DEGENERATE WEAK CONVERGENCE OF ROW SUMS FOR ARRAYS OF RANDOM ELEMENTS IN STABLE TYPE p BANACH SPACES

BY

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Abstract. For an array of rowwise independent random elements $\{V_{nj}, j \geq 1, n \geq 1\}$ in a real separable, stable type p Banach space $\mathcal{X}$ and an array of constants $\{a_{nj}, j \geq 1, n \geq 1\}$, general weak laws of large numbers of the forms (i) $\sum_{j=1}^{k_n} a_{nj} V_{nj} \xrightarrow{P} 0$ and (ii) $\sum_{j=1}^{T_n} a_{nj}(V_{nj} - c_{nj}) \xrightarrow{P} 0$ are obtained where for (i), $EV_{nj} = 0, j \geq 1, n \geq 1$ and the k_n are permitted to assume the value ∞ and for (ii), $\{c_{nj}, j \geq 1, n \geq 1\}$ is a suitable array of elements in $\mathcal{X}$ and $\{T_n, n \geq 1\}$ is a sequence of positive integer-valued random variables (called random indices). In the main results, the random elements $\{V_{nj}, j \geq 1, n \geq 1\}$ are assumed to be stochastically dominated by a random element V and the hypotheses impose conditions on the growth behavior of the $\{a_{nj}, j \geq 1, n \geq 1\}$, on the tail of the distribution of $||V||$, and (for (ii)) on the marginal distributions of the random indices. The results of the form (i) are shown to be valid for a mode of convergence which is stronger than convergence in probability, viz. convergence in the Lorentz space $\mathcal{L}_{(p,\infty)}(\mathcal{X})$. It is shown via example that the stable type p hypothesis cannot be relaxed.

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1. Introduction. In this paper, for an array $\{V_{nj}, j \geq 1, n \geq 1\}$ of rowwise independent Banach space valued random elements, general weak laws of large numbers (WLLNs) and/or related convergence results will be established for weighted sums of the form $\sum_{j=1}^{k_n} a_{nj} V_{nj}$ (or $\sum_{j=1}^{T_n} a_{nj} V_{nj}$ where T_n is random). The general setting will now be described. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{X}$ be a real separable Banach space with norm $|| \cdot ||$. The *expected value* or *mean* of a random element V, denoted by EV, is defined to be the *Pettis integral* provided it exists. That is, V has expected value $EV \in \mathcal{X}$ if $f(EV) = E(f(V))$ for every $f \in \mathcal{X}^*$ where $\mathcal{X}^*$ is the (*dual*) space of all continuous linear functionals on $\mathcal{X}$.

Consider an array of constants $\{a_{nj}, j \geq 1, n \geq 1\}$ and an array of rowwise independent $\mathcal{X}$-valued random elements $\{V_{nj}, j \geq 1, n \geq 1\}$ defined on $(\Omega, \mathcal{F}, P)$. Let $\{k_n, n \geq 1)$ be a sequence in $\{1, 2, \cdots, \infty\}$ and let $\{T_n, n \geq 1\}$ be a sequence of positive integer-valued random variables. Let $\{c_{nj}, j \geq 1, n \geq 1\}$ be a "centering" array consisting of (suitably selected) elements in $\mathcal{X}$. In this paper, general WLLNs of the two forms

$$\sum_{j=1}^{k_n} a_{nj} V_{nj} \xrightarrow{P} 0 \tag{1.1}$$

$$\sum_{j=1}^{T_n} a_{nj} (V_{nj} - c_{nj}) \xrightarrow{P} 0 \tag{1.2}$$

will be established. The expressions in (1.1) and (1.2) are referred to as *weighted sums* with *weights* a_{nj}. The results of the form (1.1) assume that $EV_{nj} = 0, j \geq 1, n \geq 1$ and are shown to be valid for convergence in the Lorentz space $\mathcal{L}_{(p,\infty)}(\mathcal{X})$ where $1 < p < 2$. (Technical definitions such as this will be discussed in Section 2). This mode of convergence is stronger than convergence in probability. Of course, in (1.2), the number of terms in the sum is random, and the $\{T_n, n \geq 1\}$ are referred to as *random indices*.

Some of the early work on the WLLN problem for random elements in a real separable normed linear space was conducted by Taylor [32] wherein it is shown that a sequence of identically distributed random elements satisfying the WLLN in the weak linear topology also satisfies the WLLN in the norm topology. This was shown to be a consequence of Taylor's [32] result asserting that a sequence of identically distributed random elements in a real separable Banach space which has a Schauder basis satisfies the WLLN in the norm topology if and only if the WLLN holds in each coordinate of the basis.

Taylor [33] provided a comprehensive and unified treatment of results under whose conditions either the WLLN $\sum_{j=1}^{n} a_{nj} V_j \xrightarrow{P} 0$ or the general *strong law of large numbers* (SLLN) $\sum_{j=1}^{n} a_{nj} V_j \to 0$ almost certainly (a.c.) obtains where $\{V_n, n \geq 1\}$ is a sequence of independent, mean 0 random elements in a real separable normed linear space and $\{a_{nj}, 1 \leq j \leq n, n \geq 1\}$ is an array of constants. For the special case where the $\{a_{nj}, 1 \leq j \leq n, n \geq 1\}$ are of the form $a_{nj} = a_j/b_n$ $(0 < b_n \to \infty)$, the SLLN problem was studied by Mikosch and Norvaiša [27] and [28] and by Adler, Rosalsky, and Taylor [2] and the WLLN problem was studied by Adler, Rosalsky, and Taylor [3] for sequences of independent random elements in Banach spaces. In Adler, Rosalsky, and Taylor [3] the corresponding SLLN need not necessarily hold. For Banach space valued weighted sums of the form $\sum_{j=1}^{n} a_{nj} V_j$ or $\sum_{j=1}^{\infty} a_{nj} V_j$ or $\sum_{j=1}^{n} a_{nj} V_{nj}$ or, more generally, $\sum_{j=1}^{\infty} a_{nj} V_{nj}$, conditions for the equivalence between the WLLN in the norm topology and the WLLN in the weak linear topology were provided by Taylor and Padgett [34], Wei and Taylor [37], Howell and Taylor [19], and Wang and Bhaskara Rao [35], respectively.

Somewhat unrelated to the results of this paper is a vast literature of investigation of hypotheses which are required for the WLLN and the SLLN to be equivalent. For example, see Kuelbs and Zinn [22], de Acosta [1], Etemadi [15], Mikosch and Norvaiša

[27] and [28], Alt [5] and [6], Heinkel [18], Ledoux and Talagrand [24], and Wang, Bhaskara Rao, and Li [36].

In the current work, the Banach space $\mathcal{X}$ is assumed to satisfy the geometric condition of being of stable type p. Conditions are placed on the growth behavior of the weights $\{a_{nj}, j \geq 1, n \geq 1\}$. In the main results (Theorems 4.1 and 4.3 which yield conclusions of the form (1.1) and (1.2), respectively), the random elements $\{V_{nj}, j \geq 1, n \geq 1\}$ are assumed to be stochastically dominated by a random element V in the sense that (2.4) holds. The tail $P\{||V|| > t\}$ of the distribution of $||V||$ as $t \to \infty$ is controlled by (4.3) or (4.18). In Theorem 4.3, a condition is imposed on the marginal distributions of the random indices $\{T_n, n \geq 1\}$. Examples are provided showing that the stable type p hypotheses cannot be relaxed. Theorem 4.2 is a result similar to Theorem 4.1 except that the random elements are not assumed to be stochastically dominated by a random element.

