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REGULAR ARTICLE

Moment inequalities for *m*-negatively associated random variables and their applications

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Abstract The moment inequalities for m-NA random variables, especially the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality are established and the Khintchine–Kolmogorov convergence theorem and the three series theorem for m-NA random variables are also obtained. As one application of the moment inequalities, we study the large deviation for least squares estimator in nonlinear regression models under some general conditions. As another application, we investigate the strong consistency for least squares estimator in multiple linear regression models based on m-NA random variables.

Keywords m-Negatively associated random variables · Marcinkiewicz–Zygmund type inequality · Rosenthal type inequality · Nonlinear regression models · Multiple linear regression models

Mathematics Subject Classification 60E05 · 60F15 · 62J02 · 62J05 · 62F12

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1 Introduction

It is well known that the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality play important roles in probability limit theory and mathematical statistics. There are many sequences of random variables satisfying the Marcinkiewicz–Zygmund type inequality or Rosenthal type inequality under some suitable conditions.

The main purpose of the paper is to establish the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality for a new dependent structure–m-negatively associated random variables (m-NA, in short). In addition, we will give some applications of Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality to the strong consistency for least squares estimator in multiple linear regression models and large deviation for least squares estimator in nonlinear regression models.

Firstly, let us recall the the definitions of negatively associated random variables and m-negatively associated random variables. The concept of negatively associated random variables is as follows.

Definition 1.1 A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA, in short) if for every pair of disjoint subsets A and B of $\{1, 2, \ldots, n\}$,

$$Cov(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leq 0$$

whenever f_1 and f_2 are coordinatewise increasing and the covariance exists.

An infinite family of random variables $\{X_n, n \geq 1\}$ is NA if every finite subfamily is NA.

The concept of NA random variables was introduced by Alam and Saxena (1981) (1981) and carefully studied by Joag-Dev and Proschan (1983). As pointed out and proved by Joag-Dev and Proschan (1983), a number of well-known multivariate distributions possess the NA property, such as multinomial, convolution of unlike multionmial, multivariate hypergeometric, Dirichlet, permutation distribution, negatively correlated normal distribution, random sampling without replacement and joint distribution of ranks. For more details about NA random variables, one can refer to Matula (1992), Shao (2000), Chen et al. (2008), Ling (2008), Liang and Zhang (2010), Sung (2011), Zarei and Jabbari (2011), Wang and Hu (2012, 2014), Wang et al. (2011, 2014a, b), and so on.

Inspired by the definition of NA random variables, Hu et al. (2009) introduced the concept of m-negatively associated random variables as follows.

Definition 1.2 Let $m \geq 1$ be a fixed integer. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be m-negatively associated (m-NA, in short) if for any $n \geq 2$ and any $i_1, i_2, \ldots, i_n$ such that $|i_k - i_j| \geq m$ for all $1 \leq k \neq j \leq n$, we have that $X_{i_1}, X_{i_2}, \ldots, X_{i_n}$ are NA.

An array $\{X_{ni}, i \geq 1, n \geq 1\}$ of random variables is said to be rowwise m-NA if for every $n \geq 1$, $\{X_{ni}, i \geq 1\}$ is a sequence of m-NA random variables.

When $m = 1$, the concept of m-NA random variables reduces to the so called NA random variables. Hence, the concept of m-NA random variables is a natural extension from NA random variables. On the other hand, it is well known that if for any $n \geq 2$ and any $i_1, i_2, \ldots, i_n$ such that $|i_k - i_j| \geq m$ for all $1 \leq k \neq j \leq n$, we have that $X_{i_1}, X_{i_2}, \ldots, X_{i_n}$ are independent, then we say that $\{X_n, n \geq 1\}$ are m-dependence. Hence, m-NA is weaker than m-dependence and NA. Studying the probability inequalities, moment inequalities, probability limiting behavior of m-NA random variables and their applications in many stochastic models are of great interest.

Recently, Hu et al. (2009) established the Kolmogorov exponential inequality for m-NA random variables. By using the Kolmogorov exponential inequality, they further investigated the complete convergence for arrays of rowwise m-NA random variables. However, the moment inequalities for m-NA random variables haven't been proved. The main purpose of the paper is to establish the moment inequalities for m-NA random variables. In addition, we will give some applications of the moment inequalities to some stochastic models, especially in multiple linear regression models and nonlinear regression models.

The following lemmas for NA random variables will be used to prove the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality for m-NA random variables.

Lemma 1.1 *Let random variables $X_1, X_2, \ldots, X_n$ be NA, $f_1, f_2, \ldots, f_n$ be all nondecreasing (or all nonincreasing) functions, then random variables $f_1(X_1), f_2(X_2), \ldots, f_n(X_n)$ are NA.*

Proof This follows by Property 6 of Joag-Dev and Proschan (1983) and the definition of NA random variables immediately. □

Lemma 1.2 (cf. Shao 2000) *Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for some $p \geq 1$ and every $n \geq 1$. Then for every $n \geq 1$,*

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \leq C_p \sum_{i=1}^{n} E|X_i|^p, \quad for \quad 1 \leq p \leq 2, \tag{1.1}$$

and

$$E\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right|^p\right) \leq D_p \left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}, \quad for \quad p > 2, \tag{1.2}$$

where $C_p = 2^{3-p}$ and $D_p = 2\left(\frac{15p}{\ln p}\right)^p$.

