ON THE STRONG LAW OF LARGE NUMBERS FOR SEQUENCES OF PAIRWISE NEGATIVE QUADRANT DEPENDENT RANDOM VARIABLES

BY

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Abstract

For a sequence of pairwise negative quadrant dependent random variables $\{X_n,\ n \geq 1\}$, conditions are given under which normed and centered partial sums converge to 0 almost certainly. As special cases, new results are obtained for weighted sums $\{\sum_{j=1}^{n} a_j X_j,\ n \geq 1\}$ where $\{a_n,\ n \geq 1\}$ is a sequence of positive constants and the $\{X_n,\ n \geq 1\}$ are also identically distributed. A result of Matuła [19] is obtained by taking $a_n \equiv 1$. Moreover, it is shown that a pairwise negative quadrant dependent sequence (which is not a sequence of independent random variables) can be constructed having any specified continuous marginal distributions. Illustrative examples are provided, two of which show that the pairwise negative quadrant dependence assumption cannot be dispensed with.

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1. Introduction. A sequence of random variables $\{X_n,\ n \geq 1\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ is said to obey the general *strong law of large numbers* (SLLN) with centering constants $\{c_n,\ n \geq 1\}$ and positive norming constants $\{B_n,\ n \geq 1\}$ if

$$\frac{\sum_{j=1}^{n}(X_j - c_j)}{B_n} \longrightarrow 0 \text{ almost certainly (a.c.).} \tag{1.1}$$

In the current work, the main result (Theorem 3.1) establishes a SLLN for a sequence of pairwise negative quadrant dependent random variables. This concept of dependence was introduced by Lehmann [13] and will be defined below.

In many stochastic models, an independence assumption among the random variables in the model is not a reasonable assumption since they may be "repelling" in the sense that increases in any of the random variables often correspond to decreases in the others. Thus the assumption of pairwise negative quadrant dependence is often more suitable than the classical assumption of independence.

Definition 1.1. A sequence of random variables $\{X_n,\ n \geq 1\}$ is said to be *pairwise negative quadrant dependent* (PNQD) if for all $i, j \geq 1$ $(i \neq j)$ and all $x, y \in \mathcal{R}$,

$$P\{X_i \leq x,\ X_j \leq y\} \leq P\{X_i \leq x\} P\{X_j \leq y\}.$$

It is well known and easy to prove that $\{X_n,\ n \geq 1\}$ is PNQD if and only if for all $i, j \geq 1$ $(i \neq j)$ and all $x, y \in \mathcal{R}$,

$$P\{X_i > x,\ X_j > y\} \leq P\{X_i > x\} P\{X_j > y\}.$$

It is of course immediate that if $\{X_n,\ n \geq 1\}$ is a sequence of pairwise independent (*a fortiori*, independent) random variables, then $\{X_n,\ n \geq 1\}$ is PNQD.

The concept of a finite set of random variables being PNQD can be defined in a manner completely analogous to the definition provided by Definition 1.1.

Matuła [19] proved for a sequence of PNQD identically distributed (i.d.) random variables $\{X_n,\ n \geq 1\}$ with $E|X_1| < \infty$ that

$$\frac{\sum_{j=1}^{n}(X_j - EX_j)}{n} \longrightarrow 0 \text{ a.c.} \tag{1.2}$$

This result extends of course both the classical Kolmogorov SLLN (for sequences of independent i.d. (i.i.d.) random variables) and the more recent Etemadi [6] SLLN (for sequences of pairwise i.i.d. random variables) to sequences of PNQD i.d. random variables. Matuła's [19] SLLN will follow immediately from Theorem 4.1(ii) below.

As will be made transparent by Remark 3.1(i), the SLLN (3.3) provided by Theorem 3.1(i) is a PNQD version of a result of Heyde [9] which dealt with independent summands. Heyde's [9] theorem asserts for independent $\{X_n,\ n \geq 1\}$ and constants $0 < B_n \uparrow \infty$ that the condition

$$\sum_{n=1}^{\infty} E\left(\frac{X_n^2}{X_n^2 + B_n^2}\right) < \infty \tag{1.3}$$

is sufficient for the SLLN (1.1) with centering constants $c_n = E(X_n I(|X_n| \le B_n))$, $n \ge 1$. Heyde's work is related to previous work of Szynal [24]. See also Lévy [14]. As was noted by Heyde [9], it is possible to avoid the presence of centering constants involving truncated expectations only by imposing additional conditions. While the proof of Heyde's [9] theorem is based on the "Khintchine-Kolmogorov convergence theorem, Kronecker lemma approach", the proof of Theorem 3.1 is based on the "method of subsequences" and, more specifically, on the general approach developed by Etemadi [6], [7], and [8].

Besides the work of Matuła [19] discussed above, there is an interesting literature of investigation on the SLLN problem for partial sums of PNQD sequences or row sums of PNQD rows of an array; see Bozorgnia, Patterson, and Taylor [5], Matuła [20], Patterson and Taylor [21], Amini, Azarnoosh, and Bozorgnia [3], Kim and Baek [11], Liang and Su [16], Liu, Gan, and Chen [17], Amini and Bozorgnia [4], Liang [15], Kim and Kim [12], and Taylor, Patterson, and Bozorgnia [25]. In fact, most of the above papers assumed that the underlying random variables obey a dependence structure stronger than PNQD. More specifically, Matuła [20], Liang and Su [16], Liu, Gan, and Chen [17], and Liang [15] considered *negatively associated* random variables and Bozorgnia, Patterson, and Taylor [5], Patterson and Taylor [21], Amini, Azarnoosh, and Bozorgnia [3], Amini and Bozorgnia [4], and Taylor, Patterson, and Bozorgnia [25] considered *negatively dependent* random variables. These two stronger dependence notions will not be discussed in this paper. Moreover, Bozorgnia, Patterson, and Taylor [5], Patterson and Taylor [21], Liang and Su [16], Amini and Bozorgnia [4], Liang [15], and Taylor, Patterson, and Bozorgnia [25] obtained complete convergence results for normed and centered sums and these results immediately yield the corresponding SLLNs.