For convenience, technical definitions will be consolidated into Section 2. The lemmata which are needed to establish the main results will be presented in Section 3. The main results will be established in Section 4. Finally, the symbol C denotes throughout a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

2. Preliminary Definitions. Technical definitions relevant to the current work will be discussed in this section.

Let $0 < p \leq 2$ and let $\{\theta_n, n \geq 1\}$ be independent and identically distributed stable random variables each with characteristic function $\phi(t) = \exp\{-|t|^p\}, -\infty < t < \infty$. The real separable Banach space $\mathcal{X}$ is said to be of *stable type* p if $\sum_{n=1}^{\infty} \theta_n v_n$ converges a.c. whenever $\{v_n, n \geq 1\} \subseteq \mathcal{X}$ with $\sum_{n=1}^{\infty} ||v_n||^p < \infty$. Equivalent characterizations of a Banach space being of stable type p, properties of stable type p Banach spaces, as well as various relationships between the conditions "Rademacher type p" and "stable type p" may be found in Woyczyński [39], Marcus and Woyczyński [26], Rosiński [30], and Pisier [29]. Some of these properties and relationships will now be summarized. Every real separable Banach space $\mathcal{X}$ is of stable type p for all $0 < p < 1$. Moreover

(i) for $0 < p < 2$, $\mathcal{X}$ is of stable type p if and only if $\mathcal{X}$ is of Rademacher type p' for some $p' \in (p, 2]$.

(ii) $\mathcal{X}$ is of stable type 2 if and only if $\mathcal{X}$ is of Rademacher type 2.

Consequently, if $\mathcal{X}$ is of stable type p for some $1 \leq p \leq 2$, then

$\mathcal{X}$ is of Rademacher type p,
$\mathcal{X}$ is of stable type q for all $0 < q < p$,
$\mathcal{X}$ is of stable type p_1 for some $p_1 \in (p, 2]$ if $p < 2$.

For $q \geq 2$, the $\mathcal{L}_q$-spaces and ℓ_q-spaces are of stable type 2 while for $1 \leq q < 2$, the $\mathcal{L}_q$-spaces and ℓ_q-spaces are of stable type p for all $0 < p < q$ but are not of stable type

q. Every real separable Hilbert space and real separable finite dimensional Banach space is of stable type 2.

Let $p(\mathcal{X}) = \sup\{p \in (0,2] : \mathcal{X}$ is of stable type $p\}$. Then $1 \le p(\mathcal{X}) \le 2$ and it follows from the above discussion that if $\mathcal{X}$ is of stable type $p < 2$, then $p(\mathcal{X}) > p$ and $\mathcal{X}$ is of stable type q for all $0 < q < p(\mathcal{X})$.

For a random element V and $0 < p < \infty$, it proves convenient to introduce the notation

$$\Lambda_p(V) = \sup_{t>0} t^p P\{||V|| > t\}.$$

Note that for $a \in R, v \in \mathcal{X}$, and random elements V_1, V_2

$$\Lambda_p(aV_1) = |a|^p \Lambda_p(V_1) \tag{2.1}$$

$$\Lambda_p(aV_1 + v) \le 2^p(||v||^p + |a|^p \Lambda_p(V_1)) \tag{2.2}$$

and

$$\Lambda_p(V_1 + V_2) \le 2^p(\Lambda_p(V_1) + \Lambda_p(V_2)). \tag{2.3}$$

For $0 < p < \infty$, let $\Lambda_p(\mathcal{X})$ denote the Lorentz space $\mathcal{L}_{(p,\infty)}(\mathcal{X})$ consisting of the collection of all random elements V in $\mathcal{X}$ for which $\Lambda_p(V) < \infty$. Relation (2.3) establishes that $\Lambda_p(\cdot)$ is a quasi-norm on $\Lambda_p(\mathcal{X})$ (provided random elements equal a.c. are identified). Random elements $\{V_n, n \ge 1\} \subseteq \Lambda_p(\mathcal{X})$ are said to converge to 0 in $\Lambda_p(\mathcal{X})$ (denoted $V_n \stackrel{\Lambda_p(\mathcal{X})}{\to} 0$) if $\Lambda_p(V_n) = o(1)$. As usual, for $0 < p < \infty, \mathcal{L}_p(\mathcal{X})$ denotes the collection of all random elements V in $\mathcal{X}$ for which $E||V||^p < \infty$, and random elements $\{V_n, n \ge 1\} \subseteq \mathcal{L}_p(\mathcal{X})$ converge to 0 in $\mathcal{L}_p(\mathcal{X})$ (denoted $V_n \stackrel{\mathcal{L}_p(\mathcal{X})}{\to} 0$) if $E||V_n||^p = o(1)$. It is well known and easy to show for $0 < r < p < \infty$ that:

(i) $\mathcal{L}_p(\mathcal{X}) \subseteq \Lambda_p(\mathcal{X}) \subseteq \mathcal{L}_r(\mathcal{X})$.

(ii) If $V_n \stackrel{\mathcal{L}_p(\mathcal{X})}{\to} 0$, then $V_n \stackrel{\Lambda_p(\mathcal{X})}{\to} 0$ and $V_n \stackrel{P}{\to} 0$.

(iii) If $V_n \stackrel{\Lambda_p(\mathcal{X})}{\to} 0$, then $V_n \stackrel{\mathcal{L}_r(\mathcal{X})}{\to} 0$ (and consequently $V_n \stackrel{\Lambda_r(\mathcal{X})}{\to} 0$ and $V_n \stackrel{P}{\to} 0$).

The reader may refer to Bennett and Rudnick [8], Bergh and Löfström [9], Butzer and Berens [12], Calderón [13], Hunt [20], Lindenstrauss and Tzafriri [25], and Stein and Weiss [31] for a thorough treatment of Lorentz space theory.

Random elements $\{V_{nj}, j \ge 1, n \ge 1\}$ are said to be *stochastically dominated* by a random element V if for some finite constant D,

$$P\{||V_{nj}|| > t\} \le DP\{||DV|| > t\}, \ \ t \ge 0, \ \ j \ge 1, \ \ n \ge 1. \tag{2.4}$$

This condition is, of course, automatic with $V = V_{11}$ and $D = 1$ if the $\{V_{nj}, j \ge 1, n \ge 1\}$ are identically distributed. It follows from Lemma 5.2.2 of Taylor [33], p. 123 (or Lemma 3 of Wei and Taylor [37]) that stochastic dominance can be accomplished by the array of random elements having a bounded absolute r^{th} moment ($r > 0$). Specifically, if $\sup_{n\geq 1, j\geq 1} E||V_{nj}||^r < \infty$ for some $r > 0$, then there exists a random element V with $E||V||^p < \infty$ for all $0 < p < r$ such that (2.4) holds with $D = 1$. (The proviso that $r > 1$ in Lemma 5.2.2 of Taylor [33], p. 123 (or Lemma 3 of Wei and Taylor [37]) is not needed as was pointed out by Adler, Rosalsky, and Taylor [4].)

3. Preliminary Lemmata. In this section, lemmata needed to establish the results in this paper will be presented. Some of them may be of independent interest.

The first lemma, due to Etemadi [16], provides a maximal inequality for a sum of independent random elements and will be used in the proof of a maximal inequality (Lemma 3.3) for random elements in stable type p Banach spaces. Lemma 3.1 may also be found in Billingsley [10], p. 288. For some related results see Etemadi [17]. In Proposition 1.1.1 of Kwapień and Woyczyński [23], p. 15, it is shown that Lemma 3.1 holds with the constant 4 replaced by 3. However the version of Lemma 3.1 stated below is adequate for our purposes.

