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More on complete convergence for arrays ☆

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Abstract

A complete convergence theorem for arrays of rowwise independent random variables was proposed by Hu et al. (Statist. Probab. Lett. 38 (1998) 27). Two years later, Hu and Volodin (Statist. Probab. Lett. 47 (2000) 209) imposed one additional condition in addendum to the paper. In this paper, we prove the complete convergence theorem without the additional condition.

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1. Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins (1947) as follows. A sequence $\{U_n, n \geqslant 1\}$ of random variables *converges completely* to the constant θ if

$$\sum_{n=1}^{\infty} P(|U_n - \theta| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0.$$

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In view of the Borel–Cantelli lemma, this implies that $U_n \to \theta$ almost surely. The converse is true if $\{U_n, n \geqslant 1\}$ are independent random variables. Hsu and Robbins (1947) proved that the sequence of arithmetic means of independent and identically distributed random variables converges completely to the expected value if the variance of the summands is finite. Erdös (1949) proved the converse. We refer to Gut (1994) for a survey on results on complete convergence related to strong laws and published before the 1990s.

The result of Hsu–Robbins–Erdös has been generalized and extended in several directions. Some of these generalizations are in a Banach space setting. A sequence of Banach space valued random elements is said to *converge completely* to the 0 element of the Banach space if the corresponding sequence of norms converges completely to 0.

The papers Hu et al. (1998, 1999) unify and extend the ideas of previously obtained results on complete convergence. That was the reason why Adler (2000) stated the following about the paper Hu et al. (1998): "This result is quite optimal". The papers Hu et al. (1998, 1999) are devoted to an extension of the Hsu–Robbins theorem to general arrays of rowwise independent but not necessarily identically distributed random variables or Banach space valued random elements. The complete convergence result for row sums with the corresponding rates of convergence is obtained. In the main results of Hu et al. (1998, 1999), no assumptions are made concerning the existence of expected values or absolute moments of the random variables or elements and no assumptions are made concerning the geometry of the underlying Banach space.

To be more precise, Hu et al. (1998) stated the following complete convergence theorem for arrays of rowwise independent random variables $\{X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$, where $\{k_n, n \geqslant 1\}$ is a sequence of positive integers.

Theorem 1. *Let $\{X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$ be an array of rowwise independent random variables and $\{a_n, n \geqslant 1\}$ a sequence of positive constants such that*

$$\sum_{n=1}^{\infty} a_n = \infty. \tag{1}$$

Suppose that for every $\varepsilon > 0$ and some $\delta > 0$:

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon) < \infty$,
(ii) *there exists $J \geqslant 2$ such that*

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} E X_{ni}^2 I(|X_{ni}| \leqslant \delta) \right)^J < \infty,$$

(iii) $\sum_{i=1}^{k_n} E X_{ni} I(|X_{ni}| \leqslant \delta) \to 0$ *as* $n \to \infty$.

Then $\sum_{n=1}^{\infty} a_n P(|\sum_{i=1}^{k_n} X_{ni}| > \varepsilon) < \infty$ for all $\varepsilon > 0$.

The proof of Hu et al. (1998) is mistakenly based on the fact that the assumptions of Theorem 1 imply that

$$\sum_{i=1}^{k_n} X_{ni} \to 0 \text{ in probability} \tag{2}$$

as $n \to \infty$. Two years later, Hu and Volodin (2000) mentioned that condition (2) does not necessarily follow from the assumptions of Theorem 1. So they replaced condition (1) by the following condition:

the sequence $\{a_n, n \geqslant 1\}$ is bounded away from zero, that is, $\liminf_{n\to\infty} a_n > 0$.

In this case the assumptions of Theorem 1 will imply the convergence in (2).

Moreover, they also presented a simple counterexample where all assumptions of Theorem 1 are satisfied, but condition (2) is not true. Another counterexample to the same statement was presented in the paper Hu et al. (2003).

What is important to mention is that both counterexamples are counterexamples to the proof of Hu et al. (1998), but not to the result. In both examples the conclusion of Theorem 1, that is the complete convergence result remains true. So Hu and Volodin (2000) left an open problem whether Theorem 1 is true for any sequence $\{a_n, n \geqslant 1\}$.