A major survey article concerning a general "theory of negative dependence" was prepared by Pemantle [22]. That article discussed the relationship between various definitions of "negative dependence", outlines some possible directions that the theory can take, and provides some interesting conjectures.

The plan of this paper is as follows. Some needed technical results will be presented in Section 2 including a result showing that a PNQD sequence (which is not a sequence of independent random variables) can be constructed having any specified continuous marginal distributions. Theorem 3.1 and its two corollaries will be established in Section 3. The special case of weighted sums of PNQD i.d. random variables will be considered in Section 4 and four illustrative examples will be presented in Section 5; the last two will show that in each part of Theorem 3.1 the PNQD hypothesis cannot be dispensed with.

2. Preliminaries.

Some technical results concerning PNQD sequences will be presented in this section. The ensuing lemma follows from Lemma 1 of Lehmann [13]; a more direct proof of it was provided by Matuła [19].

Lemma 2.1. (Lehmann [13], Matuła [19]). Let $\{X_n,\ n \ge 1\}$ be a sequence of PNQD random variables and for each $n \ge 1$, let $f_n : \mathcal{R} \longrightarrow \mathcal{R}$. If the sequence $\{f_n,\ n \ge 1\}$ consists of only nondecreasing functions or only nonincreasing functions, then $\{f_n(X_n),\ n \ge 1\}$ is a sequence of

PNQD random variables.

The next lemma is well known (see, e.g., Patterson and Taylor [21]) but we are not able to track down its origin. In any event, its proof is immediate from Lemmas 1 and 3 of Lehmann [13].

Lemma 2.2. Let $\{X_n,\ n \geq 1\}$ be a sequence of PNQD integrable random variables. Then

$$\mathrm{Var}\left(\sum_{j=1}^{n} X_j\right) \leq \sum_{j=1}^{n} \mathrm{Var} X_j, \quad n \geq 1.$$

The last lemma is a special case of Lemma 2 of Adler and Rosalsky [1].

Lemma 2.3. (Adler and Rosalsky [1]). Let X be a random variable and let $\{c_n,\ n \geq 1\}$ be a sequence of positive constants such that

$$\left(\max_{1 \leq j \leq n} c_j^2\right) \sum_{j=n}^{\infty} c_j^{-2} = \mathcal{O}(n) \quad \text{and} \quad \sum_{n=1}^{\infty} P\left\{|X| > c_n\right\} < \infty.$$

Then

$$\sum_{n=1}^{\infty} c_n^{-2} E\left(X^2 I(|X| \leq c_n)\right) < \infty.$$

A collection of n PNQD random variables arrises by sampling without replacement from a set of n real numbers (see, e.g., Bozorgnia, Patterson, and Taylor [5]). The following proposition shows that a sequence of PNQD random variables can be constructed having any specified marginal distributions.

Proposition 2.1. Let $\{F_n,\ n \geq 1\}$ be a sequence of (right-continuous) distribution functions. Then there exists a PNQD sequence of random variables $\{X_n,\ n \geq 1\}$ such that the distribution function of X_n is $F_n,\ n \geq 1$. Moreover, if the $\{F_n,\ n \geq 1\}$ are all continuous, then the PNQD sequence $\{X_n,\ n \geq 1\}$ can be chosen so that for all $k \geq 1$, $\{X_n,\ n \geq k\}$ is not a sequence of independent random variables.

Proof. Let $\{Z_n,\ n \geq 1\}$ be a sequence of i.i.d. $N(0,1)$ random variables, and consider the sequence $\{Z_n - Z_{n+1},\ n \geq 1\}$ of identically distributed $N(0,2)$ random variables. Let H be the $N(0,2)$ distribution function. For $n \geq 1$, let

$$F_n^{-1}(t) = \inf\{x \in [-\infty, \infty] :\ F_n(x) \geq t\}, \quad 0 \leq t \leq 1.$$

Observe that for all $n \geq 1,\ t \in [0,1]$, and $x \in [-\infty, \infty]$,

$$t \leq F_n(x) \ \text{ if and only if } \ F_n^{-1}(t) \leq x. \tag{2.1}$$

Set

$$X_n = F_n^{-1}\left(H(Z_n - Z_{n+1})\right), \quad n \geq 1.$$

Since

$$\mathrm{Cov}\left(Z_n - Z_{n+1}, Z_{n+1} - Z_{n+2}\right) = -1, \quad n \geq 1, \tag{2.2}$$

$\{Z_n - Z_{n+1},\ n \geq 1\}$ is a sequence of PNQD random variables as was shown by Joag-Dev and Prochan [10], and so it follows from Lemma 2.1 that $\{X_n,\ n \geq 1\}$ is also a sequence of PNQD random variables since the sequence of composite functions $\{F_n^{-1} \circ H,\ n \geq 1\}$ is a sequence of nondecreasing functions. Now for all $n \geq 1$, the distribution function of X_n is

$$\begin{aligned} P\{X_n \leq x\} &= P\left\{F_n^{-1}(H(Z_n - Z_{n+1})) \leq x\right\} \\ &= P\{H(Z_n - Z_{n+1}) \leq F_n(x)\} \quad \text{(by (2.1))} \\ &= P\left\{Z_n - Z_{n+1} \leq H^{-1}(F_n(x))\right\} \\ &= H\left(H^{-1}(F_n(x))\right) \\ &= F_n(x), \quad -\infty \leq x \leq \infty \end{aligned}$$

proving that the distribution function of X_n is $F_n,\ n \geq 1$.

