random elements in a real separable Banach space is studied in this paper. More specifically, when S_n converges almost certainly to a random element S, the tail series $T_n \equiv S - S_{n-1} = \sum_{i=n}^{\infty} V_i$ is a well-defined sequence of random elements with $T_n \to 0$ almost certainly. The main result establishes for a sequence of positive constants $\{b_n, n \geq 1\}$ with $b_j \leq \text{Const.}\, b_n$ whenever $j \geq n \geq 1$ the equivalence between the tail series weak law of large numbers $T_n/b_n \xrightarrow{P} 0$ and the limit law $\sup_{j \geq n} ||T_j||/b_n \xrightarrow{P} 0$ thereby extending a result of Nam and Rosalsky [20] to a Banach space setting while also simplifying the argument used in the earlier result. The quasi-monotonicity proviso on $\{b_n, n \geq 1\}$ cannot be dispensed with.

1 Introduction

Throughout this paper, $\{V_n, n \geq 1\}$ is a sequence of independent random elements defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in a real separable Banach space $\mathcal{X}$ with norm $|| \cdot ||$. As usual, their partial sums will be denoted by $S_n = \sum_{i=1}^{n} V_i$, $n \geq 1$.

Woyczyński [24, 25], Jain [13], Giné, Mandrekar, and Zinn [11], and Etemadi [10] among others provided conditions under which S_n converges almost certainly (a.c.) (to a random element). For example, Woyczyński [25] proved that if the Banach space $\mathcal{X}$ is of Rademacher type p $(1 \le p \le 2)$ and if

$$EV_n = 0,\ n \ge 1,\ \text{and}\ \sum_{n=1}^{\infty} E||V_n||^p < \infty,$$

then S_n converges a.c. Of course, in the (real-valued) random variable case, this result is well known and follows readily from the celebrated Kolmogorov three-series criterion (see Corollary 5.1.3 of Chow and Teicher [5, p. 117]).

When S_n converges a.c. to a random element S, then (set $S_0 = 0$)

$$T_n \equiv S - S_{n-1} = \sum_{i=n}^{\infty} V_i,\ n \ge 1$$

is a well-defined sequence of random elements (referred to as the *tail series*) with

$$T_n \to 0 \text{ a.c.} \tag{1.1}$$

In this paper, we shall be concerned with the rate in which S_n converges to S or, equivalently, in which the tail series T_n converges to 0. Recalling that (1.1) is equivalent to

$$\sup_{j\ge n} ||T_j|| \xrightarrow{P} 0,$$

Rosalsky and Rosenblatt [22] provided conditions in their Theorem 4.2 in order for the limit law

$$\frac{\sup_{j\ge n} ||T_j||}{b_n} \xrightarrow{P} 0 \tag{1.2}$$

to hold for a given sequence of positive constants $\{b_n, n \ge 1\}$ thereby extending Theorem 4 of Nam and Rosalsky [18] which pertained to only the random variable case. While the proof of Theorem 4.2 of Rosalsky and Rosenblatt [22] bears some affinity to that of its random variable counterpart (Theorem 4 of Nam and Rosalsky [18]), the overall argument was different and substantially simpler. Thus, Theorem 4.2 of Rosalsky and Rosenblatt [22] not only extended but, also, simplified the special case established by Nam and Rosalsky [18]. The limit law (1.2) in the random variable case was apparently first investigated by Nam and Rosalsky [18]. Of course, the Nam and Rosalsky [18] and Rosalsky and Rosenblatt [22] results are of greatest interest when $b_n = o(1)$. Rosalsky and Rosenblatt [22] provided examples illustrating the sharpness of their work.

The sequence of random elements $\{V_n,\ n \ge 1\}$ is said to obey the tail series *weak law of large numbers* (WLLN) (resp., *strong law of large numbers* (SLLN)) with respect to the norming constants $b_n > 0$ if the tail series T_n is well defined and

$$\frac{T_n}{b_n} \xrightarrow{P} 0 \tag{1.3}$$

$$\left(\text{resp., } \frac{T_n}{b_n} \to 0 \text{ a.c.}\right). \tag{1.4}$$

Again, these are of greatest interest when $b_n = o(1)$.

When $0 < b_n \downarrow$, Rosalsky and Rosenblatt [22] observed that the tail series SLLN (1.4) implies the limit law (1.2) and that the tail series SLLN (1.4) is equivalent to the apparently stronger limit law

$$\frac{\sup_{j\ge n} ||T_j||}{b_n} \to 0 \text{ a.c.} \tag{1.5}$$

(These observations had previously been made by Nam and Rosalsky [18] for the random variable case.) Nam and Rosalsky [18] provided an example in the random variable case wherein (1.2) holds with $0 < b_n \downarrow$ but (1.4) fails. Moreover, Nam and Rosalsky [20] provided an example in the random variable case showing that without the monotonicity condition on $\{b_n,\ n \geq 1\}$, the tail series SLLN (1.4) does not imply either of the limit laws (1.2) or (1.5).