Throughout the paper, let c, c_1, c_2, C denote positive constants not depending on n, which may be different in various places. Let $C_1(p), C_2(p), \ldots$ be positive constants depending only on p, and $C(m, p), C_1(m, p), C_2(m, p), \ldots$ be positive constants depending only on m and p. $a_n = O(b_n)$ stands for $a_n \leq Cb_n$, where $\{a_n, n \geq 1\}$ and

$\{b_n, n \geq 1\}$ are sequences of nonnegative real numbers. Denote $\log x = \ln \max(x, e)$, $x^+ = xI(x > 0)$, $x^- = -xI(x < 0)$. $\lfloor x \rfloor$ denotes the integer part of x.

This work is organized as follows: the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality for m-NA random variables are provided in Sect. 2. The large deviation for least squares estimator in nonlinear regression models is provided in Sect. 3 and the strong consistency for least squares estimator in multiple linear regression models is established in Sect. 4, respectively.

2 Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality

In this section, we will establish the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality for m-NA random variables, which can be applied to establish the Khintchine–Kolmogorov convergence theorem and the three series theorem. In addition, there inequalities can be applied to prove the consistency and asymptotic normality in many stochastic models.

To prove the main results of the paper, we need the following lemma, which will be used frequently throughout the paper.

Lemma 2.1 *Let $\{X_n, n \geq 1\}$ be a sequence of m-NA random variables. If $\{f_n(\cdot), n \geq 1\}$ are all nondecreasing (or nonincreasing) functions, then random variables $\{f_n(X_n), n \geq 1\}$ are m-NA.*

This lemma can be obtained directly by the definition of m-NA random variables and Lemma 1.1. So the details are omitted.

The next one is the maximal type moment inequality for m-NA random variables, namely, Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality.

Theorem 2.1 *Let $\{X_n, n \geq 1\}$ be a sequence of m-NA random variables with $EX_n = 0$ and $E|X_n|^p < \infty$ for some $p \geq 1$ and every $n \geq 1$. Then for every $n \geq m$,*

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leq C_{m,p}\sum_{i=1}^{n} E|X_i|^p, \quad \text{for} \quad 1 \leq p \leq 2, \tag{2.1}$$

and

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \leq D_{m,p}\left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\}, \quad \text{for} \quad p > 2, \tag{2.2}$$

where $C_{m,p} = 4m^p$ and $D_{m,p} = 2^{p+1}m^p\left(\frac{15p}{\ln p}\right)^p$.

Proof For fixed $n \geq m$, let $r = \lfloor \frac{n}{m} \rfloor$. For $1 \leq k \leq n$, let $s = \lfloor \frac{k}{m} \rfloor$. Define

$$Y_i = \begin{cases} X_i, & 1 \leq i \leq k, \\ 0, & i > k. \end{cases}$$

Noting that $\sum_{i=1}^{k} X_i = \sum_{j=1}^{m} \sum_{i=0}^{s} Y_{mi+j}$, we have

$$\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_i\right| \le \sum_{j=1}^{m} \max_{0\le s\le r} \sum_{i=0}^{s} \left|Y_{mi+j}\right|.$$

It follows by the inequality above and C_r-inequality that

$$\begin{aligned}E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_i\right|^p\right) &\le E\left(\sum_{j=1}^{m} \max_{0\le s\le r} \sum_{i=0}^{s} \left|Y_{mi+j}\right|\right)^p \\ &\le m^{p-1}\sum_{j=1}^{m} E\left(\max_{0\le s\le r} \sum_{i=0}^{s} \left|Y_{mi+j}\right|\right)^p \\ &\le (2m)^{p-1}\sum_{j=1}^{m} E\left(\max_{0\le s\le r} \sum_{i=0}^{s} Y_{mi+j}^{+}\right)^p \\ &\quad + (2m)^{p-1}\sum_{j=1}^{m} E\left(\max_{0\le s\le r} \sum_{i=0}^{s} Y_{mi+j}^{-}\right)^p. \end{aligned} \tag{2.3}$$

By the definition of m-NA random variables, we can see that $Y_j, Y_{m+j}, \ldots, Y_{mr+j}$ are NA random variables for each $j = 1, 2, \ldots, m$. Hence, $Y_j^{+}, Y_{m+j}^{+}, \ldots, Y_{mr+j}^{+}$ and $Y_j^{-}, Y_{m+j}^{-}, \ldots, Y_{mr+j}^{-}$ are both NA random variables for each $j = 1, 2, \ldots, m$ by Lemma 1.1.

For $1 \le p \le 2$, we have by (1.1) and (2.3) that for any $n \ge m$,

$$\begin{aligned}E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_i\right|^p\right) &\le (2m)^{p-1}C_p\sum_{j=1}^{m}\sum_{i=0}^{r} E\left|Y_{mi+j}^{+}\right|^p \\ &\quad + (2m)^{p-1}C_p\sum_{j=1}^{m}\sum_{i=0}^{r} E\left|Y_{mi+j}^{-}\right|^p \\ &= (2m)^{p-1}C_p\sum_{j=1}^{m}\sum_{i=0}^{r} E\left|Y_{mi+j}\right|^p \\ &\le C_{m,p}\sum_{i=1}^{n} E\left|X_i\right|^p,\end{aligned}$$

which implies (2.1)

For $p > 2$, we have by (1.2) and (2.3) that for any $n \ge m$,

$$E\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} X_i\right|^p\right) \le (2m)^{p-1}D_p\sum_{j=1}^{m}\left[\sum_{i=0}^{r} E\left|Y_{mi+j}^{+}\right|^p + \left(\sum_{i=0}^{r} E\left|Y_{mi+j}^{+}\right|^2\right)^{p/2}\right]$$

$$+(2m)^{p-1}D_p\sum_{j=1}^{m}\left[\sum_{i=0}^{r}E\left|Y_{mi+j}^{-}\right|^p+\left(\sum_{i=0}^{r}E\left|Y_{mi+j}^{-}\right|^2\right)^{p/2}\right]$$

$$\leq 2^p m^{p-1}D_p\sum_{j=1}^{m}\left[\sum_{i=0}^{r}E\left|Y_{mi+j}\right|^p+\left(\sum_{i=0}^{r}EY_{mi+j}^2\right)^{p/2}\right]$$

$$\leq D_{m,p}\left[\sum_{i=1}^{n}E|X_i|^p+\left(\sum_{i=1}^{n}EX_i^2\right)^{p/2}\right], \tag{2.4}$$

which implies (2.2). This completes the proof of the theorem. □

By using Theorem 2.1, we can get the following Khintchine–Kolmogorov type convergence theorem and the three series theorem for m-NA random variables. The proofs are standard, so we omit them.

