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A mean convergence theorem and weak law for arrays of random elements in martingale type p Banach spaces

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Abstract

For weighted sums of the form $S_n = \sum_{j=1}^{k_n} a_{nj}(V_{nj} - c_{nj})$ where $\{a_{nj}, 1 \leqslant j \leqslant k_n < \infty, n \geqslant 1\}$ are constants, $\{V_{nj}, 1 \leqslant j \leqslant k_n, n \geqslant 1\}$ are random elements in a real separable martingale type p Banach space, and $\{c_{nj}, 1 \leqslant j \leqslant k_n, n \geqslant 1\}$ are suitable conditional expectations, a mean convergence theorem and a general weak law of large numbers are established. These results take the form $\|S_n\| \xrightarrow{\mathscr{L}_r} 0$ and $S_n \xrightarrow{P} 0$, respectively. No conditions are imposed on the joint distributions of the $\{V_{nj}, 1 \leqslant j \leqslant k_n, n \geqslant 1\}$. The mean convergence theorem is proved assuming that $\{\|V_{nj}\|^r, 1 \leqslant j \leqslant k_n, n \geqslant 1\}$ is $\{|a_{nj}|^r\}$-uniformly integrable whereas the weak law is proved under a Cesàro type condition which is weaker than Cesàro uniform integrability. The sharpness of the results is illustrated by an example. The current work extends that of Gut (1992) and Hong and Oh (1995).

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1. Introduction

Consider an array of constants $\{a_{nj}, 1 \leqslant j \leqslant k_n < \infty, n \geqslant 1\}$ and an array of random elements $\{V_{nj}, 1 \leqslant j \leqslant k_n, n \geqslant 1\}$ defined on a probability space $(\Omega, \mathscr{F}, P)$ and taking values in a real separable Banach space $\mathscr{X}$ with norm $\|\cdot\|$. Let $\{c_{nj}, 1 \leqslant j \leqslant k_n, n \geqslant 1\}$ be a "centering" array consisting of (suitably selected) conditional expectations. In this paper, a mean convergence theorem and a general weak law of large numbers (WLLN)

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will be established. These convergence results are, respectively, of the form

$$\left\| \sum_{j=1}^{k_n} a_{nj}(V_{nj} - c_{nj}) \right\| \xrightarrow{\mathscr{L}_r} 0 \tag{1.1}$$

and

$$\sum_{j=1}^{k_n} a_{nj}(V_{nj} - c_{nj}) \xrightarrow{P} 0. \tag{1.2}$$

Of course, (1.1) implies (1.2) by the Markov inequality. The expressions in (1.1) and (1.2) are referred to as *weighted sums* with *weights* $\{a_{nj}, 1 \leqslant j \leqslant k_n,\ n \geqslant 1\}$.

The hypotheses to the main results (Theorems 1 and 2) impose conditions on the growth behavior of the weights $\{a_{nj}, 1 \leqslant j \leqslant k_n,\ n \geqslant 1\}$ and on the marginal distributions of the random variables $\{\|V_{nj}\|, 1 \leqslant j \leqslant k_n,\ n \geqslant 1\}$. These two types of conditions are conjoined in Theorem 1 by a uniform integrability type condition formulated by Ordóñez Cabrera (1994). For the study of weak convergence of weighted sums of random variables, Ordóñez Cabrera (1994) was the first to recognize the advantage of imposing this type of condition which relates the distributions of the summands to their respective weights. The random elements in the array under our consideration are not assumed to be rowwise independent. Indeed, no conditions are imposed on the joint distributions of the random elements comprising the array. However, the Banach space $\mathscr{X}$ is assumed to be of martingale type p. (Technical definitions such as this will be discussed in Section 2.)