Lemma 3.1 (Etemadi [16]). Let $\{V_j, 1 \leq j \leq n\}$ be independent random elements in a real separable Banach space. Then

$$P\left\{\max_{1\leq k\leq n}\left|\left|\sum_{j=1}^{k} V_j\right|\right| > t\right\} \leq 4 \max_{1\leq k\leq n} P\left\{\left|\left|\sum_{j=1}^{k} V_j\right|\right| > \frac{t}{4}\right\}, t > 0.$$

The next lemma in its original form is due to Rosiński [30]. The version presented below is a slight modification of Rosiński's [30] result and is due to Pisier [29]. (See Theorem 4.12 and Remark (ii) after it in Pisier [29].)

Lemma 3.2. If a real separable Banach space is of stable type p ($1 < p < 2$), then there exists a finite constant C_p such that for every finite collection $\{V_1, \cdots, V_n\}$ of independent mean 0 random elements

$$\Lambda_p\left(\sum_{j=1}^{n} V_j\right) \leq C_p \sum_{j=1}^{n} \Lambda_p(V_j). \tag{3.1}$$

Remark 3.1. The inequality (3.1) holds for an a.c. convergent series of independent mean 0 random elements. That is, if $\sum_{j=1}^{\infty} V_j$ converges a.c. to a random element where $\{V_n, n \geq 1\}$ are independent mean 0 random elements in a real separable, stable type p ($1 < p < 2$) Banach space, then

$$\Lambda_p\left(\sum_{j=1}^{\infty} V_j\right) \leq C_p \sum_{j=1}^{\infty} \Lambda_p(V_j).$$

This follows from (3.1) and the observation that (see, e.g., Theorem 8.1.3 of Chow and Teicher [14], p. 278)

$$P\left\{\left|\left|\sum_{j=1}^{\infty} V_j\right|\right| > t\right\} \leq \liminf_{n\to\infty} P\left\{\left|\left|\sum_{j=1}^{n} V_j\right|\right| > t\right\}.$$

The details are left to the reader.

Lemma 3.3. Let $\{V_n, n \geq 1\}$ be independent mean 0 random elements in a real separable, stable type p $(1 < p < 2)$ Banach space. Then for all $n \geq 1$,

$$P\left\{\max_{1\leq k\leq n}\left\|\sum_{j=1}^{k} V_j\right\| > t\right\} \leq \frac{C}{t^p}\sum_{j=1}^{n}\Lambda_p(V_j), \quad t > 0$$

where C is a constant independent of n.

Proof. By Lemma 3.1, for $n \geq 1$ and $t > 0$

$$\begin{aligned}
&P\left\{\max_{1\leq k\leq n}\left\|\sum_{j=1}^{k} V_j\right\| > t\right\} \leq \frac{4^{p+1}}{t^p}\max_{1\leq k\leq n}\left(\frac{t}{4}\right)^p P\left\{\left\|\sum_{j=1}^{k} V_j\right\| > \frac{t}{4}\right\}\\
&\leq \frac{4^{p+1}}{t^p}\max_{1\leq k\leq n}\Lambda_p\left(\sum_{j=1}^{k} V_j\right)\\
&\leq \frac{4^{p+1}}{t^p}\max_{1\leq k\leq n} C_p\sum_{j=1}^{k}\Lambda_p(V_j) \qquad \text{(by Lemma 3.2)}\\
&= \frac{4^{p+1}C_p}{t^p}\sum_{j=1}^{n}\Lambda_p(V_j). \qquad \square
\end{aligned}$$

In the last lemma, no independence or moment conditions are imposed on the random elements $\{W_{nj}, j \geq 1, n \geq 1\}$.

Lemma 3.4. Let $\{W_{nj}, j \geq 1, n \geq 1\}$ be an array of random elements in a real separable Banach space. Let $\{T_n, n \geq 1\}$ be positive integer-valued random variables and $\{\alpha_n, n \geq 1\}$ be constants such that for some $0 < \lambda < \infty$

$$P\left\{\frac{T_n}{\alpha_n} > \lambda\right\} = o(1). \tag{3.2}$$

Let $\{d_{nj}, j \geq 1, n \geq 1\}$ be an array of positive constants such that

$$\sum_{j=1}^{[\lambda\alpha_n]} P\{||W_{nj}|| > d_{nj}\} = o(1). \tag{3.3}$$

If

$$\max_{1\leq k\leq[\lambda\alpha_n]}\left\|\sum_{j=1}^{k}(W'_{nj} - EW'_{nj})\right\| \xrightarrow{P} 0 \tag{3.4}$$

where $W'_{nj} = W_{nj}I(||W_{nj}|| \leq d_{nj}), \;\; j \geq 1, \;\; n \geq 1$, then

$$\sum_{j=1}^{T_n}(W_{nj} - EW'_{nj}) \xrightarrow{P} 0.$$

Proof. Note at the outset that $E||W'_{nj}|| < \infty, \quad j \geq 1, \quad n \geq 1$ and so (see, e.g., Taylor [33], p. 40) the $\{W'_{nj}, j \geq 1, n \geq 1\}$ all have expected values. For arbitrary $\epsilon > 0$,

$$
\begin{aligned}
&P\left\{\left|\left|\sum_{j=1}^{T_n}(W_{nj} - EW'_{nj})\right|\right| > \epsilon\right\} \\
&\leq P\left\{\left[\left|\left|\sum_{j=1}^{T_n}(W_{nj} - EW'_{nj})\right|\right| > \epsilon\right][T_n \leq \lambda\alpha_n]\right\} + P\{T_n > \lambda\alpha_n\} \\
&\leq P\left\{\left[\left|\left|\sum_{j=1}^{T_n}(W_{nj} - EW'_{nj})\right|\right| + \left|\left|\sum_{j=1}^{T_n}(W'_{nj} - EW'_{nj})\right|\right| > \epsilon\right][T_n \leq \lambda\alpha_n]\right\} + o(1) \\
&\hspace{20em}\text{(by (3.2))} \\
&\leq P\left\{\left[\left|\left|\sum_{j=1}^{T_n}(W_{nj} - W'_{nj})\right|\right| > \frac{\epsilon}{2}\right][T_n \leq \lambda\alpha_n]\right\} \\
&\quad + P\left\{\left[\left|\left|\sum_{j=1}^{T_n}(W'_{nj} - EW'_{nj})\right|\right| > \frac{\epsilon}{2}\right][T_n \leq \lambda\alpha_n]\right\} + o(1) \\
&\leq P\left\{\bigcup_{j=1}^{[\lambda\alpha_n]}[W_{nj} \neq W'_{nj}]\right\} + P\left\{\max_{1\leq k\leq[\lambda\alpha_n]}\left|\left|\sum_{j=1}^{k}(W'_{nj} - EW'_{nj})\right|\right| > \frac{\epsilon}{2}\right\} + o(1) \\
&\leq \sum_{j=1}^{[\lambda\alpha_n]} P\{||W_{nj}|| > d_{nj}\} + o(1) \quad \text{(by (3.4))} \\
&= o(1) \qquad \text{(by (3.3))}. \qquad \square
\end{aligned}
$$

