Bull. Korean Math. Soc. **38** (2001), No. 4, pp. 763–772

ON CONVERGENCE OF SERIES OF INDEPENDENT RANDOM VARIABLES

Soo Hak Sung and Andrei I. Volodin

Abstract. The rate of convergence for an almost surely convergent series $S_n = \sum_{i=1}^{n} X_i$ of independent random variables is studied in this paper. More specifically, when S_n converges almost surely to a random variable S, the tail series $T_n \equiv S - S_{n-1} = \sum_{i=n}^{\infty} X_i$ is a well-defined sequence of random variables with $T_n \to 0$ almost surely. Conditions are provided so that for a given positive sequence $\{b_n, n \geq 1\}$, the limit law $\sup_{k\geq n} |T_k|/b_n \xrightarrow{P} 0$ holds. This result generalizes a result of Nam and Rosalsky [4].

1. Introduction

Throughout this paper, $\{X_n, n \geq 1\}$ is a sequence of independent random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. As usual, their partial sums will be denoted by $S_n = \sum_{i=1}^{n} X_i, n \geq 1$. If S_n converges almost surely (a.s.) to a random variable S, then (set $S_0 = 0$)

$$T_n \equiv S - S_{n-1} = \sum_{i=n}^{\infty} X_i,\ n \geq 1$$

is a well-defined sequence of random variables (referred to as the *tail series*) with

$$T_n \to 0 \text{ a.s.} \tag{1.1}$$

Received April 6, 2001.

2000 Mathematics Subject Classification: 60F05, 60F15.

Key words and phrases: convergence in probability, tail series, independent random variables, weak law of large numbers, almost sure convergence.

The first author was supported by Korea Research Foundation Grant(KRF-2000-015-DP0049).

In this paper, we shall be concerned with the rate in which S_n converges to S or, equivalently, in which the tail series T_n converges to 0. Recalling that (1.1) is equivalent to

$$\sup_{k\geq n}|T_k| \xrightarrow{P} 0,$$

Nam and Rosalsky [4] proved the limit law

$$\frac{\sup_{k\geq n}|T_k|}{b_n} \xrightarrow{P} 0 \tag{1.2}$$

if $\{X_n, n\geq 1\}$ is a sequence of independent random variables with $EX_n = 0, n\geq 1$, and $\{b_n, n\geq 1\}$ is a sequence of positive constants such that

$$\sum_{i=n}^{\infty} E|X_i|^p = o(b_n^p) \tag{1.3}$$

for some $1 < p \leq 2$. Rosalsky and Rosenblatt [7] extended Nam and Rosalsky's result to Banach spaces of Rademacher type p. Rosalsky and Rosenblatt [8] extended Nam and Rosalsky's result to the setting of martingale differences. The main purpose of this paper is to generalize Nam and Rosalsky's result. More specifically, we will provide more general conditions than (1.3), under which the limit law (1.2) holds.

2. Preliminary lemmas

Some lemmas are needed to establish the main results. The first lemma is due to von Bahr and Esseen [1].

LEMMA 1 (von Bahr and Esseen [1]). *Let $X_1,\cdots,X_n$ be random variables such that $E\{X_{m+1}|S_m\} = 0$ for $0\leq m\leq n-1$, where $S_0 = 0$ and $S_m = \sum_{i=1}^m X_i$ for $1\leq m\leq n$. Then*

$$E|S_n|^p \leq 2\sum_{i=1}^{n} E|X_i|^p \text{ for all } 1\leq p\leq 2.$$

Note that Lemma 1 holds when $X_1,\cdots,X_n$ are independent random variables with $EX_m = 0$ for $1\leq m\leq n$. Nam, Rosalsky, and Volodin [6] proved the following maximal inequality for the tail series using Etemadi's [3] maximal inequality for the partial sums. Furthermore, the maximal inequalities for the partial sums and the tail sums hold for Banach space valued random variables.