Corollary 2.1 (Khintchine–Kolmogorov type convergence theorem) *Let* $\{X_n, n \geq 1\}$ *be a sequence of m-NA random variables. Assume that*

$$\sum_{n=1}^{\infty} Var(X_n) < \infty,$$

then $\sum_{n=1}^{\infty}(X_n - EX_n)$ *converges a.s..*

Corollary 2.2 (Three series theorem) *Let* $\{X_n, n \geq 1\}$ *be a sequence of m-NA random variables. For some* $c > 0$*, let* $X_n^{(c)} = -cI(X_n < -c) + X_n I(|X_n| \leq c) + cI(X_n > c)$. *If*

$$\sum_{n=1}^{\infty} P(|X_n| > c) < \infty,$$

$$\sum_{n=1}^{\infty} EX_n^{(c)} \quad \text{converges},$$

$$\sum_{n=1}^{\infty} Var\left(X_n^{(c)}\right) < \infty,$$

then $\sum_{n=1}^{\infty} X_n$ *converges almost surely.*

Remark 2.1 Let $\{a_n, n \geq 1\}$ be a sequence of real numbers. Under the conditions of Theorem 2.1, we have for $n \geq m$ that

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k}a_iX_i\right|^p\right) \leq \begin{cases} 2^{p-1}C_{m,p}\sum_{i=1}^{n}E|a_iX_i|^p, & \text{for } 1 \leq p \leq 2,\\ 2^p D_{m,p}\left[\sum_{i=1}^{n}E|a_iX_i|^p+\left(\sum_{i=1}^{n}Ea_i^2X_i^2\right)^{p/2}\right], & \text{for } p > 2. \end{cases} \tag{2.5}$$

Actually, for fixed $n \geq m$, $\{a_i^+ X_i, 1 \leq i \leq n\}$ and $\{a_i^- X_i, 1 \leq i \leq n\}$ are both m-NA random variables from Lemma 2.1. Note that $a_{ni} = a_{ni}^+ - a_{ni}^-$, we have by C_r-inequality that

$$E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_i X_i\right|^p\right) \leq 2^{p-1} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_i^+ X_i\right|^p\right) + 2^{p-1} E\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_i^- X_i\right|^p\right). \tag{2.6}$$

Note that $|a_i|^p = \left(a_i^+\right)^p + \left(a_i^-\right)^p$, the desired result (2.5) follows by (2.1), (2.2) and (2.6) immediately.

3 Large deviation for least squares estimator in nonlinear regression models

In the previous section, we established the Marcinkiewicz–Zygmund type inequality and Rosenthal type inequality for m-NA random variables. As one application of the moment inequalities for m-NA random variables, we will study the large deviation for least squares estimator in nonlinear regression models under some general conditions.

3.1 Brief review

Nonlinear regression models are widely used for modeling of stochastic phenomena and the method of least squares plays a central role in the inference of parameters in nonlinear regression models. The study of asymptotic properties of the least squares estimator of parameters occurring in nonlinear regression models has been the subject of investigation since it is in general difficult to obtain the exact distribution of the least squares estimator for any fixed sample. For more details about the asymptotic properties of the least squares estimator for nonlinear regression models, one can refer to Jennrich (1969), Malinvaud (1970), Ivanov and Leonenko (1989), Ivanov (1997), and so on.

In this section, we investigate the large deviation results of the least squares estimator in the nonlinear regression model by using the moment inequalities that we established in Sect. 2. Consider the nonlinear regression model

$$X_n = g_n(\theta) + \xi_n, \quad n \geq 1, \tag{3.1}$$

where $\{X_n, n \geq 1\}$ is observed, $\{g_n(\theta), n \geq 1\}$ is a known sequence of continuous functions possibly nonlinear in $\theta \in \Theta$, a closed interval on the real line, and $\{\xi_n, n \geq 1\}$ is a sequence of random errors with mean zero. Let

$$Q_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} w_i^2[X_i - g_i(\theta)]^2,$$

where $\{w_i\}$ is a known sequence of positive numbers. An estimator θ_n is said to be a least squares estimator of θ if it minimizes $Q_n(\theta)$ over $\theta \in \Theta$, i.e.

$$Q_n(\theta_n) = \inf_{\theta\in\Theta} Q_n(\theta).$$

Note that $Q(x_1, \ldots, x_n; \theta) = Q_n(\theta)$ is defined on $\mathbb{R}^n \times \Theta$, where Θ is compact. Further $Q(\mathbf{x}; \theta)$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a Borel measurable function of $\mathbf{x}$ for any fixed $\theta \in \Theta$ and a continuous function of θ for any fixed $\mathbf{x} \in \mathbb{R}^n$. By Lemma 3.3 of Schmetterer (1974), there exists a Borel measurable map $\theta_n : \mathbb{R}^n \to \Theta$ such that

$$Q(\mathbf{x}; \theta_n(\mathbf{x})) = \inf_{\theta\in\Theta} Q_n(\theta).$$

In the following, we consider this measurable version as the least squares estimator θ_n.

