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A note on complete convergence for arrays

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Abstract

We extend and generalize some recent results on complete convergence for independent non-identically distributed random variables (cf. Duncan and Szynal, 1984; Gut, 1992; Hu et al., 1989). In the main result no assumptions are made concerning the existence of expected values or absolute moments of the random variables. Some well-known results from the literature can be easily obtained from our theorem. © 1998 Elsevier Science B.V. All rights reserved

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1. Introduction

The concept of complete convergence introduced by Hsu and Robbins (1947) is as follows. A sequence $\{U_n,\ n\geqslant 1\}$ of random variables *converges completely* to the constant C if $\sum_{n=1}^{\infty} P\{|U_n - C| > \varepsilon\} < \infty$ for all $\varepsilon > 0$. Moreover, it was proved there that the sequence of arithmetic means of i.i.d. random variables converges completely to the expected value if the variance of the summands is finite. That result has been generalized and extended in several directions (cf. Duncan and Szynal, 1984; Gut, 1992; Hu et al., 1989; Rohatgi, 1971).

Our result combines some ideas from the above papers. The results from those papers can be obtained from our theorem as will be shown in Section 4. Our theorem has been generalized in Hu et al. (1995) with quite a different proof.

The present note is organized as follows. In Section 2, we provide the well-known Hoffmann-Jørgensen inequality and a simple symmetrization lemma which will be used in the proof of our main result. Section 3 is devoted to an extension of the Hsu–Robbins theorem to general triangular arrays of rowwise independent, but not necessarily identically distributed random variables. Here rowwise independence means that random variables within each row are independent, but that no independence is assumed between the rows. In the

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main result no assumptions are made concerning the existence of expected values or absolute moments of the random variables. As it was mentioned above, we give in Section 4 simplified proofs of some known results.

2. Preliminary lemmata

Let $\{(X_{nk},\ 1\leqslant k\leqslant k_n),\ n\geqslant 1\}$ be an array of rowwise independent, but not necessarily identically distributed, random variables and let $S_n=\sum_{k=1}^{k_n}X_{nk}$. If $k_n=\infty$ we will assume that the series converges a.s. An array is said to be *symmetric* if X_{nk} is symmetrically distributed for all k and all n. For a random variable Y, its symmetrization is $Y^s=Y-\widetilde{Y}$, where $\widetilde{Y}$ is an independent copy of Y.

In the following, we shall use a well-known inequality and a simple symmetrization lemma.

Hoffman-Jørgensen (1974) inequality. *If an array is symmetric, then for any $j\geqslant 1$ and all $t>0$*

$$P\{|S_n|>3^j t\}\leqslant C_j P\left\{\sup_{1\leqslant k\leqslant k_n}|X_{nk}|>t\right\}+D_j(P\{|S_n|>t\})^{2^j},$$

where C_j and D_j are positive constants depending only on j.

Lemma. *If $\{Y_n,\ n\geqslant 1\}$ is a sequence of random variables with $Y_n\to 0$ in probability, as $n\to\infty$, then for all $t>0$ and sufficiently large n*

$$P\{|Y_n|>t\}\leqslant 2P\{|Y_n^s|>t/2\}.$$

Proof. Let med(Y) denote any median of a random variable Y. Since $Y_n\to 0$ in probability, med$(Y_n)\to 0$ as $n\to\infty$ (Loève, 1977, p. 277, Corollary 1). The conclusion now follows immediately from the weak symmetrization inequalities (Loève, 1977, p. 257). □

3. The main result

With the preliminaries accounted for, the main theorem can be presented.

Theorem. *Let $\{(X_{nk},\ 1\leqslant k\leqslant k_n),\ n\geqslant 1\}$ be an array of rowwise independent random variables and $\{c_n,\ n\geqslant 1\}$ a sequence of positive constants such that $\sum_{n=1}^{\infty}c_n=\infty$. Suppose that for all $\varepsilon>0$ and some $\delta>0$:*

(i) $\sum_{n=1}^{\infty}c_n\sum_{k=1}^{k_n}P\{|X_{nk}|>\varepsilon\}<\infty$,

(ii) *there exists $J\geqslant 2$ such that*

$$\sum_{n=1}^{\infty}c_n\left(\sum_{k=1}^{k_n}EX_{nk}^2 I\{|X_{nk}|\leqslant\delta\}\right)^J<\infty,$$

(iii) $\sum_{k=1}^{k_n}EX_{nk}I\{|X_{nk}|\leqslant\delta\}\to 0$ *as* $n\to\infty$.