Hu et al. (2003) gave a first partial solution to this question. They proved the following result.

Theorem 2. *Let* $\{X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$ *and* $\{a_n, n \geqslant 1\}$ *be as in Theorem* 1 *except that* (ii) *and* (iii) *are replaced by* (ii)′ *and* (iii)′, *respectively*:

(ii)′ *there exists* $J \geqslant 2$ *such that*

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} Var(X_{ni} I(|X_{ni}| \leqslant \delta)) \right)^J < \infty,$$

(iii)′ $\max_{1 \leqslant i \leqslant k_n} |\sum_{l=1}^{i} EX_{nl} I(|X_{nl}| \leqslant \delta)| \to 0$ *as* $n \to \infty$.

Then $\sum_{n=1}^{\infty} a_n P(|\sum_{i=1}^{k_n} X_{ni}| > \varepsilon) < \infty$ *for all* $\varepsilon > 0$.

Next partial solution was given by Kuczmaszewska (2004).

Theorem 3. *Let* $\{X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$ *and* $\{a_n, n \geqslant 1\}$ *be as in Theorem* 1 *except that* (iii) *is replaced by*

(iii)″ $\mathrm{med}(\sum_{i=1}^{k_n} X_{ni} I(|X_{ni}| \leqslant \delta)) \to 0$ *as* $n \to \infty$.

Then $\sum_{n=1}^{\infty} a_n P(|\sum_{i=1}^{k_n} X_{ni}| > \varepsilon) < \infty$ *for all* $\varepsilon > 0$.

In this paper, we solve this question completely. We prove Theorem 1 as stated. Our proof is different from those of Hu et al. (1998) and Kuczmaszewska (2004), and it does not use a symmetrization procedure and therefore convergence in probability statements.

2. Preliminaries

The following two lemmas will be used to prove our main result. Lemma 1 is well known; Marcinkiewicz–Zygmund inequality (see Chow and Teicher, 1978, p. 356) when $1 \leqslant p \leqslant 2$, Rosenthal's inequality (Rosenthal, 1970) when $p > 2$.

Lemma 1. *If $X_1, \ldots, X_n$ are independent random variables with $EX_i = 0, 1 \leqslant i \leqslant n$, then there exists a positive constant C_p depending only on p such that*

(i) *for* $1 \leqslant p \leqslant 2$,

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leqslant C_p \sum_{i=1}^{n} E|X_i|^p,$$

(ii) *for* $p > 2$,

$$E\left|\sum_{i=1}^{n} X_i\right|^p \leqslant C_p \left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}.$$

Hu et al. (2003) proved the following lemma which is a version of the famous Hoffmann–Jørgensen inequality for independent, but not necessarily symmetric, random variables. Cf., for example Ledoux and Talagrand (1991, Proposition 6.7), where the "noniterated case" ($j = 1$) of this result is proved.

Lemma 2. *If $X_1, \ldots, X_n$ are independent random variables, then for every integer $j \geqslant 1$ and $t > 0$*

$$P(|S_n| > 6^j t) \leqslant C_j P\left(\max_{1 \leqslant i \leqslant n} |X_i| > \frac{t}{4^{j-1}}\right) + D_j \max_{1 \leqslant i \leqslant n} \left[P\left(|S_i| > \frac{t}{4^j}\right)\right]^{2^j},$$

where C_j and D_j are positive constants depending only on j, and $S_i = \sum_{l=1}^{i} X_l$ for $1 \leqslant i \leqslant n$.

With the preliminaries accounted for we can now prove the result.

3. Proof of Theorem 1

Let $\varepsilon > 0$ be given. Without loss of generality, we may assume $0 < \varepsilon \leqslant \delta$. Take an integer j such that $2^j \geqslant J$. Define $X'_{ni} = X_{ni} I(|X_{ni}| \leqslant \varepsilon/(4^{j-1} \cdot 6 \cdot 6^j))$, $X''_{ni} = X_{ni} I(|X_{ni}| > \delta)$, and