For the random variable case, Nam and Rosalsky [20] proved apropos of the tail series of independent summands that the tail series WLLN (1.3) and the apparently stronger limit law (1.2) are indeed equivalent when $0 < b_n \downarrow$ thereby establishing the validity of a conjecture posed by Nam and Rosalsky [18]. When $0 < b_n \downarrow$, it is interesting to note that the Nam and Rosalsky [20] argument for the implication (1.3) $\Longrightarrow$ (1.2) is substantially more involved than that of Nam and Rosalsky [18] for the implication (1.4) $\Longrightarrow$ (1.5). Nam and Rosalsky [20] showed via a random variable example that the tail series WLLN (1.3) does not imply the limit law (1.2) if the monotonicity condition on $\{b_n,\ n \geq 1\}$ is dispensed with.

The main purpose of the current work is to extend (in Theorem 1 below) the Nam and Rosalsky [20] equivalence between the tail series WLLN (1.3) and the limit law (1.2) from the random variable case to the case of Banach space valued random elements. In Theorem 1 it is assumed that the sequence of positive constants $\{b_n, n \geq 1\}$ is *quasi-monotone decreasing* in the sense that there exists a constant $0 < C < \infty$ such that

$$b_j \leq C b_n \text{ whenever } j \geq n \geq 1. \tag{1.6}$$

Of course if $b_n \downarrow$, then (1.6) holds with $C = 1$. The proof of Theorem 1 is completely different and considerably simpler than that of its earlier random variable counterpart. Indeed, the earlier argument does not carry over to a Banach space setting unless one modifies it by introducing a symmetrization procedure which is not at all necessary in view of the proof of Theorem 1 presented below.

There has been a substantial literature of investigation on the limiting behavior of tail series in the random variable case following the ground breaking work of Chow and Teicher [4] wherein a tail series law of the iterated logarithm (LIL) was obtained. Barbour [1] then established a tail series analogue of the Lindeberg-Feller version of the central limit theorem. Numerous other investigations on the tail series LIL problem have followed; see Heyde [12], Wellner [23], Kesten [14], Budianu [3], Chow, Teicher, Wei, and Yu [6], Klesov [15], Rosalsky [21], and Mikosch [17] for such work. Klesov [15, 16], Mikosch [17], and Nam and Rosalsky [19] studied the tail series SLLN problem. The only work that the authors are aware of on the limiting behavior of tail series with Banach space valued summands is that of Dianliang [7, 8] on the tail series LIL and that of Rosalsky and Rosenblatt [22] on the limit law (1.2).

2 Preliminary Lemmas

Some lemmas are needed to establish the main result (Theorem 1). Lemma 1 provides a maximal inequality for a sum of independent random elements and is due to Etemadi [9]. Lemma 1 may also be found in Billingsley [2, p. 288]. For some related results see Etemadi [10].

Lemma 1 *(Etemadi [9]) Let $n \geq 1$ and let $\{V_i,\ 1 \leq i \leq n\}$ be independent random elements in a real separable Banach space. Then setting $S_j = \sum_{i=1}^{j} V_i,\ 1 \leq j \leq n$,*

$$P\left\{\max_{1 \leq j \leq n} ||S_j|| > t\right\} \leq 4 \max_{1 \leq j \leq n} P\left\{||S_j|| > \frac{t}{4}\right\},\ t > 0.$$

Lemma 2 *Let $1 \leq k \leq n$ and let $\{V_i, k \leq i \leq n\}$ be independent random elements in a real separable Banach space. Then setting $S_{j,n} = \sum_{i=j}^{n} V_i, \ \ k \leq j \leq n,$*

$$P\left\{\max_{k\leq j\leq n} ||S_{j,n}|| > t\right\} \leq 4 \max_{k\leq j\leq n} P\left\{||S_{j,n}|| > \frac{t}{4}\right\}, \ t > 0.$$

Proof. Set $S_j^{(n)} = \sum_{i=1}^{j} V_{n+1-i}, \ 1 \leq j \leq n+1-k$. Note at the outset that

$$\{S_{j,n}, j = k, \cdots, n\} = \{S_j^{(n)}, j = n+1-k, \cdots, 1\}.$$

Then for $t > 0$

$$\begin{aligned} P\left\{\max_{k\leq j\leq n} ||S_{j,n}|| > t\right\} &= P\left\{\max_{1\leq j\leq n+1-k} ||S_j^{(n)}|| > t\right\} \\ &\leq 4 \max_{1\leq j\leq n+1-k} P\left\{||S_j^{(n)}|| > \frac{t}{4}\right\} \text{ (by Lemma 1)} \\ &= 4 \max_{k\leq j\leq n} P\left\{||S_{j,n}|| > \frac{t}{4}\right\}. \quad \square \end{aligned}$$