Lemma 2 (Nam, Rosalsky, and Volodin [6]). *Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $\sum_{n=1}^{\infty} X_n$ converges a.s. Then setting $T_n = \sum_{i=n}^{\infty} X_i, n \geq 1$,*

$$P\{\sup_{k\geq n} |T_k| > \epsilon\} \leq 4 \sup_{k\geq n} P\{|T_k| > \frac{\epsilon}{4}\},\ \epsilon > 0.$$

Lemma 3. *Let $\{X_i, n \leq i \leq N\}$ be a sequence of independent random variables with $EX_i = 0, n \leq i \leq N$. Let $\{g_i(x), n \leq i \leq N\}$ be a sequence of functions defined on $[0, \infty)$ such that*

$$0 \leq g_i(0) \leq g_i(x), \quad \frac{g_i(x)}{x} \uparrow, \quad \frac{g_i(x)}{x^p} \downarrow \ \text{on } (0,\infty), n \leq i \leq N \tag{2.1}$$

for some $1 \leq p \leq 2$. Let $\{b_i, n \leq i \leq N\}$ be a sequence of positive constants. Then for $\epsilon > 0$

$$P\Big\{\Big|\sum_{i=n}^{N} \frac{X_i}{b_i}\Big| > \epsilon\Big\} \leq \Big(\frac{2^{2p+1}}{\epsilon^p} + \frac{2^2}{\epsilon}\Big) \sum_{i=n}^{N} \frac{Eg_i(|X_i|)}{g_i(b_i)}.$$

Proof. Define $X_i' = X_i I(|X_i| \leq b_i), X_i'' = X_i I(|X_i| > b_i)$ for $n \leq i \leq N$. Noting that $EX_i = 0$ for $n \leq i \leq N$, it follows that

$$\sum_{i=n}^{N} \frac{X_i}{b_i} = \sum_{i=n}^{N} \frac{X_i' - EX_i'}{b_i} + \sum_{i=n}^{N} \frac{X_i'' - EX_i''}{b_i}.$$

By using the Markov's inequality and Lemma 1, we have

$$\begin{aligned}
&P\Big\{\Big|\sum_{i=n}^{N} \frac{X_i}{b_i}\Big| > \epsilon\Big\} \\
&\leq P\Big\{\Big|\sum_{i=n}^{N} \frac{X_i' - EX_i'}{b_i}\Big| > \frac{\epsilon}{2}\Big\} + P\Big\{\Big|\sum_{i=n}^{N} \frac{X_i'' - EX_i''}{b_i}\Big| > \frac{\epsilon}{2}\Big\} \\
&\leq \frac{2^p}{\epsilon^p} E\Big|\sum_{i=n}^{N} \frac{X_i' - EX_i'}{b_i}\Big|^p + \frac{2}{\epsilon} E\Big|\sum_{i=n}^{N} \frac{X_i'' - EX_i''}{b_i}\Big| \\
&\leq \frac{2^{p+1}}{\epsilon^p} \sum_{i=n}^{N} E\Big|\frac{X_i' - EX_i'}{b_i}\Big|^p + \frac{2}{\epsilon} \sum_{i=n}^{N} E\Big|\frac{X_i'' - EX_i''}{b_i}\Big| \\
&\leq \frac{2^{2p+1}}{\epsilon^p} \sum_{i=n}^{N} \frac{E|X_i'|^p}{b_i^p} + \frac{2^2}{\epsilon} \sum_{i=n}^{N} \frac{E|X_i''|}{b_i}.
\end{aligned} \tag{2.2}$$

It follows from (2.1) that on the set $\{x : |x| \le b_i\}$ we have $|x|^p/b_i^p \le g_i(|x|)/g_i(b_i)$. Thus we have

$$\sum_{i=n}^{N} \frac{E|X_i'|^p}{b_i^p} \le \sum_{i=n}^{N} \frac{Eg_i(|X_i'|)}{g_i(b_i)} \le \sum_{i=n}^{N} \frac{Eg_i(|X_i|)}{g_i(b_i)}. \tag{2.3}$$

On the set $\{x : |x| > b_i\}$, we have $|x|/b_i \le g_i(|x|)/g_i(b_i)$. Thus we get

$$\sum_{i=n}^{N} \frac{E|X_i''|}{b_i} \le \sum_{i=n}^{N} \frac{Eg_i(|X_i''|)}{g_i(b_i)} \le \sum_{i=n}^{N} \frac{Eg_i(|X_i|)}{g_i(b_i)}. \tag{2.4}$$

The result follows by (2.2), (2.3), and (2.4). □

Remark 1. When $p = 1$, Lemma 3 holds without the independence condition. However, the independence condition is necessary if $1 < p \le 2$. Such a counter-example can be easily obtained.

The following lemma was proved by Nam and Rosalsky [5].