Let θ_0 be the true parameter and suppose $\theta_0 \in$ interior of Θ. The following large deviation result is proved by Lemma 1 of Ivanov (1976), where the errors ξ_n are independent and identically distributed (i.i.d.) random variables and $w_i \equiv 1$.

Theorem A *Let $\{\xi_n, n \geq 1\}$ be i.i.d. random variables with $E|\xi_1|^p < \infty$ for some $p \geq 2$. Suppose that there exist some constants $0 < c_1 \leq c_2 < \infty$ such that*

$$c_1(\theta_1 - \theta_2)^2 \leq \frac{1}{n}\sum_{i=1}^{n}[g_i(\theta_1) - g_i(\theta_2)]^2 \leq c_2(\theta_1 - \theta_2)^2,$$

for all $\theta_1, \theta_2 \in \Theta$ and for all $n \geq 1$. Then for every $\rho > 0$ and for all $n \geq 1$, it has

$$P\left(n^{1/2}|\theta_n - \theta_0| > \rho\right) \leq c\rho^{-p}, \tag{3.2}$$

where c is a positive constant independent of n and ρ.

Prakasa Rao (1984) extended Theorem A for i.i.d. random variables to the case of φ-mixing and α-mixing random variables. Hu (2002) also obtained the result (3.2) for the martingale differences, the φ-mixing sequence and the NA sequence with $\sup_{n\geq 1} E|\xi_n|^p < \infty$ for some $p > 2$. Hu (2004) established the following large deviation

$$P\left(n^{1/2}|\theta_n - \theta_0| > \rho\right) \leq cn^{1-p/2}\rho^{-p} \tag{3.3}$$

for the martingale differences, the φ-mixing sequence, the NA sequence and the weakly stationary linear process with $\sup_{n\geq 1} E|\xi_n|^p < \infty$ for some $1 < p \leq 2$.

Recently, Yang and Hu (2014) obtained some large deviation results for the least squares estimator θ_n in a nonlinear regression model (3.1) under some general conditions.

The main purpose of this section is to establish the large deviation results for the least squares estimator θ_n in model (3.1) based on m-NA errors by using the moment inequality that we obtained in Sect. 2.

In this section, let $\{\xi_n, n \geq 1\}$ be a sequence of m-NA random variables. Denote

$$\Gamma_{p,n} = \sum_{i=1}^{n} E|\xi_i|^p, \quad \Delta_{p,n} = \left(\sum_{i=1}^{n} (E|\xi_i|^p)^{2/p} \right)^{p/2}, \quad n \geq 1.$$

3.2 Main results

Our main results in this section are as follows.

Theorem 3.1 *In model* (3.1)*, suppose that there exist positive constants* c_1, c_2, c_3, c_4 *such that*

$$c_1|\theta_1 - \theta_2| \leq |g_i(\theta_1) - g_i(\theta_2)| \leq c_2|\theta_1 - \theta_2|, \quad \textit{for all} \quad \theta_1, \theta_2 \in \Theta \quad \textit{and} \quad i \geq 1, \tag{3.4}$$

and

$$c_3 \leq w_i \leq c_4, \quad \textit{for all} \quad i \geq 1. \tag{3.5}$$

If $E|\xi_n|^p < \infty$*,* $n \geq 1$ *for some* $p > 2$*, then for all* $\rho > 0$ *and* $n \geq 1$*,*

$$P\left(n^{1/2}|\theta_n - \theta_0| > \rho\right) \leq \frac{C(m, p)(\Gamma_{p,n} + \Delta_{p,n})}{n^{p/2}} \rho^{-p}, \tag{3.6}$$

where $C(m, p)$ *is a positive constant depending only on* m *and* p*.*

Theorem 3.2 *In model* (3.1)*, suppose that there exist positive constants* c_1, c_2, c_3, c_4 *such that* (3.4) *and* (3.5) *hold. If* $E|\xi_n|^p < \infty$*,* $n \geq 1$ *for some* $p \in (1, 2]$*, then for all* $\rho > 0$ *and* $n \geq 1$*,*

$$P\left(n^{1/2}|\theta_n - \theta_0| > \rho\right) \leq \frac{C(m, p)\Gamma_{p,n}}{n^{p/2}} \rho^{-p}, \tag{3.7}$$

where $C(m, p)$ *is a positive constant depending only on* m *and* p*.*

As applications of Theorems 3.1 and 3.2, we can establish the complete consistency for the least squares estimator θ_n in a nonlinear regression model (3.1) by taking $\rho = n^{1/2}\varepsilon$, where $\varepsilon > 0$ is arbitrary.

Corollary 3.1 *Let the conditions in Theorem 3.1 hold. If* $\sup_{n\geq 1} E|\xi_n|^p < \infty$ *for some* $p > 2$*, then for all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P\left(|\theta_n - \theta_0| > \varepsilon\right) < \infty, \tag{3.8}$$

that is to say, $\theta_n \to \theta_0$ *completely as* $n \to \infty$.

Corollary 3.2 *Let the conditions in Theorem 3.2 hold. If* $E|\xi_n|^p = O(n^{-\delta})$ *for some* $\delta > 2 - p$ *and* $1 < p \leq 2$*, then for all* $\varepsilon > 0$*, (3.8) still holds.*

Remark 3.1 In our Theorems 3.1 and 3.2, the condition "$\sup_{n\geq 1} E|\xi_n|^p < \infty$" is not needed. If the condition "$\sup_{n\geq 1} E|\xi_n|^p < \infty$" is satisfied, then (3.6) and (3.7) imply (3.2) and (3.3), respectively. So our results of Theorems 3.1 and 3.2 generalize and improve the corresponding ones of Theorem 2.1 of Hu (2002) and Theorem 2.1 of Hu (2004), respectively.