Next, suppose that the $\{F_n,\ n \geq 1\}$ are all continuous. Then for all $n \geq 1$,

$$F_n\left(F_n^{-1}(t)\right) = t, \quad t \in [0, 1]. \tag{2.3}$$

Now if X_n and X_{n+1} are independent for some $n \geq 1$, then

$$H^{-1}(F_n(X_n)) \quad \text{and} \quad H^{-1}(F_{n+1}(X_{n+1}))$$

are also independent. Note that

$$\begin{aligned} H^{-1}(F_n(X_n)) &= H^{-1}\left(F_n(F_n^{-1}(H(Z_n - Z_{n+1})))\right) \\ &= H^{-1}(H(Z_n - Z_{n+1})) \quad \text{(by (2.3))} \\ &= Z_n - Z_{n+1} \end{aligned}$$

and, similarly,

$$H^{-1}(F_{n+1}(X_{n+1})) = Z_{n+1} - Z_{n+2}.$$

Thus in view of (2.2), X_n and X_{n+1} cannot be independent. Consequently, for all $k \geq 1$, $\{X_n,\ n \geq k\}$ is not a sequence of independent random variables. □

Finally, the symbol C denotes a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

3. Mainstream. With the preliminaries accounted for, the main result may be established.

Theorem 3.1. Let $\{X_n,\ n \geq 1\}$ be a sequence of PNQD random variables and let $\{B_n,\ n \geq 1\}$ be a sequence of positive constants. Suppose that

$$\sum_{n=1}^{\infty} B_n^{-2} \int_0^{B_n} x P\{|X_n| > x\}\, dx < \infty. \tag{3.1}$$

(i) If

$$\frac{B_n}{n} \uparrow, \quad B_{n+1} \sim B_n, \quad \text{and} \quad \frac{n}{B_n} E(|X_n| I(|X_n| \leq B_n)) = \mathcal{O}(1), \tag{3.2}$$

then the SLLN

$$\frac{\sum_{j=1}^{n}\left(X_j - E(X_j I(|X_j| \le B_j))\right)}{B_n} \longrightarrow 0 \ \text{ a.c.} \tag{3.3}$$

obtains.

(ii) If

$$B_n \uparrow \infty, \quad B_{n+1} = \mathcal{O}(B_n), \quad \text{and} \quad \frac{\sum_{j=1}^{n} E\left(|X_j| I(|X_j| \le B_j)\right)}{B_n} \longrightarrow 0, \tag{3.4}$$

then the SLLN

$$\frac{\sum_{j=1}^{n} X_j}{B_n} \longrightarrow 0 \ \text{ a.c.} \tag{3.5}$$

obtains.

Proof. Note at the outset that for any random variable X and constant $B > 0$, integration by parts yields

$$2\int_0^B xP\{|X| > x\}dx = E\left(X^2 I(|X| \le B)\right) + B^2 P\{|X| > B\}$$

whence (3.1) is equivalent to the pair of conditions

$$\sum_{n=1}^{\infty} \frac{E\left(X_n^2 I(|X_n| \le B_n)\right)}{B_n^2} < \infty \tag{3.6}$$

and

$$\sum_{n=1}^{\infty} P\left\{|X_n| > B_n\right\} < \infty. \tag{3.7}$$

Set for $n \ge 1$,

$$Y_n = X_n I(|X_n| \le B_n) + B_n I(X_n > B_n) - B_n I(X_n < -B_n),$$

$$Y_n^{(+)} = X_n^+ I(X_n \le B_n) + B_n I(X_n > B_n),$$

$$Y_n^{(-)} = X_n^- I(X_n \ge -B_n) + B_n I(X_n < -B_n).$$

Then $Y_n = Y_n^{(+)} - Y_n^{(-)}, \ n \ge 1$. Define

$$S_n = \sum_{j=1}^{n} Y_j, \quad S_n^{(+)} = \sum_{j=1}^{n} Y_j^{(+)}, \quad S_n^{(-)} = \sum_{j=1}^{n} Y_j^{(-)}, \quad n \ge 1.$$

It follows from Lemma 2.1 that $\{Y_n^{(+)}, \ n \ge 1\}$ and $\{Y_n^{(-)}, \ n \ge 1\}$ are sequences of PNQD random variables.