$X'''_{ni} = X_{ni} I(\varepsilon/(4^{j-1} \cdot 6 \cdot 6^j) < |X_{ni}| \leqslant \delta)$ for $1 \leqslant i \leqslant k_n$ and $n \geqslant 1$. Then

$$\begin{aligned}\sum_{n=1}^{\infty} a_n P\left(\left|\sum_{i=1}^{k_n} X_{ni}\right| > \varepsilon\right) &\leqslant \sum_{n=1}^{\infty} a_n P\left(\left|\sum_{i=1}^{k_n} (X'_{ni} - EX'_{ni})\right| > \frac{\varepsilon}{3}\right) \\ &\quad + \sum_{n=1}^{\infty} a_n P\left(\left|\sum_{i=1}^{k_n} X''_{ni} + \sum_{i=1}^{k_n} (EX'_{ni} + EX'''_{ni})\right| > \frac{\varepsilon}{3}\right) \\ &\quad + \sum_{n=1}^{\infty} a_n P\left(\left|\sum_{i=1}^{k_n} (X'''_{ni} - EX'''_{ni})\right| > \frac{\varepsilon}{3}\right) \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III}.\end{aligned}$$

For I, we will apply Lemma 2 to the random variable $X'_{ni} - EX'_{ni}$. Noting that

$$\max_{1 \leqslant i \leqslant k_n} |X'_{ni} - EX'_{ni}| \leqslant \varepsilon/(4^{j-1} \cdot 3 \cdot 6^j),$$

we have by Lemma 2 and Markov's inequality that

$$\begin{aligned}&P\left(\left|\sum_{i=1}^{k_n} (X'_{ni} - EX'_{ni})\right| > \frac{\varepsilon}{3}\right) \\ &\quad \leqslant C_j P\left(\max_{1 \leqslant i \leqslant k_n} |X'_{ni} - EX'_{ni}| > \frac{\varepsilon}{4^{j-1} \cdot 3 \cdot 6^j}\right) \\ &\qquad + D_j \max_{1 \leqslant i \leqslant k_n} \left[P\left(\left|\sum_{l=1}^{i} (X'_{nl} - EX'_{nl})\right| > \frac{\varepsilon}{4^j \cdot 3 \cdot 6^j}\right)\right]^{2^j} \\ &\quad \leqslant D_j \max_{1 \leqslant i \leqslant k_n} \left[P\left(\left|\sum_{l=1}^{i} (X'_{nl} - EX'_{nl})\right| > \frac{\varepsilon}{4^j \cdot 3 \cdot 6^j}\right)\right]^J \\ &\quad \leqslant D_j \left(\frac{4^j \cdot 3 \cdot 6^j}{\varepsilon}\right)^{2J} \max_{1 \leqslant i \leqslant k_n} \left(E\left|\sum_{l=1}^{i} (X'_{nl} - EX'_{nl})\right|^2\right)^J \\ &\quad \leqslant D_j \left(\frac{4^j \cdot 3 \cdot 6^j}{\varepsilon}\right)^{2J} \left(\sum_{l=1}^{k_n} E|X'_{nl}|^2\right)^J \\ &\quad \leqslant D_j \left(\frac{4^j \cdot 3 \cdot 6^j}{\varepsilon}\right)^{2J} \left(\sum_{i=1}^{k_n} EX_{ni}^2 I(|X_{ni}| \leqslant \delta)\right)^J.\end{aligned}$$

Hence $I < \infty$ by (ii).

For II, we have by (iii) that there exists an integer N such that $|\sum_{i=1}^{k_n}(EX'_{ni}+EX'''_{ni})|<\varepsilon/6$ if $n\geqslant N$. For $n\geqslant N$,

$$\begin{aligned}
P\left(\left|\sum_{i=1}^{k_n} X''_{ni}+\sum_{i=1}^{k_n}(EX'_{ni}+EX'''_{ni})\right|>\frac{\varepsilon}{3}\right)
&\leqslant P\left(\left|\sum_{i=1}^{k_n} X''_{ni}\right|+\left|\sum_{i=1}^{k_n}(EX'_{ni}+EX'''_{ni})\right|>\frac{\varepsilon}{3}\right)\\
&\leqslant P\left(\left|\sum_{i=1}^{k_n} X''_{ni}\right|>\frac{\varepsilon}{6}\right)\\
&\leqslant \sum_{i=1}^{k_n} P(|X_{ni}|>\delta).
\end{aligned}$$

It follows that $\mathrm{II}<\infty$ by (i).