3.3 Proofs

The proofs of Theorems 3.1 and 3.2 are similar to the corresponding ones of Yang and Hu (2014). For convenience of the reader, we will present the complete proof. In order to prove Theorems 3.1 and 3.2, we need the following useful lemma, which can be found in Hu (2004).

Lemma 3.1 (cf. Hu 2004) *Let* $(\Omega, \mathscr{F}, P)$ *be a probability space,* $[T_1, T_2]$ *be a closed interval on the real line. Assume that* $V(\theta) = V(\omega, \theta)$ $(\theta \in [T_1, T_2], \omega \in \Omega)$ *is a stochastic process such that* $V(\omega, \theta)$ *is continuous for all* $\omega \in \Omega$*. If there exist positive numbers* $\alpha > 0$*,* $r > 0$ *and* $C = C(T_1, T_2) < \infty$ *such that*

$$E|V(\theta_1) - V(\theta_2)|^r \leq C|\theta_1 - \theta_2|^{1+\alpha}, \quad \textit{for all} \quad \theta_1, \theta_2 \in [T_1, T_2],$$

then

$$P\left(\sup_{\theta_0 \leq \theta_1, \theta_2 \leq \theta_0 + \varepsilon} |V(\theta_1 - V(\theta_2)| \geq a\right) \leq \frac{8C}{(\alpha - \gamma + 2)(\alpha - \gamma + 3)} \left(\frac{8\gamma}{\gamma - 2}\right)^r \frac{\varepsilon^{\alpha+1}}{a^r}$$

for any $\varepsilon > 0$*,* $a > 0$*,* $\theta_0, \theta_0 + \varepsilon \in [T_1, T_2]$ *and* $\gamma \in (2, 2 + \alpha)$.

Proof of Theorem 3.1 For fixed $n \geq 1$, denote

$$\psi_n(\theta_1, \theta_2) = \frac{1}{n} \sum_{i=1}^{n} w_i^2 [g_i(\theta_1) - g_i(\theta_2)]^2,$$

$$V_n(\theta) = \frac{1}{n^{1/2}} \sum_{i=1}^{n} \xi_i [g_i(\theta) - g_i(\theta_0)], \quad U_n(\theta) = \frac{V_n(\theta)}{n^{1/2}\psi_n(\theta, \theta_0)}, \quad \theta \neq \theta_0.$$

Without loss of generality, we assume that $w_i = 1$ for all $i \geq 1$. The general case follows from similar arguments in view of (3.5). By (3.4), we can get that

$$c_1^2(\theta_1 - \theta_2)^2 \leq \psi_n(\theta_1, \theta_2) \leq c_2^2(\theta_1 - \theta_2)^2 \tag{3.9}$$

for all $\theta_1, \theta_2 \in \Theta$ and $n \geq 1$.

For any $\varepsilon > 0$, denote $A_{n\varepsilon} = \{|\theta_n - \theta_0| > \varepsilon\}$. For any $\omega \in A_{n\varepsilon}$, we can see that $\theta_n \neq \theta_0$, and thus

$$\begin{aligned}\sum_{i=1}^n \xi_i^2 = \sum_{i=1}^n [X_i - g_i(\theta_0)]^2 &\geq \sum_{i=1}^n [X_i - g_i(\theta_n)]^2 \\ &= \sum_{i=1}^n \xi_i^2 - 2nU_n(\theta_n)\psi_n(\theta_n, \theta_0) + n\psi_n(\theta_n, \theta_0),\end{aligned}$$

which implies that

$$\psi_n(\theta_n, \theta_0)(1 - 2U_n(\theta_n)) \leq 0. \tag{3.10}$$

Noting that $\psi_n(\theta_n, \theta_0) > 0$, we have by (3.10) that $U_n(\theta_n) \geq 1/2$. Hence

$$A_{n\varepsilon} = \{|\theta_n - \theta_0| > \varepsilon\} \subset \{U_n(\theta_n) \geq 1/2\},$$

which yields that for any $\varepsilon > 0$,

$$P(|\theta_n - \theta_0| > \varepsilon) \leq P\left(\sup_{|\theta - \theta_0| > \varepsilon} |U_n(\theta)| \geq 1/2\right). \tag{3.11}$$

If we take $\varepsilon = \rho n^{-1/2}$ in (3.11), where $\rho > 0$ is arbitrary, then we have

$$\begin{aligned}&P\left(n^{1/2}|\theta_n - \theta_0| > \rho\right) \\ &\leq P\left(\sup_{|\theta - \theta_0| > \rho n^{-1/2}} |U_n(\theta)| \geq 1/2\right) \\ &\leq P\left(\sup_{|\theta - \theta_0| > \rho} |U_n(\theta)| \geq 1/2\right) + P\left(\sup_{\rho n^{-1/2} < |\theta - \theta_0| \leq \rho} |U_n(\theta)| \geq 1/2\right). \end{aligned} \tag{3.12}$$

In view of (3.9), we have

$$P\left(\sup_{|\theta - \theta_0| > \rho} |U_n(\theta)| \geq 1/2\right) \leq P\left(\sup_{|\theta - \theta_0| > \rho} \frac{|V_n(\theta)|}{n^{1/2}\psi_n^{1/2}(\theta, \theta_0)} \geq \frac{1}{2}c_1\rho\right). \tag{3.13}$$

It follows from Cauchy's inequality that

$$\left(\frac{V_n(\theta)}{n^{1/2}\psi_n^{1/2}(\theta,\theta_0)}\right)^2=\left(\frac{1}{n}\sum_{i=1}^{n}\xi_i\left[\frac{g_i(\theta)-g_i(\theta_0)}{\psi_n^{1/2}(\theta,\theta_0)}\right]\right)^2\le\frac{1}{n}\sum_{i=1}^{n}\xi_i^2,\quad\forall\quad\theta\ne\theta_0. \tag{3.14}$$