Lemma 4. *Let $\{X_n, n \ge 1\}$ be a sequence of independent and symmetric random variables with $\sum_{n=1}^{\infty} X_n$ converges a.s. Then the tail series $\{T_n = \sum_{i=n}^{\infty} X_i, n \ge 1\}$ is a well-defined sequence of random variables and for every $\epsilon > 0$ and $n \ge 1$ the inequality*

$$P\Big\{\sup_{k\ge n} |T_k| > \epsilon\Big\} \le 2P\{|T_n| > \epsilon\}$$

holds.

The following lemma gives conditions so that the tail series weak law of large numbers

$$\frac{T_n}{b_n} \xrightarrow{P} 0 \tag{2.5}$$

and the limit law

$$\frac{\sup_{k\ge n} |T_k|}{b_n} \xrightarrow{P} 0 \tag{2.6}$$

are equivalent, where $T_n = \sum_{i=n}^{\infty} X_i, n \ge 1$.

Lemma 5. *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent random variables with* $\sum_{n=1}^{\infty} X_n$ *converges a.s. Let* $\{b_n, n \geq 1\}$ *be a sequence of positive constants. Assume that either* $\{b_n, n \geq 1\}$ *is a sequence of non-increasing or* $\{X_n, n \geq 1\}$ *is a sequence of symmetric random variables. Then* (2.5) *and* (2.6) *are equivalent.*

Proof. Since (2.6) clearly implies (2.5), it needs to show that (2.5) implies (2.6). When $\{b_n\}$ is a sequence of non-increasing, Nam and Rosalsky [5] proved that (2.5) implies (2.6). We now let $\{X_n\}$ is a sequence of independent and symmetric random variables. By Lemma 4, we have

$$P\Big\{\frac{\sup_{k\geq n}|T_k|}{b_n} > \epsilon\Big\} \leq 2P\Big\{\frac{|T_n|}{b_n} > \epsilon\Big\}.$$

Hence (2.5) implies (2.6). □

3. Main results

With the preliminaries accounted for, the main results of this paper may be established. Theorem 1 gives conditions so that the series $\sum_{n=1}^{\infty} X_n$ converges a.s.

Theorem 1. *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent random variables with* $EX_n = 0, n \geq 1$, *and let* $\{b_n, n \geq 1\}$ *be a sequence of positive constants. Let* $\{g_n(x), n \geq 1\}$ *be a sequence of functions defined on* $[0, \infty)$ *such that*

$$0 \leq g_n(0) \leq g_n(x), \quad 0 < g_n(x) \uparrow \quad \text{as} \ \ n \uparrow \infty \ \ \text{for each} \ \ x > 0 \tag{3.1}$$

and

$$\frac{g_n(x)}{x}\uparrow, \quad \frac{g_n(x)}{x^p}\downarrow \quad \text{on} \ \ (0,\infty), n \geq 1 \tag{3.2}$$

for some $1 < p \leq 2$. *If*

$$\sum_{n=1}^{\infty} Eg_n(|X_n|) < \infty, \tag{3.3}$$

then $\{T_n, n \geq 1\}$ *is a well-defined sequence of random variables, and for all* $n \geq 1$ *and all* $\epsilon > 0$,

$$P\Big\{\frac{\sup_{k\geq n}|T_k|}{b_n} > \epsilon\Big\} \leq \Big(\frac{2^{4p+3}}{\epsilon^p} + \frac{2^6}{\epsilon}\Big)\sum_{i=n}^{\infty}\frac{Eg_i(|X_i|)}{g_i(b_n)}.$$

Proof. First, we show that T_n is well-defined random variable, i.e., $\sum_{n=1}^\infty X_n$ converges a.s. Let $X'_n = X_n I(|X_n| \le 1), n \ge 1$. Noting that (3.2) implies $g_n(x) \uparrow$ on $(0, \infty), n \ge 1$, we have

$$\begin{aligned}\sum_{n=1}^\infty P\{|X_n| > 1\} &\le \sum_{n=1}^\infty P\{g_n(|X_n|) \ge g_n(1)\} \\ &\le \sum_{n=1}^\infty \frac{Eg_n(|X_n|)}{g_n(1)} \\ &\le \frac{1}{g_1(1)} \sum_{n=1}^\infty Eg_n(|X_n|) < \infty\end{aligned}$$

by (3.1) and (3.3). It follows from (3.2) that on the set $\{x : |x| \le 1\}$ we have $|x|^2 \le |x|^p \le g_n(|x|)/g_n(1)$. Thus we have

$$\begin{aligned}\sum_{n=1}^\infty Var(X'_n) &\le \sum_{n=1}^\infty E|X'_n|^2 \\ &\le \sum_{n=1}^\infty \frac{1}{g_n(1)} Eg_n(|X'_n|) \\ &\le \frac{1}{g_1(1)} \sum_{n=1}^\infty Eg_n(|X_n|) < \infty.\end{aligned}$$