Theorems 1 and 2 are extensions to a martingale type p Banach space setting of results of Gut (1992) and Hong and Oh (1995), respectively, which were proved for arrays of (real-valued) random variables. Theorem 1 (which establishes the $\mathscr{L}_r$ convergence result (3.2) and hence the WLLN (3.3)) concerns arrays of random elements satisfying a uniform integrability type condition of Ordóñez Cabrera (1994) which is more general than the Cesàro uniform integrability condition employed by Gut (1992). In Theorem 2, the WLLN (3.8) is proved assuming a Cesàro type condition of Hong and Oh (1995) which is weaker than Cesàro uniform integrability. As will be apparent, the proofs of Theorems 1 and 2 owe much to those earlier articles. An example is provided to illustrate the sharpness of the results.

Finally, the symbol C denotes throughout a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

2. Preliminary definitions

Technical definitions relevant to the current work will be discussed in this section.

For a random element V and sub σ-algebra $\mathscr{G}$ of $\mathscr{F}$, the *conditional expectation* $E(V|\mathscr{G})$ was introduced by Scalora (1961) and is defined analogously to that in the random variable case. See Scalora (1961) for a complete development including Banach space valued martingales and martingale convergence theorems.

A real separable Banach space $\mathscr{X}$ is said to be of *martingale type p* ($1 \leqslant p \leqslant 2$) if there exists a finite constant C such that for all martingales $\{S_n,\ n \geqslant 1\}$ with values in $\mathscr{X}$,

$$\sup_{n \geqslant 1} E\|S_n\|^p \leqslant C \sum_{n=1}^{\infty} E\|S_n - S_{n-1}\|^p \tag{2.1}$$

where $S_0 \equiv 0$. It can be shown using classical methods from martingale theory (see Pisier, 1975, 1986) that if $\mathscr{X}$ is of martingale type p, then for all $1 \leqslant r < \infty$ there exists a finite constant C' such that for all $\mathscr{X}$-valued martingales $\{S_n,\ n \geq 1\}$

$$E \sup_{n\geq 1} \|S_n\|^r \leq C'E \left(\sum_{n=1}^{\infty} \|S_n - S_{n-1}\|^p \right)^{r/p}. \tag{2.2}$$

Clearly every real separable Banach space is of martingale type 1, while for $1 \leq p < \infty$ the $\mathscr{L}_p$-spaces and ℓ_p-spaces are of martingale type $p \wedge 2$ (see, e.g., Woyczyński (1975, Section 2, 1976, Section 4) or Schwartz (1981, Lectures 18 and 19)). It follows from the Hoffmann-Jørgensen and Pisier (1976) characterization of Rademacher type p Banach spaces that if a Banach space is of martingale type p, then it is of Rademacher type p. But the notion of martingale type p is only superficially similar to that of Rademacher type p and has a geometric characterization in terms of smoothness. For details, the reader may refer to the work of Pisier, Schwartz, and Woyczyński cited above.

An array of random variables $\{X_{nj}, 1 \leq j \leq k_n < \infty,\ n \geq 1\}$ is said to be *Cesàro uniformly integrable* if

$$\lim_{a\to\infty} \sup_{n\geq 1} \frac{1}{k_n} \sum_{j=1}^{k_n} E|X_{nj}|I(|X_{nj}| > a) = 0.$$

This condition was introduced by Chandra (1989). For random elements $\{V_{nj}, 1 \leq j \leq k_n < \infty,\ n \geq 1\}$, it is immediate that $\{\|V_{nj}\|^p, 1 \leq j \leq k_n < \infty,\ n \geq 1\}$ being Cesàro uniformly integrable for some $p > 0$ is a weaker condition than the $\{V_{nj}, 1 \leq j \leq k_n, n \geq 1\}$ being *stochastically dominated* by a random element V (i.e., for some finite constant D, $\sup_{n\geq 1,\ 1\leq j\leq k_n} P\{\|V_{nj}\| > t\} \leq DP\{\|DV\| > t\}$ for all $t \geq 0$) with $E\|V\|^p < \infty$. Relationships between Cesàro uniform integrability and the WLLN, the strong law of large numbers, and $\mathscr{L}_p$ convergence for random variables are studied in the papers by Chandra (1989), Chandra and Goswami (1992), and Bose and Chandra (1993), respectively.