Noting that $p/2>1$, we have by Minkowski's inequality, Markov's inequality and (3.13)–(3.14) that

$$\begin{aligned}P\left(\sup_{|\theta-\theta_0|>\rho}|U_n(\theta)|\ge 1/2\right)&\le\left(\frac{4}{nc_1^2\rho^2}\right)^{p/2}\left(\sum_{i=1}^{n}(E|\xi_i|^p)^{2/p}\right)^{p/2}\\&\doteq\frac{C_1(p)\Delta_{p,n}}{n^{p/2}}\rho^{-p}.\end{aligned} \tag{3.15}$$

For $m=0,1,2,\ldots,\lfloor n^{1/2}\rfloor$, denote

$$\theta(m)=\theta_0+\frac{\rho}{n^{1/2}}+\frac{m\rho}{\lfloor n^{1/2}\rfloor},\quad\rho_m=\theta(m)-\theta_0.$$

It follows from (3.9) again that

$$\begin{aligned}P\left(\sup_{\rho n^{-1/2}\le\theta-\theta_0\le\rho}|U_n(\theta)|\ge\frac{1}{2}\right)&\le\sum_{m=0}^{\lfloor n^{1/2}\rfloor-1}P\left(\sup_{\rho_m\le\theta-\theta_0\le\rho_{m+1}}|U_n(\theta)|\ge\frac{1}{2}\right)\\&\le\sum_{m=0}^{\lfloor n^{1/2}\rfloor-1}P\left(\sup_{\rho_m\le\theta-\theta_0\le\rho_{m+1}}|V_n(\theta)|\ge\frac{1}{2}c_1^2\rho_m^2n^{1/2}\right).\end{aligned} \tag{3.16}$$

Noting that

$$\sup_{\rho_m\le\theta-\theta_0\le\rho_{m+1}}|V_n(\theta)|\le|V_n(\theta(m))|+\sup_{\theta(m)\le\theta_1,\theta_2\le\theta(m+1)}|V_n(\theta_2)-V_n(\theta_1)|,$$

we have

$$\begin{aligned}&P\left(\sup_{\rho_m\le\theta-\theta_0\le\rho_{m+1}}|V_n(\theta)|\ge\frac{1}{2}c_1^2\rho_m^2n^{1/2}\right)\\&\le P\left(|V_n(\theta(m))|\ge\frac{1}{4}c_1^2\rho_m^2n^{1/2}\right)\\&\quad+P\left(\sup_{\theta(m)\le\theta_1,\theta_2\le\theta(m+1)}|V_n(\theta_2)-V_n(\theta_1)|\ge\frac{1}{4}c_1^2\rho_m^2n^{1/2}\right).\end{aligned} \tag{3.17}$$

By Markov's inequality, (3.4) and Remark 2.1, we can see that

$$
\begin{aligned}
&P\left(|V_n(\theta(m))| \geq \frac{1}{4}c_1^2\rho_m^2 n^{1/2}\right)\\
&\leq \left(\frac{4}{c_1^2\rho_m^2 n^{1/2}}\right)^p \frac{C_1(m,p)}{n^{p/2}}\sum_{i=1}^n E|\xi_i|^p|g_i(\theta(m))-g_i(\theta_0)|^p\\
&\quad+\left(\frac{4}{c_1^2\rho_m^2 n^{1/2}}\right)^p \frac{C_1(m,p)}{n^{p/2}}\left(\sum_{i=1}^n E\xi_i^2[g_i(\theta(m))-g_i(\theta_0)]^2\right)^{\frac{p}{2}}\\
&\leq \left(\frac{4}{c_1^2\rho_m^2 n^{1/2}}\right)^p \frac{C_2(m,p)}{n^{p/2}}|\theta(m)-\theta_0|^p\left\{\sum_{i=1}^n E|\xi_i|^p+\left(\sum_{i=1}^n(E|\xi_i|^p)^{2/p}\right)^{p/2}\right\}\\
&\doteq \frac{C_3(m,p)(\Gamma_{p,n}+\Delta_{p,n})}{n^p}\rho_m^{-p}, \qquad (3.18)
\end{aligned}
$$

and for all $\theta_1, \theta_2 \in \Theta$,

$$
E|V_n(\theta_2)-V_n(\theta_1)|^p \leq \frac{C_4(m,p)}{n^{p/2}}(\Gamma_{p,n}+\Delta_{p,n})|\theta_2-\theta_1|^p \doteq C(n,m,p)|\theta_2-\theta_1|^p. \qquad (3.19)
$$

Applying Lemma 3.1 with $r = p = 1+\alpha$, $C = C(n,m,p)$, $\varepsilon = \rho/\lfloor n^{1/2}\rfloor$, $a = \frac{1}{4}c_1^2\rho_m^2 n^{1/2}$ and $\gamma \in (2, p+1)$, we can get that

$$
\begin{aligned}
&P\left(\sup_{\theta(m)\leq\theta_1,\theta_2\leq\theta(m+1)}|V_n(\theta_2)-V_n(\theta_1)| \geq \frac{1}{4}c_1^2\rho_m^2 n^{1/2}\right)\\
&= P\left(\sup_{\theta(m)\leq\theta_1,\theta_2\leq\theta(m)+\rho/\lfloor n^{1/2}\rfloor}|V_n(\theta_2)-V_n(\theta_1)| \geq \frac{1}{4}c_1^2\rho_m^2 n^{1/2}\right)\\
&\leq \frac{C_5(m,p)(\Gamma_{p,n}+\Delta_{p,n})}{n^{3p/2}}\rho^p\rho_m^{-2p}. \qquad (3.20)
\end{aligned}
$$