On the set $\{x : |x| > 1\}$ we have $|x| \le g_n(|x|)/g_n(1)$. Using this and the fact that $EX_n = 0$, we get

$$\begin{aligned}\sum_{n=1}^\infty |EX'_n| &= \sum_{n=1}^\infty |EX_n I(|X_n| > 1)| \\ &\le \sum_{n=1}^\infty E|X_n| I(|X_n| > 1) \\ &\le \sum_{n=1}^\infty \frac{Eg_n(|X_n| I(|X_n| > 1))}{g_n(1)} \\ &\le \frac{1}{g_1(1)} \sum_{n=1}^\infty Eg_n(|X_n|) < \infty.\end{aligned}$$

Thus $\sum_{n=1}^\infty X_n$ converges a.s. by the Kolmogorov's three-series theorem.

Now we find an upper bound of $P\{\sup_{k\geq n}|T_k|/b_n > \epsilon\}$. Observe that, by Theorem 8.1.3 of Chow and Teicher [2], for $\epsilon > 0$

$$P\Big\{|T_k| > \epsilon\Big\} = P\Big\{\lim_{M\to\infty}\Big|\sum_{i=k}^{M} X_i\Big| > \epsilon\Big\} \leq \liminf_{M\to\infty} P\Big\{\Big|\sum_{i=k}^{M} X_i\Big| > \epsilon\Big\}.$$

Thus, it follows by Lemma 2 and Lemma 3 with $b_i = b_n, k \leq i \leq M$, that

$$\begin{aligned}
P\Big\{\frac{\sup_{k\geq n}|T_k|}{b_n} > \epsilon\Big\} &\leq 4\sup_{k\geq n} P\Big\{\frac{|T_k|}{b_n} > \frac{\epsilon}{4}\Big\} \\
&\leq 4\sup_{k\geq n}\liminf_{M\to\infty} P\Big\{\frac{\big|\sum_{i=k}^{M} X_i|}{b_n} > \frac{\epsilon}{4}\Big\} \\
&\leq 4\sup_{k\geq n}\liminf_{M\to\infty}\Big(\frac{2^{2p+1}}{(\epsilon/4)^p} + \frac{2^2}{\epsilon/4}\Big)\sum_{i=k}^{M}\frac{Eg_i(|X_i|)}{g_i(b_n)} \\
&= \Big(\frac{2^{4p+3}}{\epsilon^p} + \frac{2^6}{\epsilon}\Big)\sum_{i=n}^{\infty}\frac{Eg_i(|X_i|)}{g_i(b_n)}.
\end{aligned}$$

Hence the proof is completed. □

THEOREM 2. *Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables with $EX_n = 0, n \geq 1$, and let $\{g_n(x), n \geq 1\}$ be a sequence of functions defined on $[0,\infty)$ satisfying* (3.1) *and* (3.2). *Let $\{b_n, n \geq 1\}$ be a sequence of positive constants such that*

$$\sum_{i=n}^{\infty} Eg_i(|X_i|) = o(g_n(b_n)). \tag{3.4}$$

Then $\{T_n, n \geq 1\}$ is a well-defined sequence of random variables obeying (2.6)

Proof. Since (3.4) implies (3.3), T_n is well-defined by Theorem 1. Also, Theorem 1 implies that

$$\begin{aligned}
P\Big\{\frac{\sup_{k\geq n}|T_k|}{b_n} > \epsilon\Big\} &\leq \Big(\frac{2^{4p+3}}{\epsilon^p} + \frac{2^6}{\epsilon}\Big)\sum_{i=n}^{\infty}\frac{Eg_i(|X_i|)}{g_i(b_n)} \\
&\leq \Big(\frac{2^{4p+3}}{\epsilon^p} + \frac{2^6}{\epsilon}\Big)\frac{1}{g_n(b_n)}\sum_{i=n}^{\infty} Eg_i(|X_i|) = o(1).
\end{aligned}$$

Thus the proof is completed. □

Nam and Rosalsky [4] proved the following corollary which is the special case $g_n(x) = |x|^p, x \geq 0, n \geq 1$, of Theorem 2.