The notion of Cesàro uniform integrability was extended by Ordóñez Cabrera (1994) who introduced the following condition. Let $\{a_{nj}, 1 \leq j \leq k_n < \infty,\ n \geq 1\}$ be an array of constants. An array of random variables $\{X_{nj}, 1 \leq j \leq k_n, n \geq 1\}$ is said to be $\{a_{nj}\}$*-uniformly integrable* if

$$\lim_{a\to\infty} \sup_{n\geq 1} \sum_{j=1}^{k_n} |a_{nj}|E|X_{nj}|I(|X_{nj}| > a) = 0.$$

Of course, $\{a_{nj}\}$-uniform integrability reduces to Cesàro uniform integrability when $a_{nj} = k_n^{-1}, 1 \leq j \leq k_n, n \geq 1$. Ordóñez Cabrera (1994) obtained $\mathscr{L}_r$ convergence results for weighted sums of random variables (and random elements) under an $\{|a_{nj}|^r\}$-uniform integrability condition. Theorem 1 establishes the $\mathscr{L}_r$ convergence result (3.2) for weighted sums $\sum_{j=1}^{k_n} a_{nj}V_{nj}$ of random elements in a martingale type p Banach space where $\{\|V_{nj}\|^r, 1 \leq j \leq k_n < \infty, n \geq 1\}$ is $\{|a_{nj}|^r\}$-uniformly integrable.

3. Mainstream

With these preliminaries accounted for, the first theorem may now be stated and proved.

Theorem 1. *Let* $1 \leq r \leq p \leq 2$ *and let* $\{V_{nj}, 1 \leq j \leq k_n < \infty,\ n \geq 1\}$ *be an array of random elements in a real separable, martingale type p Banach space and suppose that the array* $\{\|V_{nj}\|^r, 1 \leq j \leq k_n,\ n \geq 1\}$ *is* $\{|a_{nj}|^r\}$*-uniformly integrable where* $\{a_{nj},\ 1 \leq j \leq k_n,\ n \geq 1\}$ *is an array of constants satisfying*

$$\sum_{j=1}^{k_n} |a_{nj}|^p = o(1). \tag{3.1}$$

Then

$$\left\| \sum_{j=1}^{k_n} a_{nj}(V_{nj} - E(V_{nj}|\mathscr{F}_{n,j-1})) \right\| \xrightarrow{\mathscr{L}_r} 0 \tag{3.2}$$

and, a fortiori, the WLLN

$$\sum_{j=1}^{k_n} a_{nj}(V_{nj} - E(V_{nj}|\mathscr{F}_{n,j-1})) \xrightarrow{P} 0 \tag{3.3}$$

obtains where $\mathscr{F}_{nj} = \sigma(V_{ni},\ 1 \leqslant i \leqslant j),\ 1 \leqslant j \leqslant k_n,\ n \geqslant 1$ *and* $\mathscr{F}_{n0} = \{\emptyset, \Omega\},\ n \geqslant 1$.

Proof. Let $M > 0$ and set

$$W_{nj} = (V_{nj} - \mu_{nj})I(\|V_{nj} - \mu_{nj}\| > M), \quad 1 \leqslant j \leqslant k_n,\ \ n \geqslant 1,$$

where $\mu_{nj} = E(V_{nj}|\mathscr{F}_{n,j-1}), 1 \leqslant j \leqslant k_n, n \geqslant 1$. Then arguing as in Gut (1992)

$$\begin{aligned} E&\left\| \sum_{j=1}^{k_n} a_{nj}(V_{nj} - E(V_{nj}|\mathscr{F}_{n,j-1})) \right\|^r \\ &\leqslant C'E\left(\sum_{j=1}^{k_n} (|a_{nj}|\ \|V_{nj} - \mu_{nj}\|)^p \right)^{r/p} \quad \text{(by (2.2))} \\ &\leqslant CM^r \left(\sum_{j=1}^{k_n} |a_{nj}|^p \right)^{r/p} + C\sum_{j=1}^{k_n} |a_{nj}|^r E\|W_{nj}\|^r \quad (\text{since } r/p \leqslant 1). \end{aligned} \tag{3.4}$$