Noting that $\rho_0 = \rho n^{-1/2}$, $\rho_m > m\rho n^{-1/2}$ and $p > 2$, we have by (3.16), (3.17), (3.18) and (3.20) that,

$$
P\left(\sup_{\rho n^{-1/2}\leq\theta-\theta_0\leq\rho}|U_n(\theta)| \geq \frac{1}{2}\right) \leq \frac{C_6(m,p)(\Gamma_{p,n}+\Delta_{p,n})}{n^{p/2}}\rho^{-p}. \qquad (3.21)
$$

Similarly,

$$
P\left(\sup_{\rho n^{-1/2}\leq\theta_0-\theta\leq\rho}|U_n(\theta)| \geq \frac{1}{2}\right) \leq \frac{C_7(m,p)(\Gamma_{p,n}+\Delta_{p,n})}{n^{p/2}}\rho^{-p}. \qquad (3.22)
$$

The desired result (3.6) follows from (3.12), (3.15), (3.21) and (3.22) immediately. This completes the proof of the theorem. □

Proof of Theorem 3.2 The proof is similar to that of Theorem 3.1. Noting that $1 < p \leq 2$, we have by (3.4), (3.9) and the C_r inequality that

$$\begin{aligned}\left|\frac{V_n(\theta)}{n^{1/2}\psi_n^{1/2}(\theta,\theta_0)}\right|^p &\leq n^{-p}n^{p-1}\sum_{i=1}^n|\xi_i|^p\frac{|g_i(\theta)-g_i(\theta_0)|^p}{\psi_n^{p/2}(\theta,\theta_0)}\\ &\leq \frac{C_2(p)}{n}\sum_{i=1}^n|\xi_i|^p, \quad \forall \quad \theta\neq\theta_0. \end{aligned} \tag{3.23}$$

Combining (3.13) and (3.23), we can see that

$$P\left(\sup_{|\theta-\theta_0|>\rho}|U_n(\theta)|\geq 1/2\right)\leq\left(\frac{2}{c_1\rho}\right)^p\frac{C_2(p)}{n}\sum_{i=1}^n E|\xi_i|^p \doteq \frac{C_3(p)\Gamma_{p,n}}{n}\rho^{-p}. \tag{3.24}$$

Similarly to the proofs of (3.18) and (3.19), we have by Markov's inequality that

$$P\left(|V_n(\theta(m))|\geq\frac{1}{4}c_1^2\rho_m^2n^{1/2}\right)\leq\frac{C_8(m,p)\Gamma_{p,n}}{n^p}\rho_m^{-p}, \tag{3.25}$$

and

$$E|V_n(\theta_2)-V_n(\theta_1)|^p\leq\frac{C_9(m,p)\Gamma_{p,n}}{n^{p/2}}|\theta_2-\theta_1|^p \doteq C(n,m,p)|\theta_2-\theta_1|^p$$

for all $\theta_1,\theta_2\in\Theta$ and $n\geq 1$.

Applying Lemma 3.1 with $r=p=1+\alpha$, $C=C(n,m,p)$, $\varepsilon=\rho/\lfloor n^{1/2}\rfloor$, $a=\frac{1}{4}c_1^2\rho_m^2n^{1/2}$ and $\gamma\in(2,p+1)$ again, and similarly to the proof of (3.20), we can get that

$$P\left(\sup_{\theta(m)\leq\theta_1,\theta_2\leq\theta(m+1)}|V_n(\theta_2)-V_n(\theta_1)|\geq\frac{1}{4}c_1^2\rho_m^2n^{1/2}\right)\leq\frac{C_{10}(m,p)\Gamma_{p,n}}{n^{3p/2}}\rho^p\rho_m^{-2p}. \tag{3.26}$$

Similarly to the proof of (3.21), we have by (3.25) and (3.26) that

$$P\left(\sup_{\rho n^{-1/2}\leq\theta-\theta_0\leq\rho}|U_n(\theta)|\geq\frac{1}{2}\right)\leq\frac{C_{11}(m,p)\Gamma_{p,n}}{n^{p/2}}\rho^{-p}. \tag{3.27}$$

Similarly,

$$P\left(\sup_{\rho n^{-1/2} \le \theta_0 - \theta \le \rho} |U_n(\theta)| \ge \frac{1}{2} \right) \le \frac{C_{12}(m, p)\Gamma_{p,n}}{n^{p/2}} \rho^{-p}. \tag{3.28}$$

The desired result (3.7) follows from (3.12), (3.24), (3.27) and (3.28) immediately. This completes the proof of the theorem. □

Proof of Corollary 3.1 The condition $\sup_{n\ge 1} E|\xi_n|^p < \infty$ implies that $\Gamma_{p,n} + \Delta_{p,n} \le Cn^{p/2}$. For any $\varepsilon > 0$, taking $\rho = n^{1/2}\varepsilon$ in Theorem 3.1, we have by (3.6) and $p > 2$ that

$$\sum_{n=1}^{\infty} P\left(|\theta_n - \theta_0| > \varepsilon\right) \le C(m, p) \sum_{n=1}^{\infty} \frac{1}{n^{p/2}} < \infty,$$

which implies (3.8). The proof is completed. □

Proof of Corollary 3.2 The condition $E|\xi_n|^p = O(n^{-\delta})$ implies that $\Gamma_{p,n} \le Cn^{1-\delta}$. For any $\varepsilon > 0$, taking $\rho = n^{1/2}\varepsilon$ in Theorem 3.1, we have by (3.7) and $\delta > 2 - p$ that

$$\sum_{n=1}^{\infty} P\left(|\theta_n - \theta_0| > \varepsilon\right) \le C(m, p) \sum_{n=1}^{\infty} \frac{1}{n^{\delta+p-1}} < \infty,$$

which implies (3.8). The proof is completed. □

4 The strong consistency for least squares estimator in multiple linear regression models

As another application of the moment inequalities for m-NA random variables, we will study the strong consistency for least squares estimator in multiple linear regression models based on m-NA random variables.