Corollary 1. *Let* $\{X_n, n \geq 1\}$ *be a sequence of independent random variables with* $EX_n = 0, n \geq 1$, *and let* $\{b_n, n \geq 1\}$ *be a sequence of positive constants such that*

$$\sum_{i=n}^{\infty} E|X_i|^p = o(b_n^p) \tag{3.5}$$

for some $1 < p \leq 2$. *Then* $\{T_n, n \geq 1\}$ *is a well-defined sequence of random variables obeying* (2.6).

The following example illustrates the sharpness of Theorem 2. It shows that if o in (3.4) is replaced by O, then Theorem 2 can fail. It also shows that if $p > 2$, then Theorem 2 can fail.

Example 1. Let $\{Y_n, n \geq 1\}$ be sequence of independent and identically distributed $N(0,1)$ random variables, and let $\{a_n, n \geq 1\}$ be a sequence of positive constants such that $\sum_{n=1}^{\infty} a_n^2 < \infty$. Let $X_n = a_n Y_n, b_n = \sqrt{\sum_{i=n}^{\infty} a_i^2}, n \geq 1$, and $g_n(x) = |x|^p (p > 1), x \geq 0, n \geq 1$. Then $\{X_n, n \geq 1\}$ is a sequence of independent and symmetric random variables with $EX_n = 0, n \geq 1$. Since $\sum_{n=1}^{\infty} Var(X_n) = \sum_{n=1}^{\infty} a_n^2 < \infty$, T_n is well defined.

To show that the limit law $\sup_{k \geq n} |T_k|/b_n \xrightarrow{P} 0$ does not hold, it is enough to show by Lemma 5 that the tail series weak law of large numbers does not hold. To do this, let $S_{n,k} = \sum_{i=n}^{k} X_i, k \geq n \geq 1$, and let $\phi_{n,k}(t)$ be the characteristic function of $S_{n,k}$. Then $S_{n,k} \sim N(0, \sum_{i=n}^{k} a_i^2)$, and so

$$\phi_{n,k}(t) = \exp\Big\{ -\frac{1}{2}t^2 \sum_{i=n}^{k} a_i^2 \Big\} \to \exp\Big\{ -\frac{1}{2}t^2 \sum_{i=n}^{\infty} a_i^2 \Big\}$$

as $k \to \infty$. Since $\lim_{k\to\infty} \phi_{n,k}(t)$ is the characteristic function of $N(0, \sum_{i=n}^{\infty} a_i^2)$, it follows by the Lévy continuity theorem that $T_n \sim N(0,$

$\sum_{i=n}^{\infty} a_i^2$). Thus T_n/b_n has standard normal distribution, which implies that the tail series weak law of large numbers does not hold.

Now we let $a_n = 1/2^n, n \geq 1$. Note that

$$\sum_{i=n}^{\infty} Eg_i(|X_i|) = E|Y_1|^p \sum_{i=n}^{\infty} \frac{1}{2^{ip}} \sim C_1 \frac{1}{2^{np}},$$
$$g_n(b_n) = \Big(\sum_{i=n}^{\infty} \frac{1}{2^{2i}} \Big)^{p/2} \sim C_2 \frac{1}{2^{np}}$$

for some constants $C_1 > 0$ and $C_2 > 0$. Thus

$$\sum_{i=n}^{\infty} Eg_i(|X_i|) = O(g_n(b_n)),$$

and as shown before T_n is well-defined, but the limit law $\sup_{k \geq n} |T_k|/b_n \xrightarrow{P} 0$ does not hold.

On the other hand, if we take $a_n = 1/n, n \geq 1$, then

$$\sum_{i=n}^{\infty} Eg_i(|X_i|) = E|Y_1|^p \sum_{i=n}^{\infty} \frac{1}{i^p} \sim C_3 \frac{1}{n^{p-1}},$$
$$g_n(b_n) = \Big(\sum_{i=n}^{\infty} \frac{1}{i^2} \Big)^{p/2} \sim C_4 \frac{1}{n^{p/2}}$$

for some constants $C_3 > 0$ and $C_4 > 0$. Thus, when $p > 2$, the limit law $\sup_{k \geq n} |T_k|/b_n \xrightarrow{P} 0$ does not hold although (3.4) holds and T_n is well-defined.

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Soo Hak Sung, Department of Applied Mathematics, PaiChai University, Taejon 302-735, Korea
E-mail: sungsh@mail.paichai.ac.kr

Andrei I. Volodin, Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada
E-mail: volodin@math.uregina.ca