Let $b > 1$ be arbitrary. For each $k \ge 1$, let $N(k)$ be the smallest integer $n \ge 1$ for which $B_n \ge b^k$ and for each $n \ge 1$, let $K(n)$ be the smallest integer $k \ge 1$ for which $N(k) \ge n$. Now under (3.2), for all k such that $N(k) \ge 2$,

$$1 \le \frac{B_{N(k)}}{b^k} < \frac{B_{N(k)}}{B_{N(k)-1}} \longrightarrow 1 \tag{3.8}$$

and hence under (3.2)

$$\frac{N(k+1)}{N(k)} \le \frac{B_{N(k+1)}}{B_{N(k)}} = (1 + o(1))\,\frac{b^{k+1}}{b^k} = (1 + o(1))\,b. \tag{3.9}$$

On the other hand, under (3.4), for all k such that $N(k) \geq 2$,

$$1 \leq \frac{B_{N(k)}}{b^k} < \frac{B_{N(k)}}{B_{N(k)-1}} = \mathcal{O}(1) \tag{3.10}$$

implying

$$\frac{B_{N(k+1)}}{B_{N(k)}} \leq \frac{Cb^{k+1}}{b^k} = Cb. \tag{3.11}$$

In view of (3.8), (3.10), (3.9), and (3.11), both

$$1 \leq \frac{B_{N(k)}}{b^k} \leq C, \quad k \geq 1 \tag{3.12}$$

and

$$\frac{B_{N(k+1)}}{B_{N(k)}} \leq C, \quad k \geq 1 \tag{3.13}$$

prevail under either (3.2) or (3.4).

Now for arbitrary $\varepsilon > 0$, using Chebyshev's inequality, the PNQD property of the sequence $\{Y_n^{(+)},\ n \geq 1\}$, Lemma 2.2, and (3.12)

$$\begin{aligned}
&\sum_{k=1}^{\infty} P\left\{\left|\frac{S_{N(k)}^{(+)} - ES_{N(k)}^{(+)}}{B_{N(k)}}\right| > \varepsilon\right\} \\
&\leq C \sum_{k=1}^{\infty} \frac{\mathrm{Var} S_{N(k)}^{(+)}}{B_{N(k)}^2} \\
&\leq C \sum_{k=1}^{\infty} \frac{1}{B_{N(k)}^2} \sum_{n=1}^{N(k)} \mathrm{Var} Y_n^{(+)} \\
&\leq C \sum_{k=1}^{\infty} \frac{1}{B_{N(k)}^2} \sum_{n=1}^{N(k)} E(Y_n^{(+)})^2 \\
&= C \sum_{k=1}^{\infty} \frac{1}{B_{N(k)}^2} \sum_{n=1}^{N(k)} \left(E\left(X_n^+ I(X_n \leq B_n)\right)^2 + B_n^2 P\{X_n > B_n\}\right) \\
&= C \sum_{n=1}^{\infty} \left(E\left(X_n^+ I(X_n \leq B_n)\right)^2 + B_n^2 P\{X_n > B_n\}\right) \sum_{k=K(n)}^{\infty} \frac{1}{B_{N(k)}^2} \\
&\leq C \sum_{n=1}^{\infty} \left(E\left(X_n^+ I(X_n \leq B_n)\right)^2 + B_n^2 P\{X_n > B_n\}\right) \sum_{k=K(n)}^{\infty} \frac{1}{b^{2k}} \\
&\leq C \sum_{n=1}^{\infty} \frac{E\left(X_n^+ I(X_n \leq B_n)\right)^2 + B_n^2 P\{X_n > B_n\}}{b^{2K(n)}} \\
&\leq C \sum_{n=1}^{\infty} \frac{E\left(X_n^+ I(X_n \leq B_n)\right)^2 + B_n^2 P\{X_n > B_n\}}{B_{N(K(n))}^2} \\
&\leq C \sum_{n=1}^{\infty} \frac{E\left(X_n^+ I(X_n \leq B_n)\right)^2 + B_n^2 P\{X_n > B_n\}}{B_n^2} \\
&< \infty \quad \text{(by (3.6) and (3.7))}.
\end{aligned}$$

Hence by the Borel-Cantelli lemma and the arbitrariness of ε,

$$\frac{S_{N(k)}^{(+)} - ES_{N(k)}^{(+)}}{B_{N(k)}} \longrightarrow 0 \ \text{ a.c.} \tag{3.14}$$

Next, if n is such that $K(n) \geq 2$, then $N(K(n)-1) < n \leq N(K(n))$ and noting that $S_n^{(+)}$ is a sum of nonnegative random variables and recalling (3.13),

$$\begin{aligned} &\frac{S_n^{(+)} - ES_n^{(+)}}{B_n} \\ &\leq \frac{S_{N(K(n))}^{(+)} - ES_{N(K(n))}^{(+)}}{B_n} + \frac{ES_{N(K(n))}^{(+)} - ES_{N(K(n)-1)}^{(+)}}{B_n} \\ &\leq \frac{B_{N(K(n))}\left|S_{N(K(n))}^{(+)} - ES_{N(K(n))}^{(+)}\right|}{B_{N(K(n)-1)}B_{N(K(n))}} + \frac{\sum_{j=N(K(n)-1)+1}^{N(K(n))} EY_j^{(+)}}{B_{N(K(n)-1)}} \\ &\leq \frac{C\left|S_{N(K(n))}^{(+)} - ES_{N(K(n))}^{(+)}\right|}{B_{N(K(n))}} + \frac{\sum_{j=N(K(n)-1)+1}^{N(K(n))} EY_j^{(+)}}{B_{N(K(n)-1)}} \end{aligned} \tag{3.15}$$

and, similarly,

$$\begin{aligned} &\frac{S_n^{(+)} - ES_n^{(+)}}{B_n} \\ &\geq -\frac{\left|S_{N(K(n)-1)}^{(+)} - ES_{N(K(n)-1)}^{(+)}\right|}{B_{N(K(n)-1)}} - \frac{\sum_{j=N(K(n)-1)+1}^{N(K(n))} EY_j^{(+)}}{B_{N(K(n)-1)}}. \end{aligned} \tag{3.16}$$