4. Mainstream. With the preliminaries accounted for, the first main result, Theorem 4.1, may be presented. Theorem 4.1 establishes for an array of rowwise independent and stochastically dominated mean 0 random elements in a stable type p $(1 < p < 2)$ Banach space the convergence in $\Lambda_p(\mathcal{X})$ limit law (4.4) which is stronger than the corresponding WLLN. Its proof owes much to that of Theorem 3.2 of Howell and Taylor [19] where the WLLN was obtained for the particular case $k_n = n, n \geq 1$. However, in Theorem 4.1, the $\{k_n, n \geq 1\}$ are permitted to assume the value ∞. (Of course, if $k_n = \infty$ for any $n \geq 1$, it must be established that the series $\sum_{j=1}^{\infty} a_{nj}V_{nj}$ converges a.c. to a random element.) Note that it follows from (2.4) and (4.3) that $E||V_{nj}||^r < \infty$ for all $0 < r < p, 1 \leq j \leq k_n, n \geq 1$. Moreover, it follows from the discussion of stochastic dominance in Section 2 that a sufficient condition for the existence of a random element V satisfying (2.4) and (4.3) is that $\sup_{n\geq1,1\leq j\leq k_n} E||V_{nj}||^r < \infty$ for some $r > p$. Professor Robert L. Taylor has so kindly pointed out to the authors via example that $\sup_{n\geq1,1\leq j\leq k_n} E||V_{nj}||^p < \infty$ does not ensure the existence of a random element V satisfying (2.4) and (4.3).

Theorem 4.1. Let $\{V_{nj}, 1 \leq j \leq k_n \leq \infty, n \geq 1\}$ be an array of rowwise independent mean 0 random elements in a real separable, stable type p $(1 < p < 2)$

Banach space $\mathcal{X}$. Suppose that $\{V_{nj}, 1 \le j \le k_n, n \ge 1\}$ is stochastically dominated by a random element V. Let $\{a_{nj}, 1 \le j \le k_n, n \ge 1\}$ be an array of constants such that

$$\sup_{n\ge1}\sum_{j=1}^{k_n}|a_{nj}|^p < \infty \tag{4.1}$$

and

$$\sup_{1\le j\le k_n}|a_{nj}| = o(1). \tag{4.2}$$

If

$$\lim_{t\to\infty} t^p P\{||V|| > t\} = 0, \tag{4.3}$$

then

$$\sum_{j=1}^{k_n} a_{nj}V_{nj} \overset{\Lambda_p(\mathcal{X})}{\longrightarrow} 0 \tag{4.4}$$

(and consequently $\sum_{j=1}^{k_n} a_{nj}V_{nj}$ converges to 0 in $\mathcal{L}_r(\mathcal{X})$ for all $0 < r < p$ and in probability).

Proof. Let D be as in (2.4). In view of (4.1), it will be assumed without loss of generality that

$$\sup_{n\ge1}\sum_{j=1}^{k_n}|a_{nj}|^p = 1. \tag{4.5}$$

Let $b_{nj} = |a_{nj}|^{-1}$ (where $0^{-1} = \infty$) and

$$U_{nj} = V_{nj}I(||V_{nj}|| \le b_{nj}), U'_{nj} = V_{nj}I(||V_{nj}|| > b_{nj}), 1 \le j \le k_n, n \ge 1.$$

Since $E||U_{nj}|| < \infty, 1 \le j \le k_n, n \ge 1$, the $\{U_{nj}, 1 \le j \le k_n, n \ge 1\}$ all have expected values. Then since $EV_{nj} = 0, 1 \le j \le k_n, n \ge 1$, the $\{U'_{nj}, 1 \le j \le k_n, n \ge 1\}$ all have expected values as well. Also note that (4.3) ensures that $\Lambda_p(V) < \infty$.

It will first be shown that if $k_n = \infty$ for any $n \ge 1$, then for that n the series $\sum_{j=1}^{\infty} a_{nj}V_{nj}$ and $\sum_{j=1}^{\infty} a_{nj}(U_{nj} - EU_{nj})$ converge a.c. to random elements in $\mathcal{X}$. Let $\epsilon > 0$ be arbitrary. For $m \ge 1$,

$$\begin{aligned}
&\sup_{M>m} E\left|\left|\sum_{j=1}^{M} a_{nj}V_{nj} - \sum_{j=1}^{m} a_{nj}V_{nj}\right|\right| = \sup_{M>m} E\left|\left|\sum_{j=m+1}^{M} a_{nj}V_{nj}\right|\right| \\
&\le \sup_{M>m}\left(\epsilon + \int_\epsilon^\infty t^p P\left\{\left|\left|\sum_{j=m+1}^{M} a_{nj}V_{nj}\right|\right| > t\right\} t^{-p}dt\right) \\
&\le \sup_{M>m}\left(\epsilon + \int_\epsilon^\infty \Lambda_p\left(\sum_{j=m+1}^{M} a_{nj}V_{nj}\right) t^{-p}dt\right) \\
&\le \sup_{M>m}\left(\epsilon + \frac{C_p}{\epsilon^{p-1}(p-1)}\sum_{j=m+1}^{M}|a_{nj}|^p\Lambda_p(V_{nj})\right) \text{ (by Lemma 3.2 and (2.1))}
\end{aligned}$$

$$\leq \epsilon + \frac{C_p D^{p+1} \Lambda_p(V)}{\epsilon^{p-1}(p-1)} \sum_{j=m+1}^{\infty} |a_{nj}|^p \quad \text{(by (2.4) and (2.1))}$$

$$\to \epsilon \text{ as } m \to \infty \quad \text{(by (4.5))}.$$

Since $\epsilon > 0$ is arbitrary,

$$\lim_{m\to\infty} \sup_{M>m} E\left|\left| \sum_{j=1}^{M} a_{nj}V_{nj} - \sum_{j=1}^{m} a_{nj}V_{nj} \right|\right| = 0$$

whence there exists a random element S_n in $\mathcal{X}$ such that

$$\lim_{m\to\infty} E\left|\left| \sum_{j=1}^{m} a_{nj}V_{nj} - S_n \right|\right| = 0.$$

This implies that

$$\sum_{j=1}^{m} a_{nj}V_{nj} \xrightarrow{P} S_n \text{ as } m \to \infty.$$

Since convergence in probability and convergence a.c. are equivalent for sums of independent random elements in a real separable Banach space (see Itô and Nisio [21]), $\sum_{j=1}^{\infty} a_{nj}V_{nj}$ converges a.c. to a random element in $\mathcal{X}$.

Next, if $k_n = \infty$ for some $n \geq 1$, then recalling that $EV_{nj} = 0, j \geq 1$

$$\sum_{j=1}^{\infty} ||a_{nj}EU_{nj}|| = \sum_{j=1}^{\infty} |a_{nj}| \; ||EU'_{nj}||$$

$$\leq \sum_{j=1}^{\infty} |a_{nj}| E||V_{nj}|| I(||V_{nj}|| > b_{nj})$$

$$= \sum_{j=1}^{\infty} |a_{nj}| \int_{b_{nj}}^{\infty} P\{||V_{nj}|| > t\}dt + \sum_{j=1}^{\infty} |a_{nj}| b_{nj} P\{||V_{nj}|| > b_{nj}\}$$

(by integration by parts)

$$\leq D \sum_{j=1}^{\infty} |a_{nj}| \int_{b_{nj}}^{\infty} t^p P\{||DV|| > t\} t^{-p} dt + D \sum_{j=1}^{\infty} |a_{nj}|^p b_{nj}^p P\{||DV|| > b_{nj}\}$$