Next, the array $\{\|V_{nj} - \mu_{nj}\|^r, 1 \leqslant j \leqslant k_n,\ n \geqslant 1\}$ is $\{|a_{nj}|^r\}$-uniformly integrable by using the approach of Gut (1992). (The only modification needed in the argument is that Theorem 2 of Ordóñez Cabrera (1994) is used instead of Theorem 3 of Chandra (1989).) Consequently, recalling (3.1), the conclusion (3.2) follows from (3.4) by first letting $n \to \infty$ and then letting $M \to \infty$. □

Remarks 1. Theorem 1 should be compared with Theorem 6 of Ordóñez Cabrera (1994) which establishes an $\mathscr{L}_r$ convergence result ($0 < r < 1$) for weighted sums of random elements where there is not any martingale type p condition (or any other geometric condition) on the Banach space.

Corollary 1. *Let* $1 \leqslant r < p \leqslant 2$ *and let* $\{V_{nj}, 1 \leqslant j \leqslant k_n < \infty, n \geqslant 1, k_n \to \infty\}$ *be an array of random variables in a real separable Banach space which is of martingale type p and suppose that the array* $\{\|V_{nj}\|^r, 1 \leqslant j \leqslant k_n, n \geqslant 1\}$ *is Cesàro uniformly integrable. Let* $\{a_{nj}, 1 \leqslant j \leqslant k_n, n \geqslant 1\}$ *be an array of constants such that*

$$\max_{1 \leqslant j \leqslant k_n} |a_{nj}| = \mathcal{O}(k_n^{-1/r}). \tag{3.5}$$

Then (3.2) *and* (3.3) *obtain.*

Proof. Note that (3.5) and the Cesàro uniform integrability of the array $\{\|V_{nj}\|^r, 1 \leqslant j \leqslant k_n, n \geqslant 1\}$ ensure that this array is $\{|a_{nj}|^r\}$-uniformly integrable. Moreover, (3.1) is an immediate consequence of (3.5), $p > r$, and $k_n \to \infty$. Corollary 1 follows immediately from Theorem 1. □

The following example, which was inspired by another one due to Beck (1963) and Ordóñez Cabrera (1994), shows that Corollary 1 can fail if $p = r$. The example concerns the Banach space ℓ_r (where $1 \leqslant r \leqslant 2$) of absolute rth power summable real sequences $v = \{v_i, i \geqslant 1\}$ with norm $\|v\| = (\sum_{i=1}^{\infty} |v_i|^r)^{1/r}$. Let $v^{(n)}$ denote the element having 1 in its nth position and 0 elsewhere, $n \geqslant 1$.

Example 1. Let $1 \leqslant r \leqslant 2$ and consider the martingale type $p = r$ Banach space ℓ_r. Define a sequence $\{V_n, n \geqslant 1\}$ of independent random elements in ℓ_r by requiring the $\{V_n, n \geqslant 1\}$ to be independent with

$$P\{V_n = v^{(n)}\} = P\{V_n = -v^{(n)}\} = \tfrac{1}{2}, \quad n \geqslant 1.$$