Note that

$$\begin{aligned} &\frac{\sum_{j=N(K(n)-1)+1}^{N(K(n))} B_j P\{X_j > B_j\}}{B_{N(K(n)-1)}} \\ &\leq \frac{B_{N(K(n))}}{B_{N(K(n)-1)}} \sum_{j=N(K(n)-1)+1}^{N(K(n))} P\{X_j > B_j\} \\ &\leq C \sum_{j=N(K(n)-1)+1}^{\infty} P\{X_j > B_j\} \quad \text{(by (3.13))} \\ &= o(1) \quad \text{(by (3.7))}. \end{aligned} \tag{3.17}$$

Now (3.2) ensures that

$$\frac{n}{B_n} \max_{1 \leq j \leq n} E\left(X_j^+ I(X_j \leq B_j)\right) \leq \max_{1 \leq j \leq n} \frac{j}{B_j} E\left(X_j^+ I(X_j \leq B_j)\right) = \mathcal{O}(1). \tag{3.18}$$

Thus under (3.2),

$$\begin{aligned} &\frac{\sum_{j=N(K(n)-1)+1}^{N(K(n))} EY_j^{(+)}}{B_{N(K(n)-1)}} \\ &\leq \frac{N(K(n)-1)\left(\max_{1\leq j\leq N(K(n))} E\left(X_j^+ I(X_j \leq B_j)\right)\right)(N(K(n)) - N(K(n)-1))}{B_{N(K(n)-1)}N(K(n)-1)} \\ &\qquad + \frac{\sum_{j=N(K(n)-1)+1}^{N(K(n))} B_j P\{X_j > B_j\}}{B_{N(K(n)-1)}} \\ &\leq \frac{CN(K(n))\left(\max_{1\leq j\leq N(K(n))} E\left(X_j^+ I(X_j \leq B_j)\right)\right)(N(K(n)) - N(K(n)-1))}{B_{N(K(n))}N(K(n)-1)} \\ &\qquad + o(1) \qquad \text{(by (3.13) and (3.17))} \\ &\leq C\left(\frac{N(K(n))}{N(K(n)-1)} - 1\right) + o(1) \qquad \text{(by (3.18))} \\ &\leq C\left((1+o(1))b - 1\right) + o(1) \qquad \text{(by (3.9))}. \end{aligned} \tag{3.19}$$

Thus under (3.2), it follows from (3.15), (3.16), (3.14), and (3.19) that

$$-C(b-1) \le \liminf_{n\to\infty} \frac{S_n^{(+)} - ES_n^{(+)}}{B_n} \le \limsup_{n\to\infty} \frac{S_n^{(+)} - ES_n^{(+)}}{B_n} \le C(b-1) \quad \text{a.c.}$$

Then since $b > 1$ is arbitrary,

$$\lim_{n\to\infty} \frac{S_n^{(+)} - ES_n^{(+)}}{B_n} = 0 \quad \text{a.c.} \tag{3.20}$$

On the other hand, under (3.4)

$$\begin{aligned}
&\frac{\sum_{j=N(K(n)-1)+1}^{N(K(n))} EY_j^{(+)}}{B_{N(K(n)-1)}} \\
&\le \frac{B_{N(K(n))} \sum_{j=1}^{N(K(n))} E\left(X_j^+ I(X_j \le B_j)\right)}{B_{N(K(n)-1)} B_{N(K(n))}} \\
&\quad + \frac{\sum_{j=N(K(n)-1)+1}^{N(K(n))} B_j P\left\{X_j > B_j\right\}}{B_{N(K(n)-1)}} \\
&\le C o(1) + o(1) \qquad \text{(by (3.13), (3.4), and (3.17))} \\
&= o(1).
\end{aligned} \tag{3.21}$$

Thus under (3.4), it follows from (3.15), (3.16), (3.14), and (3.21) that

$$0 \le \liminf_{n\to\infty} \frac{S_n^{(+)} - ES_n^{(+)}}{B_n} \le \limsup_{n\to\infty} \frac{S_n^{(+)} - ES_n^{(+)}}{B_n} \le 0 \quad \text{a.c.;}$$

that is, (3.20) again prevails.

Similarly,

$$\lim_{n\to\infty} \frac{S_n^{(-)} - ES_n^{(-)}}{B_n} = 0 \quad \text{a.c.}$$

and so

$$\frac{S_n - ES_n}{B_n} = \frac{S_n^{(+)} - ES_n^{(+)}}{B_n} - \frac{S_n^{(-)} - ES_n^{(-)}}{B_n} \longrightarrow 0 \quad \text{a.c.} \tag{3.22}$$