(by (2.4))

$$\leq \frac{D^{p+1}\Lambda_p(V)}{p-1} \sum_{j=1}^{\infty} |a_{nj}| b_{nj}^{1-p} + D^{p+1}\Lambda_p(V) \sum_{j=1}^{\infty} |a_{nj}|^p \quad \text{(by (2.1))}$$

$$= \frac{D^{p+1}\Lambda_p(V)}{p-1} \sum_{j=1}^{\infty} |a_{nj}|^p + D^{p+1}\Lambda_p(V) \sum_{j=1}^{\infty} |a_{nj}|^p$$

$$\leq \frac{pD^{p+1}\Lambda_p(V)}{p-1} \qquad \text{(by (4.5))}$$
$$< \infty. \tag{4.6}$$

Then (arguing as in the proof that $\sum_{j=1}^{\infty} a_{nj}V_{nj}$ converges a.c.) for arbitrary $\epsilon > 0$ and all $m \geq 1$

$$\begin{aligned}
&\sup_{M>m} E\left|\left|\sum_{j=1}^{M} a_{nj}(U_{nj} - EU_{nj}) - \sum_{j=1}^{m} a_{nj}(U_{nj} - EU_{nj})\right|\right| \\
&\leq \sup_{M>n}\left(\epsilon + \frac{C_p}{\epsilon^{p-1}(p-1)}\sum_{j=m+1}^{M}|a_{nj}|^p\Lambda_p(U_{nj} - EU_{nj})\right) \\
&\leq \epsilon + \frac{C_p 2^p}{\epsilon^{p-1}(p-1)}\left(\sum_{j=m+1}^{\infty}|a_{nj}|^p||EU_{nj}||^p + \sum_{j=m+1}^{\infty}|a_{nj}|^p\Lambda_p(U_{nj})\right) \text{ (by (2.2))} \\
&\leq \epsilon + \frac{C_p 2^p}{\epsilon^{p-1}(p-1)}\left(\left(\sum_{j=m+1}^{\infty}||a_{nj}EU_{nj}||\right)^p + \sum_{j=m+1}^{\infty}|a_{nj}|^p\Lambda_p(V_{nj})\right) \\
&\leq \epsilon + \frac{C_p 2^p}{\epsilon^{p-1}(p-1)}\left(\left(\sum_{j=m+1}^{\infty}||a_{nj}EU_{nj}||\right)^p + D^{p+1}\Lambda_p(V)\sum_{j=m+1}^{\infty}|a_{nj}|^p\right) \\
&\qquad\qquad \text{(by (2.4) and (2.1))} \\
&\to \epsilon \text{ as } m \to \infty \text{ (by (4.6) and (4.5))}
\end{aligned}$$

implying that $\sum_{j=1}^{\infty} a_{nj}(U_{nj} - EU_{nj})$ converges a.c. to a random element in $\mathcal{X}$. It then follows from the a.c. convergence of $\sum_{j=1}^{\infty} a_{nj}V_{nj}$ that $\sum_{j=1}^{\infty} a_{nj}(U'_{nj} - EU'_{nj})$ converges a.c. to a random element in $\mathcal{X}$.

Now recalling (2.3),

$$\Lambda_p\left(\sum_{j=1}^{k_n} a_{nj}V_{nj}\right) \leq 2^p\left(\Lambda_p\left(\sum_{j=1}^{k_n} a_{nj}(U_{nj} - EU_{nj})\right) + \Lambda_p\left(\sum_{j=1}^{k_n} a_{nj}(U'_{nj} - EU'_{nj})\right)\right)$$

and so (4.4) will certainly hold provided it can be shown

$$\Lambda_p\left(\sum_{j=1}^{k_n} a_{nj}(U_{nj} - EU_{nj})\right) = o(1) \tag{4.7}$$

and

$$\Lambda_p\left(\sum_{j=1}^{k_n} a_{nj}(U'_{nj} - EU'_{nj})\right) = o(1). \tag{4.8}$$

Choose $q \in (p, p(\mathcal{X}))$. Then $\mathcal{X}$ is of stable type q. Let $\epsilon > 0$ be arbitrary. Now (4.3) ensures that there exists a constant $0 < B < \infty$ such that

$$\sup_{t \geq B} t^p P\{||DV|| > t\} \leq \min\left\{\frac{\epsilon(p-1)}{4D}, \frac{\epsilon^q}{D2^q}\right\}. \tag{4.9}$$

The condition (4.2) guarantees that for all large n

$$\sup_{1\le j\le k_n} |a_{nj}| \le \min\left\{\frac{1}{B}, \left(\frac{\epsilon}{2B}\right)^{\frac{q}{q-p}}\right\}. \tag{4.10}$$

Set $B_n = \inf_{1\le i\le k_n} b_{ni}, n \ge 1$. Then $B_n \to \infty$ by (4.2) and

$$\begin{aligned}
&\sum_{j=1}^{k_n} |a_{nj}|^p b_{nj}^p P\{||DV|| > b_{nj}\} \\
&\le \sup_{t\ge B_n} t^p P\{||DV|| > t\} \qquad \text{(by (4.5))} \\
&= o(1) \qquad \text{(by (4.3))}.
\end{aligned} \tag{4.11}$$

Then for all large n, arguing as in (4.6),

$$\begin{aligned}
&\sum_{j=1}^{k_n} ||a_{nj}EU_{nj}|| \\
&\le D\sum_{j=1}^{k_n} |a_{nj}| \int_{b_{nj}}^{\infty} t^p P\{||DV|| > t\} t^{-p} dt + D\sum_{j=1}^{k_n} |a_{nj}|^p b_{nj}^p P\{||DV|| > b_{nj}\} \\
&\le D\sum_{j=1}^{k_n} |a_{nj}| \frac{\epsilon(p-1)b_{nj}^{1-p}}{4D(p-1)} + \frac{\epsilon}{4} \quad \text{(by (4.10), (4.9), and (4.11))} \\
&= \sum_{j=1}^{k_n} |a_{nj}|^p \frac{\epsilon}{4} + \frac{\epsilon}{4} \\
&\le \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2} \qquad \text{(by (4.5))}.
\end{aligned} \tag{4.12}$$

Thus for all large n,

$$\sum_{j=1}^{k_n} |a_{nj}|^q ||EU_{nj}||^q \le \left(\sum_{j=1}^{k_n} ||a_{nj}EU_{nj}||\right)^q \le \frac{\epsilon^q}{2^q}. \tag{4.13}$$

Then for all large n

$$\begin{aligned}
&\Lambda_q\left(\sum_{j=1}^{k_n} a_{nj}(U_{nj} - EU_{nj})\right) \\
&\le C_q \sum_{j=1}^{k_n} \Lambda_q(a_{nj}(U_{nj} - EU_{nj})) \quad \text{(by Lemma 3.2 and Remark 3.1)} \\
&\le 2^q C_q \left(\sum_{j=1}^{k_n} |a_{nj}|^q ||EU_{nj}||^q + \sum_{j=1}^{k_n} |a_{nj}|^q \Lambda_q(U_{nj})\right) \quad \text{(by (2.2))}
\end{aligned}$$

$$
\begin{aligned}
&\le 2^q C_q \left(\frac{\epsilon^q}{2^q} + \sum_{j=1}^{k_n} |a_{nj}|^q \sup_{0<t<b_{nj}} t^q P\{||U_{nj}|| > t\} \right) \quad \text{(by (4.13))} \\
&\le C_q \left(\epsilon^q + 2^q \sum_{j=1}^{k_n} |a_{nj}|^q \max \left\{ B^q, D \sup_{B \le t \le b_{nj}} t^q P\{||DV|| > t\} \right\} \right) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (4.10) and (2.4))} \\
&\le C_q \left(\epsilon^q + 2^q \sum_{j=1}^{k_n} |a_{nj}|^q \max \left\{ B^q, D b_{nj}^{q-p} \sup_{t \ge B} t^p P\{||DV|| > t\} \right\} \right) \\
&\le C_q \left(\epsilon^q + 2^q \sum_{j=1}^{k_n} |a_{nj}|^p \max \left\{ B^q \sup_{1 \le i \le k_n} |a_{ni}|^{q-p}, \frac{\epsilon^q}{2^q} \right\} \right) \quad \text{(by (4.9))} \\
&\le C_q \left(\epsilon^q + 2^q \max \left\{ \frac{\epsilon^q}{2^q}, \frac{\epsilon^q}{2^q} \right\} \right) \quad \text{(by (4.5) and (4.10))} \\
&= 2 C_q \epsilon^q.
\end{aligned}
$$