Lemma 3 *Let $\{V_n, n \geq 1\}$ be a sequence of independent random elements in a real separable Banach space with $\sum_{n=1}^{\infty} V_n$ converging a.c. Then the tail series $\{T_n = \sum_{i=n}^{\infty} V_i, \ n \geq 1\}$ is a well-defined sequence of random elements satisfying*

$$P\left\{\sup_{j\geq n} ||T_j|| \geq t\right\} \leq 4 \sup_{j\geq n} P\left\{||T_j|| \geq \frac{t}{4}\right\}, \ t > 0, \ n \geq 1.$$

Proof. Let $1 \leq n < N < M$ and $t > 0$. For $n \leq j \leq N$, set $S_{j,M} = \sum_{i=j}^{M} V_i$. Then

$$\Big| ||S_{j,M}|| - ||T_j|| \Big| \leq ||S_{j,M} - T_j|| \stackrel{M\to\infty}{\longrightarrow} 0 \text{ a.c.}$$

implying

$$||T_j|| = \lim_{M\to\infty} ||S_{j,M}|| \text{ a.c.} \tag{2.1}$$

Now

$$\begin{aligned} &P\left\{\max_{n\leq j\leq N} ||T_j|| > t\right\} \\ &= P\left\{\max_{n\leq j\leq N} \lim_{M\to\infty} ||S_{j,M}|| > t\right\} \text{ (by (2.1))} \\ &= P\left\{\lim_{M\to\infty} \max_{n\leq j\leq N} ||S_{j,M}|| > t\right\} \\ &\leq \liminf_{M\to\infty} P\left\{\max_{n\leq j\leq N} ||S_{j,M}|| > t\right\} \end{aligned}$$

(by Theorem 8.1.3 of Chow and Teicher [5, p. 260])

$$\leq \limsup_{M\to\infty} P\left\{\max_{n\leq j\leq N} ||S_{j,M}|| > t\right\}. \tag{2.2}$$

Set $S'_{j,N} = \sum_{i=j}^{N} V'_i$ for $n \leq j \leq N$ where

$$V'_i = V_i \text{ for } n \leq i \leq N-1 \text{ and } V'_N = \sum_{k=N}^{M} V_k.$$

Then $S_{j,M} = S'_{j,N}$ for $n \le j \le N$ and so

$$\begin{aligned} P\left\{\max_{n\le j\le N} ||S_{j,M}|| > t\right\} &= P\left\{\max_{n\le j\le N} ||S'_{j,N}|| > t\right\} \\ &\le 4 \max_{n\le j\le N} P\left\{||S'_{j,N}|| > \frac{t}{4}\right\} \quad \text{(by Lemma 2)} \\ &\le 4 \max_{n\le j\le N} P\left\{||S_{j,M}|| \ge \frac{t}{4}\right\}. \end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3) yields

$$\begin{aligned} &P\left\{\max_{n\le j\le N} ||T_j|| > t\right\} \\ &\le 4 \limsup_{M\to\infty} \max_{n\le j\le N} P\left\{||S_{j,M}|| \ge \frac{t}{4}\right\} \\ &= 4 \max_{n\le j\le N} \limsup_{M\to\infty} P\left\{||S_{j,M}|| \ge \frac{t}{4}\right\} \\ &\le 4 \max_{n\le j\le N} P\left\{\lim_{M\to\infty} ||S_{j,M}|| \ge \frac{t}{4}\right\} \\ &\qquad \text{(by Theorem 8.1.3 of Chow and Teicher [5, p. 260])} \\ &= 4 \max_{n\le j\le N} P\left\{||T_j|| \ge \frac{t}{4}\right\} \quad \text{(by (2.1))} \\ &\le 4 \sup_{j\ge n} P\left\{||T_j|| \ge \frac{t}{4}\right\}. \end{aligned}$$

Then

$$P\left\{\sup_{j\ge n} ||T_j|| > t\right\} = \lim_{N\to\infty} P\left\{\max_{n\le j\le N} ||T_j|| > t\right\} \le 4 \sup_{j\ge n} P\left\{||T_j|| \ge \frac{t}{4}\right\}. \tag{2.4}$$

Now, in (2.4), replace t by $t - m^{-1}$ (where m is an integer $> t^{-1}$) and Lemma 3 then follows by letting $m \to \infty$. □

3 The Main Result

With the preliminaries accounted for, the main result of this paper may be established.