Now by (3.7) and the Borel-Cantelli lemma,

$$P\left\{\liminf_{n\to\infty} [X_n = Y_n]\right\} = 1. \tag{3.23}$$

Note that for $n \ge 1$,

$$\begin{aligned}
&\frac{ES_n}{B_n} - \frac{\sum_{j=1}^n E\left(X_j I(|X_j| \le B_j)\right)}{B_n} \\
&= \frac{\sum_{j=1}^n B_j P\left\{X_j > B_j\right\}}{B_n} - \frac{\sum_{j=1}^n B_j P\left\{X_j < -B_j\right\}}{B_n} \\
&\longrightarrow 0
\end{aligned} \tag{3.24}$$

by (3.7) and the Kronecker lemma. It thus follows from (3.22), (3.23), and (3.24) that (3.3) holds thereby completing the proof of (i). Now the last condition in (3.4) ensures that

$$\frac{\sum_{j=1}^n E\left(X_j I\left(|X_j| \le B_j\right)\right)}{B_n} \longrightarrow 0$$

which when combined with (3.3) yields (3.5) and completes the proof of (ii). $\square$

Remarks 3.1. (i) The condition (3.1) is equivalent to the key condition (1.3) in Heyde's [9] SLLN discussed in Section 1. This equivalence follows from Heyde's [9] observation that (1.3) and the pair of conditions (3.6) and (3.7) are equivalent. As was noted in the proof of Theorem 3.1, (3.1) and this pair of conditions are equivalent. Consequently, Theorem 3.1(i) is a PNQD version of Heyde's [9] theorem.

(ii) A sufficient condition for (3.1) is that

$$\sum_{n=1}^{\infty} \frac{E|X_n|^{r_n}}{B_n^{r_n}} < \infty \tag{3.25}$$

for some sequence of constants $\{r_n,\ n \geq 1\} \subseteq (0, 2]$. To see this, note that for $n \geq 1$

$$\begin{aligned} B_n^{-2} \int_0^{B_n} xP\left\{|X_n| > x\right\} dx \quad &= B_n^{-2} \int_0^{B_n} x^{2-r_n} x^{r_n - 1} P\left\{|X_n| > x\right\} dx \\ &\leq B_n^{-2} B_n^{2-r_n} \int_0^{B_n} x^{r_n - 1} P\left\{|X_n| > x\right\} dx \\ &\leq B_n^{-r_n} E|X_n|^{r_n}. \end{aligned}$$

Thus (3.1) follows from (3.25).

(iii) It may be recalled that (3.25) with $r_n \equiv 2$ is the classical Kolmogorov condition for a sequence of independent mean 0 random variables $\{X_n,\ n \geq 1\}$ to obey the SLLN (1.1) with $c_n \equiv 0$ when $B_n \uparrow \infty$. But (3.1) can hold for a sequence of independent mean 0 random variables $\{X_n,\ n \geq 1\}$ when (3.25) with $r_n \equiv 2$ fails. To see this, simply let $\{X_n,\ n \geq 1\}$ be a sequence of i.i.d. mean 0 random variables with $EX_1^2 = \infty$ and let $B_n = n(\log(n+1))^{\lambda},\ n \geq 1$ where $\lambda > 1$. Then (3.25) with $r_n \equiv 2$ fails but (3.25) with $r_n \equiv 1$ holds. Thus (3.1) holds by Remark 3.1(ii).

Corollary 3.1. Let $\{X_n,\ n \geq 1\}$ be a sequence of PNQD random variables and let $\{B_n,\ n \geq 1\}$ be a sequence of positive constants. Suppose that (3.1) holds.

(i) If (3.2) holds and

$$E\left(X_n I(|X_n| \leq B_n)\right) = o(1), \tag{3.26}$$

then the SLLN

$$\frac{\sum_{j=1}^{n} X_j}{B_n} \longrightarrow 0 \ \text{ a.c.} \tag{3.27}$$

obtains.

(ii) If

$$\frac{B_n}{n} \uparrow \infty, \quad B_{n+1} = \mathcal{O}\left(B_n\right), \quad \text{and} \quad \frac{n}{B_n} E\left(|X_n| I(|X_n| \leq B_n)\right) = o(1), \tag{3.28}$$

then the SLLN (3.27) obtains.

(iii) If

$$B_n \uparrow, \quad n = \mathcal{O}\left(B_n\right), B_{n+1} = \mathcal{O}\left(B_n\right), \quad \text{and} \quad E\left(|X_n| I(|X_n| \leq B_n)\right) = o(1), \tag{3.29}$$

then the SLLN (3.27) obtains.

Proof. **(i)** By Theorem 3.1(i), (3.3) holds. Now by (3.26) and the Cesàro mean summability theorem

$$\frac{\sum_{j=1}^{n} E\left(X_j I(|X_j| \le B_j)\right)}{n} \longrightarrow 0. \tag{3.30}$$

Since $n = \mathcal{O}\left(B_n\right)$ by the first condition of (3.2), it follows from (3.30) that

$$\frac{\sum_{j=1}^{n} E\left(X_j I(|X_j| \le B_j)\right)}{B_n} \longrightarrow 0. \tag{3.31}$$

Combining (3.3) and (3.31) yields (3.27).