Let $k_n = n$, $n \geqslant 1$ and $V_{nj} = V_j, 1 \leqslant j \leqslant n$, $n \geqslant 1$. Now for all $a \geqslant 1$, $\int_{[\|V_{nj}\|^r > a]} \|V_{nj}\|^r \, dP = 0, 1 \leqslant j \leqslant n$, $n \geqslant 1$ and so $\{\|V_{nj}\|^r, 1 \leqslant j \leqslant n, n \geqslant 1\}$ is Cesàro uniformly integrable. Let $a_{nj} = n^{-1/r}, 1 \leqslant j \leqslant n$, $n \geqslant 1$. Then (3.5) holds and

$$\left\| \sum_{j=1}^{n} a_{nj}(V_{nj} - E(V_{nj}|\mathscr{F}_{n,j-1})) \right\| = \left\| \sum_{j=1}^{n} a_{nj} V_{nj} \right\| = \frac{\|\sum_{j=1}^{n} V_j\|}{n^{1/r}} = 1 \text{ almost certainly}, \quad n \geqslant 1,$$

and so (3.2) and (3.3) both fail.

Remarks 2. Example 1 also shows that Theorem 1 can fail if (3.1) is weakened to $\sum_{j=1}^{k_n} |a_{nj}|^p = \mathcal{O}(1)$.

The second theorem imposes the uniform Cesàro type condition (3.6) which is readily seen via the Markov inequality to be weaker than the condition (as in Corollary 1) that $\{\|V_{nj}\|^r, 1 \leqslant j \leqslant k_n, n \geqslant 1\}$ is Cesàro uniformly integrable.

Theorem 2. *Let $\{V_{nj}, 1 \leqslant j \leqslant k_n < \infty, n \geqslant 1, k_n \to \infty\}$ be an array of random elements in a real separable, martingale type p $(1 \leqslant p \leqslant 2)$ Banach space. Suppose that the uniform Cesàro type condition*

$$\lim_{a \to \infty} \sup_{n \geqslant 1} \frac{1}{k_n} \sum_{j=1}^{k_n} aP\{\|V_{nj}\|^r > a\} = 0 \tag{3.6}$$

holds for some $0 < r < p$. Let $\{a_{nj}, 1 \leqslant j \leqslant k_n, n \geqslant 1\}$ be an array of constants such that

$$\max_{1 \leqslant j \leqslant k_n} |a_{nj}| = \mathcal{O}(k_n^{-1/r}). \tag{3.7}$$

Then the WLLN

$$\sum_{j=1}^{k_n} a_{nj}(V_{nj} - E(V'_{nj}|\mathscr{F}_{n,j-1})) \xrightarrow{P} 0 \tag{3.8}$$

obtains where $V'_{nj} = V_{nj} I(\|V_{nj}\|^r \leqslant k_n), \mathscr{F}_{nj} = \sigma(V_{ni}, 1 \leqslant i \leqslant j), 1 \leqslant j \leqslant k_n, n \geqslant 1$, and $\mathscr{F}_{n0} = \{\emptyset, \Omega\}, n \geqslant 1$.

Proof. For arbitrary $\varepsilon > 0$, note that for $n \geqslant 1$

$$P\left\{\left\|\sum_{j=1}^{k_n} a_{nj}V_{nj} - \sum_{j=1}^{k_n} a_{nj}V'_{nj}\right\| > \varepsilon\right\}$$

$$\leqslant P\left\{\bigcup_{j=1}^{k_n}[V_{nj} \neq V'_{nj}]\right\}$$

$$\leqslant \frac{1}{k_n}\sum_{j=1}^{k_n} k_n P\{\|V_{nj}\|^r > k_n\} = o(1) \quad \text{(by (3.6))}.$$

Thus

$$\sum_{j=1}^{k_n} a_{nj}V_{nj} - \sum_{j=1}^{k_n} a_{nj}V'_{nj} \xrightarrow{P} 0$$

and it suffices to show that

$$\sum_{j=1}^{k_n} a_{nj}V'_{nj} - \sum_{j=1}^{k_n} a_{nj}E(V'_{nj}|\mathscr{F}_{n,j-1}) \xrightarrow{P} 0.$$