Theorem 1 *Let $\{V_n, n \ge 1\}$ be a sequence of independent random elements in a real separable Banach space with $\sum_{n=1}^{\infty} V_n$ converging a.c. and tail series $T_n = \sum_{i=n}^{\infty} V_i,\ n \ge 1$, and let $\{b_n,\ n \ge 1\}$ be a sequence of positive constants which is quasi-monotone decreasing in the sense that (1.6) holds. Then the tail series WLLN*

$$\frac{T_n}{b_n} \xrightarrow{P} 0 \tag{3.1}$$

and the limit law

$$\frac{\sup_{j\ge n} ||T_j||}{b_n} \xrightarrow{P} 0 \tag{3.2}$$

are equivalent.

Proof. Since (3.2) clearly implies (3.1), it needs to be established that (3.1) implies (3.2). For arbitrary $\epsilon > 0$, it follows from Lemma 3 with t replaced by ϵb_n that

$$
\begin{aligned}
& P\left\{\frac{\sup_{j\geq n}||T_j||}{b_n} \geq \epsilon\right\} \leq 4\sup_{j\geq n} P\left\{\frac{||T_j||}{b_n} \geq \frac{\epsilon}{4}\right\} \\
& \leq 4\sup_{j\geq n} P\left\{\frac{||T_j||}{b_j} \geq \frac{\epsilon}{4C}\right\} \quad \text{(by (1.6))} \\
& = o(1) \quad \text{(by (3.1))}. \quad \Box
\end{aligned}
$$

Remark. The example of Nam and Rosalsky [20] reveals that (3.1) does not imply (3.2) without the quasi-monotonicity assumption on $\{b_n,\ n \geq 1\}$.

Corollary 1 *Let $\{V_n,\ n \geq 1\}$ be a sequence of independent random elements in a real separable Banach space with $\sum_{n=1}^{\infty} V_n$ converging a.c. and tail series $T_n = \sum_{i=n}^{\infty} V_i,\ n \geq 1$, and let $\{b_n,\ n \geq 1\}$ be a sequence of positive constants which is quasi-monotone decreasing in the sense that (1.6) holds. Let $\{g_n(x),\ n \geq 1\}$ be a sequence of nondecreasing nonnegative functions defined on $[0,\infty)$ such that for all $\epsilon > 0$, there exists a number $\delta_\epsilon > 0$ and a positive integer N_ϵ such that*

$$
n \geq N_\epsilon \Longrightarrow g_n(\epsilon b_n) \geq \delta_\epsilon g_n(b_n). \tag{3.3}
$$

If

$$
Eg_n(||T_n||) = o(g_n(b_n)), \tag{3.4}
$$

then

$$
\frac{\sup_{j\geq n}||T_j||}{b_n} \xrightarrow{P} 0.
$$

Proof. For arbitrary $\epsilon > 0$ and all $n \geq N_\epsilon$,

$$
\begin{aligned}
& P\left\{\frac{||T_n||}{b_n} \geq \epsilon\right\} \leq P\{g_n(||T_n||) \geq g_n(\epsilon b_n)\} \\
& \leq P\{g_n(||T_n||) \geq \delta_\epsilon g_n(b_n)\} \quad \text{(by (3.3))} \\
& \leq \frac{Eg_n(||T_n||)}{\delta_\epsilon g_n(b_n)} \quad \text{(by the Markov inequality)} \\
& = o(1) \quad \text{(by (3.4))}.
\end{aligned}
$$

Thus, $T_n/b_n \xrightarrow{P} 0$ and Corollary 1 then follows immediately from Theorem 1. $\Box$

The second corollary follows immediately from Corollary 1 by taking $g_n(x) = x^{p_n},\ 0 \leq x < \infty,\ n \geq 1$ where $0 < p_n = \mathcal{O}(1)$.

Corollary 2 *Let $\{V_n,\ n \geq 1\}$ be a sequence of independent random elements in a real separable Banach space with $\sum_{n=1}^{\infty} V_n$ converging a.c. and tail series $T_n = \sum_{i=n}^{\infty} V_i,\ n \geq 1$, and let $\{b_n,\ n \geq 1\}$ be a sequence of positive constants which is quasi-monotone decreasing in the sense that (1.6) holds. If*

$$
E||T_n||^{p_n} = o(b_n^{p_n}) \ \text{ where } \ 0 < p_n = \mathcal{O}(1),
$$

then

$$
\frac{\sup_{j\geq n}||T_j||}{b_n} \xrightarrow{P} 0.
$$

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