By the arbitrariness of $\epsilon > 0$,

$$
\Lambda_q \left(\sum_{j=1}^{k_n} a_{nj}(U_{nj} - EU_{nj}) \right) = o(1)
$$

implying (4.7) since $p < q$.

Next, it follows from (4.12) that for all large n

$$
\sum_{j=1}^{k_n} |a_{nj}|^p ||EU'_{nj}||^p = \sum_{j=1}^{k_n} |a_{nj}|^p ||EU_{nj}||^p \le \left(\sum_{j=1}^{k_n} ||a_{nj} EU_{nj}|| \right)^p \le \frac{\epsilon^p}{2^p}. \tag{4.14}
$$

Now to prove (4.8), note that for all large n,

$$
\begin{aligned}
&\Lambda_p \left(\sum_{j=1}^{k_n} a_{nj}(U'_{nj} - EU'_{nj}) \right) \\
&\le C_p \sum_{j=1}^{k_n} \Lambda_p(a_{nj}(U'_{nj} - EU'_{nj})) \quad \text{(by Lemma 3.2 and Remark 3.1)} \\
&\le 2^p C_p \left(\sum_{j=1}^{k_n} |a_{nj}|^p ||EU'_{nj}||^p + \sum_{j=1}^{k_n} |a_{nj}|^p \Lambda_p(U'_{nj}) \right) \quad \text{(by 2.2))} \\
&\le 2^p C_p \left(\frac{\epsilon^p}{2^p} + \sum_{j=1}^{k_n} |a_{nj}|^p \sup_{t>0} t^p P\{||U'_{nj}|| > t\} \right) \quad \text{(by (4.14))} \\
&\le C_p \left(\epsilon^p + 2^p \sum_{j=1}^{k_n} |a_{nj}|^p \big(\sup_{0<t<b_{nj}} t^p P\{||V_{nj}|| > b_{nj}\} + \sup_{t \ge b_{nj}} t^p P\{||V_{nj}|| > t\} \big) \right)
\end{aligned}
$$

$$\le C_p\left(\epsilon^p + 2^{p+1}\sum_{j=1}^{k_n}|a_{nj}|^p \sup_{t\ge b_{nj}} t^p P\{||V_{nj}||>t\}\right)$$
$$\le C_p\left(\epsilon^p + 2^{p+1}D\sum_{j=1}^{k_n}|a_{nj}|^p \sup_{t\ge b_{nj}} t^p P\{||DV||>t\}\right) \quad \text{(by (2.4))}$$
$$\le C_p\left(\epsilon^p + 2^{p+1}D \sup_{t\ge B_n} t^p P\{||DV||>t\}\right) \quad \text{(by (4.5))}$$
$$= C_p\epsilon^p + o(1) \quad \text{(by (4.3))}.$$

Since $\epsilon > 0$ is arbitrary, (4.8) holds thereby completing the proof of Theorem 4.1. □

Remarks 4.1. The connection between Theorem 4.1 and other results in the literature concerning the WLLN problem for random elements in a stable type p Banach space will now be discussed.

(i) In Theorem 2 of Wei and Taylor [38], a WLLN version of Theorem 4.1 was obtained in the case $0 < p < 2$ for a sequence of independent mean 0 random elements. The k_n are permitted to be ∞. Condition (4.3) of Theorem 4.1 is weaker than the corresponding condition $E||V||^p < \infty$ of Theorem 2 of Wei and Taylor [38].

(ii) In Theorem 2.1 of Bozorgnia and Bhaskara Rao [11], a WLLN version of Theorem 4.1 was obtained in the case $0 < p < 1$ for a sequence of pairwise independent random elements. The k_n are again permitted to be ∞. The Bozorgnia and Bhaskara Rao [11] result has the condition $E||V||^p < \infty$ instead of (4.3). Since $0 < p < 1$, the Banach space is, of course, automatically of stable type p. The random elements are not assumed to have mean 0.

(iii) As mentioned earlier, Howell and Taylor [19] obtained a WLLN version of Theorem 4.1 when $k_n = n, \ \ n \ge 1$.

(iv) For a sequence $\{V_n, n \ge 1\}$ of independent random elements, Alt [6] obtained two WLLN versions of Theorem 4.1. In Theorem 1.2 of Alt [6], the random elements are not assumed to have mean 0, the k_n are permitted to be ∞, and $0 < p < 1$ so again the stable type p condition is automatic. Alt [6] also obtains an $\mathcal{L}_p(\mathcal{X})$ convergence version of his Theorem 1.2 under an additional uniform integrability condition when $k_n < \infty, \ \ n \ge 1$. In Theorem 1.4 of Alt [6], $k_n < \infty$ for all $n \ge 1$, $1 \le p < 2$, and the random elements are not assumed to have mean 0 but are centered at the truncated expectations $EV_jI(||V_j|| \le |a_{nj}|^{-1}), \ 1 \le j \le k_n, \ n \ge 1$ in the conclusion. Moreover, Alt [6] asserts that if $1 < p < 2$ and $EV_n = 0$, $n \ge 1$, then the truncated expectations are not needed and so $\sum_{j=1}^{k_n} a_{nj}V_j \xrightarrow{P} 0$. Unfortunately, the authors cannot quite follow this assertion of Alt [6] if $\limsup_{n\to\infty} k_n = \infty$. Thus, the main difference between Alt's [6] Theorem 1.4 and Theorem 4.1 is that in Alt's [6] Theorem 1.4

the k_n are required to be finite, the random elements are centered at truncated expectations, and the convergence is in probability whereas in Theorem 4.1, the k_n are permitted to be ∞, the random elements are centered at their true expectations, and the convergence is in $\Lambda_p(\mathcal{X})$.

(v) For the $1 < p < 2$ case, a WLLN version of Theorem 4.1 is established in Theorem 3.3 of Wang, Bhaskara Rao, and Li [36] for a sequence of independent mean 0 random elements. The k_n are permitted to be ∞. However, Wang, Bhaskara Rao, and Li [36] imposed the condition $\sup_{n\geq 1} E||V_n||^p < \infty$ which isn't needed for $\sum_{j=1}^{k_n} a_{nj}V_j \xrightarrow{P} 0$ in view of Theorem 4.1.

The following example, inspired by a similar one of Beck [7], shows that the stable type p hypothesis in Theorem 4.1 cannot be replaced by the weaker hypothesis that $\mathcal{X}$ is of Rademacher type p.