Then $\sum_{n=1}^{\infty}c_nP\{|\sum_{k=1}^{k_n}X_{nk}|>\varepsilon\}<\infty$ for all $\varepsilon>0$.

Remarks. (1) It is not difficult to show that condition (i) of the theorem is also necessary for complete convergence.

(2) A special case is when all variables have mean zero and conditions (i) and (ii′) there exist $J\geqslant 2$ and $1\leqslant q\leqslant 2$ such that $\sum_{n=1}^{\infty}(\sum_{k=1}^{k_n}E|X_{nk}|^q)^J<\infty$ are satisfied. Then $\sum_{n=1}^{\infty}c_nP\{|S_n|>\varepsilon\}<\infty$ for all $\varepsilon>0$.

In view of the theorem, it is enough to show that conditions (ii) and (iii) are satisfied.

For (ii) note that

$$\left(\sum_{k=1}^{k_n} EX_{nk}^2 I\{|X_{nk}|\leqslant\delta\}\right)^J \leqslant \delta^{J(2-q)}\left(\sum_{k=1}^{k_n} E|X_{nk}|^q\right)^J.$$

For (iii) since $EX_{nk}=0$ it follows that

$$\left|\sum_{k=1}^{k_n} EX_{nk} I\{|X_{nk}|\leqslant\delta\}\right| = \left|\sum_{k=1}^{k_n} EX_{nk} I\{|X_{nk}|>\delta\}\right| \leqslant \delta^{1-q}\sum_{k=1}^{k_n} E|X_{nk}|^q \to 0$$

as $n\to\infty$, which proves (iii).

Proof of Theorem Note at the outset that in the case $k_n=\infty$ the series $\sum_{k=1}^{k_n} X_{nk}$ converges a.s. by the three series theorem (Loève, 1977, p. 249).

For $\delta>0$ define $Y_{nk}=X_{nk}I\{|X_{nk}|\leqslant\delta\}$ and let

$$S_n' = \sum_{k=1}^{k_n} Y_{nk} \quad\text{and}\quad S_n'' = \sum_{k=1}^{k_n} X_{nk}I\{|X_{nk}|>\delta\}.$$

Before estimating, we note that our assumptions imply that $S_n\to 0$ in probability as $n\to\infty$, since the degenerate convergence criterion holds (Loève, 1977, p. 329).

Next, we observe that

$$|P\{|S_n'|\geqslant\varepsilon\} - P\{|S_n|\geqslant\varepsilon\}| \leqslant \sum_{k=1}^{k_n} P\{|X_{nk}|>\delta\}\to 0 \quad \text{as } n\to\infty$$

by (i) and, hence, $S_n'\to 0$ in probability as $n\to\infty$.

Let $j=\max\{[\log_2 J]+1, \min\{j\colon \varepsilon/2\cdot 3^j<\delta\}\}$, where [ ] is the integer part function. Then by the lemma and the Hoffmann-Jørgensen inequality:

$$\begin{aligned} P\{|S_n'|\geqslant\varepsilon\} &\leqslant 2P\{|S_n'^s|\geqslant\varepsilon/2\} \\ &\leqslant 2C_j\sum_{k=1}^{k_n} P\{|X_{nk}^s|\geqslant(\varepsilon/2)\cdot 3^j\} + 2D_j(P\{|S_n'^s|\geqslant(\varepsilon/2)\cdot 3^j\})^J \\ &\leqslant 4C_j\sum_{k=1}^{k_n} P\{|X_{nk}|\geqslant(\varepsilon/4)\cdot 3^j\} + 2D_j(P\{|S_n'^s|\geqslant(\varepsilon/2)\cdot 3^j\})^J, \end{aligned}$$

for n large.