This, in turn, will certainly hold provided it can be shown that

$$E\left\|\sum_{j=1}^{k_n} a_{nj}(V'_{nj} - E(V'_{nj}|\mathscr{F}_{n,j-1}))\right\|^p = o(1). \tag{3.9}$$

To this end, set $\phi_n = \max_{1 \leqslant j \leqslant k_n} |a_{nj}|, n \geqslant 1$. Then for all $n \geqslant 1$,

$$E\left\|\sum_{j=1}^{k_n} a_{nj}(V'_{nj} - E(V'_{nj}|\mathscr{F}_{n,j-1}))\right\|^p$$

$$\leqslant C\sum_{j=1}^{k_n} E(|a_{nj}|\ \|V'_{nj} - E(V'_{nj}|\mathscr{F}_{n,j-1})\|)^p \quad \text{(by (2.1))}$$

$$\leqslant C\phi_n^p 2^{p-1}\sum_{j=1}^{k_n}(E\|V'_{nj}\|^p + E(E(\|V'_{nj}\|\ |\mathscr{F}_{n,j-1}))^p)$$

$$\leqslant C\phi_n^p 2^p \sum_{j=1}^{k_n} E\|V'_{nj}\|^p$$ (by applying Jensen's inequality for conditional expectations (see, e.g., Chow and Teicher, 1988, p. 209) to the convex function $|\cdot|^p$)

$$\leqslant C\phi_n^p k_n + C\phi_n^p k_n \sum_{i=1}^{k_n-1}((i+1)^{p/r-1} - i^{p/r-1})\sup_{n\geqslant 1}\left(k_n^{-1}\sum_{j=1}^{k_n} iP\{\|V_{nj}\|^r > i\}\right) \tag{3.10}$$

by arguing as in Hong and Oh (1995). Now by (3.7),

$$\phi_n^p k_n = o(1) \quad (\text{since } p > r \text{ and } k_n \to \infty)$$

and

$$\phi_n^p k_n \sum_{i=1}^{k_n-1} \left((i+1)^{p/r-1} - i^{p/r-1}\right) = \mathcal{O}\left(\phi_n^p k_n^{p/r}\right) = \mathcal{O}(1).$$

Thus by (3.6) and the Toeplitz lemma (see, e.g., Loève, 1977, p. 250) the expression in (3.10) is o(1), thereby proving (3.9) and completing the proof of Theorem 2. □

Remarks 3. (i) Example 1 also shows that Theorem 2 can fail if $r = p$. Moreover, Example 1 (with r replaced by the symbol r') shows that Theorem 2 can fail if (3.6) holds for all $0 < r < p$ but (3.7) fails for all $0 < r < p$.

(ii) Consider the Banach space ℓ_r where $1 \leqslant r < 2$. Let $\{k_n,\ n \geqslant 1\}, \{V_{nj}, 1 \leqslant j \leqslant n,\ n \geqslant 1\}$, and $\{a_{nj}, 1 \leqslant j \leqslant n,\ n \geqslant 1\}$ be as in Example 1. Now for any $p \in (r, 2]$, observe for the random elements $\{V_n, n \geqslant 1\}$ considered in Example 1 that

$$E\left\|\sum_{j=1}^{n} V_j\right\|^p = n^{p/r}, \quad \sum_{j=1}^{n} E\|V_j\|^p = n, \qquad n \geqslant 1,$$

whence (see Hoffmann-Jørgensen and Pisier, 1976) ℓ_r is not of Rademacher type p and, consequently, ℓ_r is not of martingale type p. Now all of the hypotheses of Theorem 1, Corollary 1, and Theorem 2 are satisfied except for the Banach spaces being of martingale type p. But in view of Example 1, the conclusions of Theorem 1, Corollary 1, and Theorem 2 fail, thereby showing that the martingale type p hypotheses cannot be dispensed with.

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