For III, we will apply Lemma 1 to the random variable $X'''_{ni}-EX'''_{ni}$. Observe that

$$\begin{aligned}
E|X'''_{ni}-EX'''_{ni}|^{2J} &\leqslant 2^{2J-1}\left(E|X'''_{ni}|^{2J}+|EX'''_{ni}|^{2J}\right)\\
&\leqslant 2^{2J}E|X'''_{ni}|^{2J}\\
&\leqslant 2^{2J}\delta^{2J}P\left(|X_{ni}|>\frac{\varepsilon}{4^{j-1}\cdot 6\cdot 6^{j}}\right)
\end{aligned}$$

and

$$E(X'''_{ni}-EX'''_{ni})^2\leqslant E|X'''_{ni}|^2\leqslant EX^2_{ni}I(|X_{ni}|\leqslant\delta).$$

It follows by Markov's inequality and Lemma 1 that

$$\begin{aligned}
P\left(\left|\sum_{i=1}^{k_n}(X'''_{ni}-EX'''_{ni})\right|>\frac{\varepsilon}{3}\right)
&\leqslant \left(\frac{3}{\varepsilon}\right)^{2J}E\left|\sum_{i=1}^{k_n}(X'''_{ni}-EX'''_{ni})\right|^{2J}\\
&\leqslant \left(\frac{3}{\varepsilon}\right)^{2J}C_{2J}\left\{\sum_{i=1}^{k_n}E|X'''_{ni}-EX'''_{ni}|^{2J}+\left(\sum_{i=1}^{k_n}E(X'''_{ni}-EX'''_{ni})^2\right)^{J}\right\}\\
&\leqslant \left(\frac{3}{\varepsilon}\right)^{2J}C_{2J}\left\{2^{2J}\delta^{2J}\sum_{i=1}^{k_n}P\left(|X_{ni}|>\frac{\varepsilon}{4^{j-1}\cdot 6\cdot 6^{j}}\right)+\left(\sum_{i=1}^{k_n}EX^2_{ni}I(|X_{ni}|\leqslant\delta)\right)^{J}\right\}.
\end{aligned}$$

Hence $\mathrm{III}<\infty$ by (i) and (ii).

4. Concluding remarks

Mention that assumptions (ii) and (iii) of Theorem 1 is somehow difficult to check. For a special case of mean zero array we can establish the following result, where these assumptions are simplified.

Proposition 1. *Let $\{X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$ be an array of rowwise independent mean zero random variables and $\{a_n, n \geqslant 1\}$ a sequence of positive constants. Next, let $\phi(x)$ be a real function such that for some $\delta > 0$:*

$$\sup_{x \geqslant \delta} \frac{x}{\phi(x)} < \infty \text{ and } \sup_{0 \leqslant x \leqslant \delta} \frac{x^2}{\phi(x)} < \infty.$$

Moreover, suppose that for all $\varepsilon > 0$:

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon) < \infty$,

(ii)$'''$ *there exists $J \geqslant 2$ such that*

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} E\phi(|X_{ni}|) \right)^J < \infty \text{ and}$$

(iii)$'''$ *if the sequence $\{a_n, n \geqslant 1\}$ is not bounded away from zero, that is, if $\liminf_{n\to\infty} a_n = 0$, then assume also that*

$$\sum_{i=1}^{k_n} E\phi(|X_{ni}|) \to 0 \text{ as } n \to \infty.$$

Then $\sum_{n=1}^{\infty} a_n P(|\sum_{i=1}^{k_n} X_{ni}| > \varepsilon) < \infty$ for all $\varepsilon > 0$.

Proof. In view of Theorem 1, it is enough to show that conditions (ii) and (iii) are satisfied.