Hence, it suffices to estimate $(P\{|S_n'^s|\geqslant\varepsilon\})^J$. Next by Markov's inequality:

$$P\{|S_n'^s|\geqslant\varepsilon\}\leqslant E(S_n'^s)^2/\varepsilon^2 = \sum_{k=1}^{k_n} E(Y_{nk}^s)^2/\varepsilon^2 \leqslant \frac{2}{\varepsilon^2}\sum_{k=1}^{k_n} EY_{nk}^2.$$

By (ii) we obtain the result. □

4. Corollaries

Recall that the array $\{(X_{nk},\ 1\leqslant k\leqslant k_n),\ n\geqslant 1\}$ of random variables is said to be

(1) *Stochastically dominated in the Cesàro sense* by a random variable X if $k_n<\infty$ and there exists a constant $D>0$ such that $\sum_{k=1}^{k_n} P\{|X_{nk}|>x\}\leqslant Dk_nP\{|X|>x\}$ for all $x>0$ and all n.

(2) *Stochastically dominated* by a random variable X if there exists a constant $D>0$ such that

$$P\{|X_{nk}|>x\}\leqslant DP\{|X|>x\}$$

for all $x>0$ and for all k and n.

As an application of the theorem we can obtain the main result of Hu et al. (1989) (cf. also Gut, 1992, Theorem 2.1). Futhermore, we can obtain the rate of convergence as follows.

Corollary 1. *Let $\{(X_{nk},\ 1\leqslant k\leqslant n),\ n\geqslant 1\}$ be an array of rowwise independent mean zero random variables which are stochastically dominated in the Cesàro sense by a random variable X, $r>1$ and $1\leqslant p<2$. If $E|X|^{rp}<\infty$ then $\sum_{n=1}^{\infty} n^{r-2}P\{|\sum_{k=1}^{n} X_{nk}|>\varepsilon n^{1/p}\}<\infty$ for any $\varepsilon>0$.*

Proof. Take $c_n=n^{r-2}, k_n=n$ and $X_{nk}/n^{1/p}$ instead of X_{nk} and check that the conditions (i) and (ii′) hold. □

Remark. The choice $r=2$ in Corollary 1 gives the main result of Hu et al. (1989).

And now we present a modified version of a result of Rohatgi (1971). Recall that a double array $\{a_{nk};\ k, n\geqslant 1\}$ of real numbers is said to be a *Toeplitz sequence* if $\lim_{n\to\infty} a_{nk}=0$ for each k and $\sum_k |a_{nk}|\leqslant C$ for each n, where C is a positive constant.

Corollary 2. *Let $\{X_{nk};\ k,n\geqslant 1\}$ be an array of rowwise independent mean zero random variables, stochastically dominated by a random variable X, and let $\{a_{nk};\ k,n\geqslant 1\}$ be a Toeplitz sequence. If*

I. $\max_k |a_{nk}|=O(n^{-r})$ *for some* $r>0$

II. $E|X|^{1+1/r}<\infty$,

then $\sum_k a_{nk}X_{nk}\to 0$ completely as $n\to\infty$.

Proof. Let $c_n=1$ and $r\geqslant 1$.

(i) $\sum_n\sum_k P\{|a_{nk}X_{nk}|>\varepsilon\}\leqslant \sum_n\sum_k P\{|a_{nk}X|>\varepsilon\}<\infty$ by Lemma 1 of Rohatgi (1971).

(ii′) Let $q=1+1/r$. Then

$$\sum_n\left(\sum_k |a_{nk}|^{1+1/r}E|X_{nk}|^{1+1/r}\right)^J\leqslant \sum_n\left(\max_k |a_{nk}|^{1/r}\sum_k |a_{nk}|E|X|^{1+1/r}\right)^J$$
$$\leqslant \sum_n (O(n^{-1}))^J<\infty\quad \text{if } J>1.$$

In the case $r<1$ the statement of Corollary 2 is trivial with $q=2$ and $Jr>1$. □

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