(ii) It follows from the first and the third condition of (3.28) that for $n > m \ge 1$

$$\begin{aligned}
&\frac{\sum_{j=1}^{n} E\left(|X_j| I(|X_j| \le B_j)\right)}{B_n} \\
&\le \frac{n \max_{1 \le j \le n} E\left(|X_j| I(|X_j| \le B_j)\right)}{B_n} \\
&\le \frac{n \max_{1 \le j \le m} E\left(|X_j| I(|X_j| \le B_j)\right)}{B_n} + \frac{n \max_{m+1 \le j \le n} E\left(|X_j| I(|X_j| \le B_j)\right)}{B_n} \\
&\le \frac{n \max_{1 \le j \le m} E\left(|X_j| I(|X_j| \le B_j)\right)}{B_n} + \max_{m+1 \le j \le n} \frac{jE\left(|X_j| I(|X_j| \le B_j)\right)}{B_j} \\
&\overset{n\to\infty}{\longrightarrow} 0 + \sup_{j \ge m+1} \frac{jE\left(|X_j| I(|X_j| \le B_j)\right)}{B_j} \\
&\overset{m\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Thus (3.4) holds and (3.27) follows from Theorem 3.1(ii).

(iii) By the second and fourth condition of (3.29) and arguing as in part (i),

$$\frac{\sum_{j=1}^{n} E\left(|X_j| I(|X_j| \le B_j)\right)}{B_n} \longrightarrow 0.$$

Thus (3.4) holds and (3.27) follows from Theorem 3.1(ii). □

The following corollary reduces to Theorem 2 of Etemadi [7] in the special case where the $\{X_n,\ n \ge 1\}$ are nonnegative and pairwise independent, $\sup_{n \ge 1} EX_n < \infty$, and $B_n \equiv n$.

Corollary 3.2. Let $\{X_n,\ n \ge 1\}$ be a sequence of PNQD integrable random variables. Suppose that (3.2), (3.6), (3.7), and

$$\frac{\sum_{j=1}^{n} E\left(X_j I(|X_j| > B_j)\right)}{B_n} \longrightarrow 0 \tag{3.32}$$

hold. Then the SLLN

$$\frac{\sum_{j=1}^{n} \left(X_j - EX_j\right)}{B_n} \longrightarrow 0 \ \text{ a.c.} \tag{3.33}$$

obtains.

Proof. Since (3.1) and the pair of conditions (3.6) and (3.7) are equivalent, Theorem 3.1(i) ensures (3.3) which when combined with (3.32) yields the conclusion (3.33). □

4. Weighted sums of PNQD i.d. random variables. As a consequence of Theorem 3.1, we will obtain in this section new SLLN results for partial sums of the form $\sum_{j=1}^{n} a_j X_j,\ n \ge 1$

where $\{X_n,\ n \geq 1\}$ is a sequence of PNQD i.d. random variables and $\{a_n,\ n \geq 1\}$ is a sequence of positive constants. The constants $a_n,\ n \geq 1$ and the partial sums $\sum_{j=1}^{n} a_j X_j,\ n \geq 1$ are referred to, respectively, as *weights* and *weighted sums*. By Lemma 2.1, $\{a_n X_n,\ n \geq 1\}$ is a sequence of PNQD random variables.

Kim and Kim [12] obtained a SLLN for weighted sums of PNQD i.d. random variables but their norming sequence $\{B_n,\ n \geq 1\}$ is very rapidly growing in that it satisfies the condition $\sum_{j=n}^{\infty} B_j^{-2} = \mathcal{O}\left(B_n^{-2}\right)$ and hence their assertion that their theorem extends Theorem 6 of Adler, Rosalsky, and Taylor [2] (from the independence case to the PNQD case) is incorrect.

Theorem 4.1. Let $\{X_n,\ n \geq 1\}$ be a sequence of PNQD i.d. random variables and let $\{a_n,\ n \geq 1\}$ and $\{B_n,\ n \geq 1\}$ be sequences of positive constants.

(i) Suppose

$$\frac{B_n}{n} \uparrow, \quad B_{n+1} \sim B_n, \tag{4.1}$$

$$\frac{B_n}{a_n} \uparrow, \quad \frac{B_n}{n a_n} \to \infty, \quad \frac{B_n}{n a_n} = \mathcal{O}\left(\inf_{j \geq n} \frac{B_j}{j a_j}\right), \tag{4.2}$$

$$\sum_{j=1}^{n} |a_j| = \mathcal{O}\left(n a_n\right), \tag{4.3}$$

$$\sum_{n=1}^{\infty} P\left\{|a_n X_1| > B_n\right\} < \infty, \tag{4.4}$$

and

$$\frac{n a_n}{B_n} E\left(|X_1| I\left(|X_1| \leq \frac{B_n}{a_n}\right)\right) = \mathcal{O}(1). \tag{4.5}$$

Then the SLLN

$$\frac{\sum_{j=1}^{n} a_j X_j}{B_n} \longrightarrow 0 \ \text{ a.c.} \tag{4.6}$$

obtains.

(ii) Suppose (4.1),

$$\left(\max_{1 \leq j \leq n} \left(\frac{B_j}{a_j}\right)^2\right) \sum_{j=n}^{\infty} \left(\frac{a_j}{B_j}\right)^2 = \mathcal{O}(n), \tag{4.7}$$

and

$$\frac{n a_n}{B_n} = \mathcal{O}(1), \quad \sum_{j=1}^{n} a_j = \mathcal{O}\left(B_n\right), \quad E X_1 = 0. \tag{4.8}$$

Then the SLLN (4.6) obtains.

(iii) Suppose (4.4), (4.7),

$$B_n \uparrow \infty, \quad B_{n+1} = \mathcal{O}\left(B_n\right), \tag{4.9}$$

and

$$\frac{\sum_{j=1}^{n} E\left(|a_j X_j| I\left(|X_j| \leq B_j / a_j\right)\right)}{B_n} \longrightarrow 0. \tag{4.10}$$

Then the SLLN (4.6) obtains.