For (ii) note that

$$\sum_{i=1}^{k_n} EX_{ni}^2 I(|X_{ni}| \leqslant \delta) = \sum_{i=1}^{k_n} E \frac{X_{ni}^2}{\phi(|X_{ni}|)} I(|X_{ni}| \leqslant \delta)\phi(|X_{ni}|)$$

$$\leqslant \left(\sup_{0 \leqslant x \leqslant \delta} \frac{x^2}{\phi(x)} \right) \sum_{i=1}^{k_n} E\phi(|X_{ni}|).$$

For (iii) since $EX_{ni} = 0$ it follows that

$$\left| \sum_{i=1}^{k_n} EX_{ni} I(|X_{ni}| \leqslant \delta) \right| = \left| \sum_{i=1}^{k_n} EX_{ni} I(|X_{ni}| > \delta) \right|$$

$$\leqslant \sum_{i=1}^{k_n} E \frac{|X_{ni}|}{\phi(|X_{ni}|)} I(|X_{ni}| > \delta)\phi(|X_{ni}|)$$

$$\leqslant \left(\sup_{x > \delta} \frac{x}{\phi(x)} \right) \sum_{i=1}^{k_n} E\phi(|X_{ni}|) \to 0. \qquad \square$$

Remarks. (1) It is obvious that if the sequence $\{a_n, n \geqslant 1\}$ is bounded away from zero, that is, if $\liminf_{n\to\infty} a_n > 0$, then assumption (iii)$'''$ is unnecessary, it follows from assumption (ii)$'''$.

(2) Take $\phi(x) = x^q, 1 \leqslant q \leqslant 2$, in order to obtain the result formulated in Remark (2) of Hu et al. (1998). Another example of function $\phi(x)$ that satisfies assumptions of Proposition 1 is $\phi(x) = x^q \mathrm{Log}^\alpha(x)$, where $\mathrm{Log}(x) = \max\{1, \log(x)\}, 1 \leqslant q \leqslant 2$, and α is arbitrary for $1 < q < 2$; $\alpha \geqslant 0$ for $q = 1$; and $\alpha \leqslant 0$ for $q = 2$.

There is an interesting problem left here, namely how it is possible to generalize Theorem 1 in the Banach space setting. There are two methods for such a generalization. The first one, called "stability result", deals with an arbitrary separable Banach space, without any geometric assumptions. We refer to the paper Hu et al. (1999) for such a result.

The second method of establishing limit theorems in Banach space setting deals with geometric assumptions on the underlying Banach space. One of the most well-known geometric assumptions is the type p.

Recall that a real separable Banach space $(B, \|\ \|)$ is said to be of (Rademacher) *type* $p, 1 \leqslant p \leqslant 2$, if there exists a positive constant C such that

$$E\left\|\sum_{i=1}^{n} X_i\right\|^p \leqslant C \sum_{i=1}^{n} E\|X_i\|^p$$

for all independent mean zero and finite pth moment random elements $X_1, \ldots, X_n$ with values in B.

Let us mention that a version of Hoffmann–Jørgensen inequality (Lemma 2) is still valid for independent, but not necessarily symmetric, random elements with values in B. For a random element X with expected value and $p > 0$ denote $\sigma_p(X) = E\|X - EX\|^p$. Theorem 2 can be generalized to Banach space setting and the following result was proved in Hu et al. (2003).

Theorem 4. *Let $\{X_{ni}, 1 \leqslant i \leqslant k_n, n \geqslant 1\}$ be an array of rowwise independent random elements taking values in a real separable Banach space $(B, \|\ \|)$ of type $p, 1 \leqslant p \leqslant 2$, and $\{a_n, n \geqslant 1\}$ a sequence of positive constants. Suppose that for every $\varepsilon > 0$ and some $\delta > 0$:*

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \varepsilon) < \infty$,
(ii) *there exists $J \geqslant 2$ such that*

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} \sigma_p(X_{ni} I(\|X_{ni}\| \leqslant \delta)) \right)^J < \infty,$$

(iii) $\max_{1 \leqslant i \leqslant k_n} \|\sum_{l=1}^{i} E X_{nl} I(\|X_{nl}\| \leqslant \delta)\| \to 0$ *as* $n \to \infty$.

Then $\sum_{n=1}^{\infty} a_n P(\|\sum_{i=1}^{k_n} X_{ni}\| > \varepsilon) < \infty$ *for all* $\varepsilon > 0$.

We would like to mention that the proof of Theorem 4 is very close to the proof of Theorem 2. Unfortunately, we were not able to adopt the proof of Theorem 1 we presented here to the Banach space setting. We expect that something can be done in the so-called $\mathrm{Ros}(p), 1 \leqslant p < \infty$ Banach spaces, cf. Ledoux and Talagrand (1991, Section 10.2).

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