Example 4.1. Let $1 < p < 2$. Consider the real separable Banach space ℓ_p of absolute p^{th} power summable real sequences $v = \{v_i, i \geq 1\}$ with norm $||v|| = (\sum_{i=1}^{\infty} |v_i|^p)^{1/p}$. Let $v^{(n)}$ denote the element having 1 in its n^{th} position and 0 elsewhere, $n \geq 1$. Define a sequence $\{V_n, n \geq 1\}$ of independent mean 0 random elements in ℓ_p by requiring the $\{V_n, n \geq 1\}$ to be independent with

$$P\{V_n = v^{(n)}\} = P\{V_n = -v^{(n)}\} = \frac{1}{2}, \quad n \geq 1.$$

Let $k_n = n, \ n \geq 1$, and $V_{nj} = V_j, 1 \leq j \leq n, \ n \geq 1$. Then (2.4) and (4.3) hold with $D = 1$ and $V = V_{11}$. Let $a_{nj} = n^{-1/p}, 1 \leq j \leq n, \ n \geq 1$. Then (4.1) and (4.2) hold but

$$\left|\left|\sum_{j=1}^{n} a_{nj}V_{nj}\right|\right| = \frac{||\sum_{j=1}^{n} V_j||}{n^{1/p}} = 1 \text{ a.c.}, \quad n \geq 1$$

and so (4.4) fails. Of course, ℓ_p is not of stable type p but it is of Rademacher type p.

The next theorem is a version of Theorem 4.1 but without the assumption that the array of random elements is stochastically dominated by a random element. Theorem 4.2 contains the additional condition (4.15) relating the marginal distributions of the random variables $\{||V_{nj}||, 1 \leq j \leq k_n, n \geq 1\}$ with the respective weights $\{a_{nj}, 1 \leq j \leq k_n, n \geq 1\}$ in the weighted sum $\sum_{j=1}^{n} a_{nj}V_{nj}$. Condition (4.16) serves the role in Theorem 4.2 of condition (4.3) in Theorem 4.1. Theorem 4.2 will only be stated as its proof follows along the lines of Theorem 4.1 and is left to the reader.

Theorem 4.2. Let $\{V_{nj}, 1 \leq j \leq k_n \leq \infty, n \geq 1\}$ be an array of rowwise independent mean 0 random elements in a real separable, stable type p $(1 < p < 2)$ Banach space $\mathcal{X}$. Let $\{a_{nj}, 1 \leq j \leq k_n, n \geq 1\}$ be an array of constants such that (4.1) and (4.2) hold. If

$$\lim_{t\to\infty} \sup_{n\geq 1} \sum_{j=1}^{k_n} |a_{nj}| t^p P\{||V_{nj}|| > t\} = 0 \tag{4.15}$$

and

$$\lim_{t\to\infty} \sup_{n\geq 1} \sup_{1\leq j\leq k_n} t^p P\{||V_{nj}|| > t\} = 0, \tag{4.16}$$

then

$$\sum_{j=1}^{k_n} a_{nj} V_{nj} \overset{\Lambda_p(\mathcal{X})}{\longrightarrow} 0.$$

Remarks 4.2. Some discussion is in order concerning the conditions of Theorem 4.2 and their relation with the conditions of Theorem 4.1.

(i) It is clear that (4.16) $\Rightarrow$ (4.15) if

$$\sup_{n\geq 1} \sum_{j=1}^{k_n} |a_{nj}| < \infty. \tag{4.17}$$

(ii) In general, the hypotheses of Theorems 4.1 and 4.2 do not imply each other. However, if $\{V_{nj}, 1 \leq j \leq k_n, n \geq 1\}$ is stochastically dominated by a random element V and if (4.3) and (4.17) hold, then (4.15) and (4.16) hold.

Proof. Stochastic domination and (4.3) immediately yield (4.16), and then (4.15) follows from Remark 4.2(i). □

(iii) If the array of random variables $\{||V_{nj}||^p, 1 \leq j \leq k_n, n \geq 1\}$ is $\{a_{nj}\}$-uniformly integrable (that is,

$$\lim_{t\to\infty} \sup_{n\geq 1} \sum_{j=1}^{k_n} |a_{nj}| E||V_{nj}||^p I(||V_{nj}|| > t) = 0),$$

then (4.15) holds.

Proof. For $n \geq 1$,

$$\sum_{j=1}^{k_n} |a_{nj}| t^p P\{||V_{nj}|| > t\} \leq \sum_{j=1}^{k_n} |a_{nj}| E||V_{nj}||^p I(||V_{nj}|| > t)$$

and (4.15) follows. □

(iv) If the array of random elements $\{V_{nj}, 1 \leq j \leq k_n, n \geq 1\}$ is $\{a_{nj}\}$-compactly uniformly p-th order integrable, then (4.15) holds.

Proof. By hypothesis, for arbitrary $\epsilon > 0$, there exists a compact subset K_ϵ of $\mathcal{X}$ such that

$$\sup_{n\geq 1} \sum_{j=1}^{k_n} |a_{nj}| E||V_{nj}||^p I(V_{nj} \notin K_\epsilon) < \epsilon.$$

In view of Remark 4.2(iii), it suffices to show that the array $\{||V_{nj}||^p, 1 \leq j \leq k_n, n \geq 1\}$ is $\{a_{nj}\}$-uniformly integrable. By the boundedness of K_ϵ,

$$t_\epsilon \equiv \sup\{||x|| : x \in K_\epsilon\} < \infty.$$

Thus for all $t \geq t_\epsilon$,

$$\sup_{n\geq 1} \sum_{j=1}^{k_n} |a_{nj}| E||V_{nj}||^p I(||V_{nj}|| > t) \leq \sup_{n\geq 1} \sum_{j=1}^{k_n} |a_{nj}| E||V_{nj}||^p I(V_{nj} \notin K_\epsilon) < \epsilon.$$

The conclusion follows since ϵ is arbitrary. □

(v) Example 4.1 also shows that the stable type p hypothesis in Theorem 4.2 cannot be replaced by the weaker hypothesis that $\mathcal{X}$ is of Rademacher type p.

For random elements in a stable type p $(1 \leq p < 2)$ Banach space, the next theorem establishes the WLLN (4.22) which involves random indices $\{T_n, n \geq 1\}$. It is not assumed that the random elements $\{V_{nj}, j \geq 1, n \geq 1\}$ have mean 0. In addition, no assumptions are made regarding the joint distributions of the random indices whose marginal distributions are constrained solely by (4.19). Nor is it assumed that the stochastic processes $\{T_n, n \geq 1\}$ and $\{V_{nj}, j \geq 1, n \geq 1\}$ are independent of each other. It should be noted that the condition (4.19) is considerably weaker than $T_n/\alpha_n \xrightarrow{P} c$ for some constant $c \in [0, \infty)$.