Consider the following multiple regression model

$$y_i = \beta_1 x_{i1} + \cdots + \beta_p x_{ip} + e_i, \quad i = 1, 2, \ldots, \tag{4.1}$$

where $x_{ij}(j = 1, 2, \ldots, p;\ i = 1, 2, \ldots)$ are known constants, $\beta_1, \ldots, \beta_p$ are unknown parameters, $y_1, y_2, \ldots$ are observable random variables, $e_1, e_2, \ldots$ are unobservable random variables. Throughout the sequel we shall let X_n denote the design matrix $(x_{ij})_{1\le i\le n, 1\le j\le p}$, and let

$$Y_n = (y_1, \ldots, y_n)', \quad \beta = (\beta_1, \ldots, \beta_p)'.$$

For $n \geq p$, the least squares estimate $b_n = (b_{n1}, \ldots, b_{np})'$ of the vector β based on the design matrix X_n and the response vector Y_n, is given by

$$b_n = (X_n' X_n)^{-1} X_n' Y_n \doteq \left(b_{n1}, b_{n2}, \ldots, b_{np}\right)', \tag{4.2}$$

provided that $X_n' X_n$ is nonsingular.

The following concept of convergence system is needed.

Let $\{\varepsilon_n, n \geq 1\}$ be a sequence of random variables. If $\{\varepsilon_n, n \geq 1\}$ satisfies the following condition, then we say $\{\varepsilon_n, n \geq 1\}$ is a convergence system: for any sequence $\{a_n, n \geq 1\}$ of real numbers, if $\sum_{n=1}^{\infty} a_n^2 < \infty$, then $\sum_{n=1}^{\infty} a_n \varepsilon_n$ converges $a.s.$

Based on the convergence system, Chen et al. (1981) obtained the following general result.

Theorem 4.1 *In model* (4.1)*, denote*

$$V_n = \left(v_{ij}^{(n)}\right)_{1 \leq i, j \leq p} = (X_n' X_n)^{-1}.$$

Assume that the following two conditions are satisfied:

(i) for the sequence $\{e_n, n \geq 1\}$ *of random errors,*

$$\{g_n e_n, n \geq 1\} \quad \textit{is a convergence system}, \tag{4.3}$$

where $\{g_n, n \geq 1\}$ *is a given sequence of real numbers such that the sequence* $\{|g_n|, n \geq 1\}$ *is positive and non-increasing;*

(ii) for fixed $j = 1, 2, \ldots, p$,

$$\lim_{n \to \infty} v_{jj}^{(n)} = 0. \tag{4.4}$$

Then we have

$$b_{nj} - \beta_j = O\left(\left(f\left(v_{jj}^{(n)}\right)\right)^{1/2} \cdot |g_n|^{-1}\right) \quad a.s., \quad j = 1, 2, \ldots, p, \tag{4.5}$$

where f *is a positive function on* $(0, \infty)$ *such that*

$$\int_0^A \frac{dt}{f(t)} < \infty \quad \textit{for some} \quad A > 0, \quad \textit{and} \quad \frac{f(t)}{t^2} \uparrow \infty \quad \textit{as} \quad t \downarrow 0. \tag{4.6}$$

By using Corollary 2.1 and Theorem 4.1, we can get the following strong consistency for least squares estimator in multiple linear regression models based on m-NA random variables.

Theorem 4.2 *Let* $\{e_n, n \geq 1\}$ *be a sequence of* m*-NA random variables with* $Ee_n = 0$ *and* $Ee_n^2 \leq \sigma^2 < \infty$ *for* $n \geq 1$. *Assume further that* (4.4) *holds. Then for any* $\delta > 1$,

$$b_{nj} - \beta_j = O\left(\left(v_{jj}^{(n)} \cdot \left|\log v_{jj}^{(n)}\right|^{\delta}\right)^{1/2}\right) \quad a.s., \quad j = 1, 2, \ldots, p. \tag{4.7}$$

Proof Firstly, we will show that $\{e_n, n \geq 1\}$ is a convergence system. Let $\{a_n, n \geq 1\}$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n^2 < \infty$. Note that for fixed $t = 0, 1, 2, \ldots, m-1$, $\{e_{t+mk}, k = 1, 2, \ldots\}$ is a sequence of NA random variables and

$$\sum_{k=1}^{\infty} Var(a_{t+mk}e_{t+mk}) = \sum_{k=1}^{\infty} E(a_{t+mk}e_{t+mk})^2 \leq \sigma^2 \sum_{n=1}^{\infty} a_n^2 < \infty. \tag{4.8}$$

Hence, we have by (4.8) and Corollary 2.1 that $\sum_{k=1}^{\infty} a_{t+mk}e_{t+mk}$ converges $a.s.$, and thus,

$$\sum_{i=m}^{\infty} a_i e_i = \sum_{t=0}^{m-1} \sum_{k=1}^{\infty} a_{t+mk}e_{t+mk} \quad \text{converges} \quad a.s.$$

Therefore, $\{e_n, n \geq 1\}$ is a convergence system.

Taking $g_n \equiv 1$ and $f(t) = t|\log t|^{\delta}$ for $\delta > 1$ and $0 < t < e^{-1}$, we can see that the conditions of Theorem 4.1 are satisfied. Hence, the desired result (4.7) follows from (4.5) immediately. □

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