Theorem 4.3. Let $\{V_{nj}, j \geq 1, n \geq 1\}$ be an array of rowwise independent random elements in a real separable, stable type p $(1 \leq p < 2)$ Banach space $\mathcal{X}$. Suppose that $\{V_{nj}, j \geq 1, n \geq 1\}$ is stochastically dominated by a random element V. Assume that

$$\lim_{t\to\infty} t^p P\{||V|| > t\} = 0 \text{ if } p > 1 \text{ and } E||V|| < \infty \text{ if } p = 1. \tag{4.15}$$

Let $\{T_n, n \geq 1\}$ be positive integer-valued random variables and $0 < \alpha_n \to \infty$ be constants such that for some $0 < \lambda < \infty$

$$P\left\{\frac{T_n}{\alpha_n} > \lambda\right\} = o(1). \tag{4.16}$$

Let $\{a_{nj}, j \geq 1, n \geq 1\}$ be an array of constants such that

$$\sup_{n\geq 1} \sum_{j=1}^{[\lambda\alpha_n]} |a_{[\alpha_n],j}|^p < \infty \tag{4.17}$$

and

$$\sup_{1\leq j\leq[\lambda\alpha_n]} |a_{[\alpha_n],j}|^p = \mathcal{O}(\alpha_n^{-1}). \tag{4.18}$$

Then the WLLN

$$\sum_{j=1}^{T_n} a_{[\alpha_n],j}\left(V_{[\alpha_n],j} - EV_{[\alpha_n],j}I(||V_{[\alpha_n],j}|| \le (\sup_{1\le j\le[\lambda\alpha_n]} |a_{[\alpha_n],j}|)^{-1})\right) \xrightarrow{P} 0 \tag{4.19}$$

obtains.

Proof. Let D be as in (2.4). Note at the outset that it follows from (4.18) that

$$\lim_{t\to\infty} t^p P\{||V|| > t\} = 0 \text{ and } E||V|| < \infty. \tag{4.20}$$

Let $\Gamma = \sup_{n\ge1}\sum_{j=1}^{[\lambda\alpha_n]} |a_{[\alpha_n],j}|^p$ and $e_n = (\sup_{1\le j\le[\lambda\alpha_n]} |a_{[\alpha_n],j}|)^{-1}, n \ge 1$. Then (4.21) and $\alpha_n \to \infty$ ensure that $e_n \to \infty$, and it follows from (4.20), (4.21), and $\alpha_n \to \infty$ that for every $q > p$

$$\sum_{j=1}^{[\lambda\alpha_n]} |a_{[\alpha_n],j}|^q = \mathcal{O}(\alpha_n^{\frac{p-q}{p}}) = o(1). \tag{4.21}$$

Let $q \in (p, p(\mathcal{X}))$. Then $\mathcal{X}$ is of stable type q. Set

$$\begin{aligned}&d_{nj} = |a_{[\alpha_n],j}|e_n,\ \ W_{nj} = a_{[\alpha_n],j}V_{[\alpha_n],j},\ U_{nj} = V_{[\alpha_n],j}I(||V_{[\alpha_n],j}|| \le e_n),\\ &W'_{nj} = W_{nj}I(||W_{nj}|| \le d_{nj}) = a_{[\alpha_n],j}U_{nj}, j \ge 1, n \ge 1.\end{aligned}$$

Since $E||U_{nj}|| \le \infty,\ \ j \ge 1,\ \ n \ge 1$, the $\{U_{nj}, j \ge 1, n \ge 1\}$ all have expected values.

Note that

$$\begin{aligned}&\sum_{j=1}^{[\lambda\alpha_n]} P\{||W_{nj}|| > d_{nj}\} \le \sum_{j=1}^{[\lambda\alpha_n]} P\{||V_{[\alpha_n],j}|| > e_n\}\\ &\le D\lambda\alpha_n P\{||DV|| > e_n\} \quad \text{(by (2.4))}\\ &\le Ce_n^p P\{||DV|| > e_n\} = o(1) \quad \text{(by (4.21) and (4.23))}.\end{aligned}$$

Thus, in view of Lemma 3.4, it only needs to be shown that (3.4) holds. To this end, for arbitrary $\epsilon > 0$

$$\begin{aligned}&P\left\{\max_{1\le k\le[\lambda\alpha_n]}\left|\left|\sum_{j=1}^{k}(W'_{nj} - EW'_{nj})\right|\right| > \epsilon\right\}\\ &\le \frac{C}{\epsilon^q}\sum_{j=1}^{[\lambda\alpha_n]} \Lambda_q(W'_{nj} - EW'_{nj}) \quad \text{(by Lemma 3.3)}\\ &\le \frac{2^qC}{\epsilon^q}\left(\sum_{j=1}^{[\lambda\alpha_n]} ||EW'_{nj}||^q + \sum_{j=1}^{[\lambda\alpha_n]} \Lambda_q(W'_{nj})\right) \quad \text{(by (2.2))}.\end{aligned} \tag{4.22}$$

Now by (2.4) and (4.23)

$$E||V_{[\alpha_n],j}|| \le D^2E||V|| < \infty,\ \ j \ge 1,\ \ n \ge 1$$

and so recalling (4.24)

$$\sum_{j=1}^{[\lambda\alpha_n]} ||EW'_{nj}||^q \le \sum_{j=1}^{[\lambda\alpha_n]} |a_{[\alpha_n],j}|^q (E||V_{[\alpha_n],j}||)^q$$

$$\le D^{2q}(E||V||)^q \sum_{j=1}^{[\lambda\alpha_n]} |a_{[\alpha_n],j}|^q = o(1).$$

For arbitrary $\delta > 0$, (4.23) ensures that there exists a constant $0 < B < \infty$ such that

$$\sup_{t \ge B} t^p P\{||DV|| > t\} \le \frac{\delta}{2D\Gamma}. \qquad 4.24)$$

Since $e_n \to \infty, e_n \ge B$ for all large n. Thus for all large n,

$$\sum_{j=1}^{[\lambda\alpha_n]} \Lambda_q(W'_{nj}) = \sum_{j=1}^{[\lambda\alpha_n]} |a_{[\alpha_n],j}|^q \sup_{0<t<e_n} t^q P\{||U_{nj}|| > t\} \quad \text{(by (2.1))}$$

$$\le \sum_{j=1}^{[\lambda\alpha_n]} |a_{[\alpha_n],j}|^q \max\left\{B^q, D \sup_{B \le t \le e_n} t^q P\{||DV|| > t\}\right\} \quad \text{(by (2.4))}$$

$$\le B^q \sum_{j=1}^{[\lambda\alpha_n]} |a_{[\alpha_n],j}|^q + D \sum_{j=1}^{[\lambda\alpha_n]} |a_{[\alpha_n],j}|^p \sup_{1 \le i \le [\lambda\alpha_n]} |a_{[\alpha_n],i}|^{q-p} e_n^{q-p} \sup_{t \ge B} t^p P\{||DV|| > t\}$$

$$\le \frac{\delta}{2} + D\Gamma \frac{\delta}{2D\Gamma} = \delta \quad \text{(by (4.24) and (4.27))}.$$

Since $\delta > 0$ is arbitrary, $\sum_{j=1}^{[\lambda\alpha_n]} \Lambda_q(W'_{nj}) = o(1)$ which in view of (4.25) and (4.26) yields (3.4) thereby completing the proof of Theorem 4.3. □

The following modification of Example 4.1 shows that the stable type p hypothesis in Theorem 4.3 cannot be replaced by the weaker hypothesis that $\mathcal{X}$ is of Rademacher type p.

Example 4.2. Let $1 \le p < 2$ and let $\{V_n, n \ge 1\}$ be the sequence of independent random elements in ℓ_p (which is of Rademacher type p) defined as in Example 4.1. Set $V_{nj} = V_j, j \ge 1, n \ge 1$. Let $a_{nj} = n^{-1/p}, 1 \le j \le n, a_{nj} = 0, j \ge n+1, n \ge 1$ and let $T_n \equiv n, \alpha_n = n, n \ge 1$, and $\lambda = 1$. Then all of the hypotheses of Theorem 4.3 are satisfied (except for the stable type p hypothesis) but (4.22) fails. The details are left to the reader.

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