(iv) Suppose (4.4), (4.7),

$$B_n \uparrow, \quad B_{n+1} = \mathcal{O}(B_n), \tag{4.11}$$

and

$$\sum_{j=1}^{n} a_j = o(B_n), \quad E|X_1| < \infty. \tag{4.12}$$

Then the SLLN (4.6) obtains.

Proof. **(i)** It follows from (4.2) that $B_n/a_n \uparrow \infty$ and (4.7) holds (see the proof of Theorem 6 of Adler, Rosalsky, and Taylor [2]). Then by (4.7), (4.4), and Lemma 2.3,

$$\sum_{n=1}^{\infty} \left(\frac{a_n}{B_n}\right)^2 E\left(X_1^2 I\left(|X_1| \le \frac{B_n}{a_n}\right)\right) < \infty. \tag{4.13}$$

Thus, in view of the equivalence between (3.1) and the pair of conditions (3.6) and (3.7), we have from (4.4) and (4.13) that

$$\sum_{n=1}^{\infty} B_n^{-2} \int_0^{B_n} xP\{|a_n X_n| > x\}\, dx < \infty. \tag{4.14}$$

Then by (4.14), (4.1), (4.5), and Theorem 3.1(i),

$$\frac{\sum_{j=1}^{n} \left(a_j X_j - E\left(a_j X_j I\left(|a_j X_j| \le B_j\right)\right)\right)}{B_n} \longrightarrow 0 \quad \text{a.c.} \tag{4.15}$$

But

$$\frac{\sum_{j=1}^{n} E\left(a_j X_j I\left(|a_j X_j| \le B_j\right)\right)}{B_n} \longrightarrow 0 \tag{4.16}$$

using (4.2), (4.3), and (4.4) and arguing exactly as in the proof of Theorem 6 of Adler, Rosalsky, and Taylor [2]. The conclusion (4.6) follows immediately from (4.15) and (4.16).

(ii) Note that (4.5) is automatic by (4.8). Now it also follows from (4.8) that $B_n/a_n \ge \delta n$ for some $\delta > 0$ and all $n \ge 1$ and since $E|X_1| < \infty$,

$$\sum_{n=1}^{\infty} P\{|a_n X_1| > B_n\} \le \sum_{n=1}^{\infty} P\{|X_1| > \delta n\} < \infty.$$

Then, arguing as in the proof of part (i), (4.15) holds and it remains to verify (4.16). To this end, note that since $B_n/a_n \to \infty$ and $E|X_1| < \infty$,

$$E\left(|X_1| I\left(|X_1| > B_n/a_n\right)\right) \longrightarrow 0$$

and hence for arbitrary $\varepsilon > 0$, there exists an integer $N \ge 2$ such that

$$E\left(|X_1| I\left(|X_1| > B_n/a_n\right)\right) \le \varepsilon \quad \text{for all} \quad n \ge N.$$

Then for all $n \geq N$,

$$\begin{aligned}
&\frac{\left|\sum_{j=1}^{n} a_j E\left(X_j I\left(|X_j| \leq B_j/a_j\right)\right)\right|}{B_n} \\
&\leq \frac{\sum_{j=1}^{n} a_j \left|E\left(X_1 I\left(|X_1| \leq B_j/a_j\right)\right)\right|}{B_n} \\
&= \frac{\sum_{j=1}^{n} a_j \left|-E\left(X_1 I\left(|X_1| > B_j/a_j\right)\right)\right|}{B_n} \quad (\text{since } EX_1 = 0\ ) \\
&\leq \frac{\sum_{j=1}^{n} a_j E\left(|X_1| I\left(|X_1| > B_j/a_j\right)\right)}{B_n} \\
&\leq \frac{\sum_{j=1}^{N-1} a_j E\left(|X_1| I\left(|X_1| > B_j/a_j\right)\right)}{B_n} + \frac{\sum_{j=N}^{n} a_j \varepsilon}{B_n} \\
&\leq o(1) + C\varepsilon \ (\text{by the first condition of (4.1) and the second condition of (4.8)}).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, (4.16) holds.

(iii) Arguing as in the proof of part (i), (4.14) holds and (4.6) then follows immediately from Theorem 3.1(ii) in view of (4.9) and (4.10).

(iv) In view of part (iii), it only needs to be verified that $B_n \uparrow \infty$ and (4.10) holds. Now $B_n \uparrow \infty$ follows immediately from the first condition of (4.11) and the first condition of (4.12). To verify (4.10), it follows from (4.12) that

$$\frac{\sum_{j=1}^{n} E\left(|a_j X_j| I\left(|X_j| \leq B_j/a_j\right)\right)}{B_n} \leq \frac{C \sum_{j=1}^{n} a_j}{B_n} = o(1). \quad \square$$

Remark 4.1. If $a_n \equiv 1$ and $B_n \equiv n$, then (4.1), (4.7), and the first two conditions of (4.8) are automatic. Thus for a sequence of PNQD i.d. random variables $\{X_n,\ n \geq 1\}$ with $E|X_1| < \infty$, noting that $\{X_n - EX_n,\ n \geq 1\}$ is PNQD by Lemma 2.1, Matuła's [19] extension (1.2) of the classical Kolmogorov SLLN follows from Theorem 4.1(ii).

5. Some interesting examples. To conclude, we present four examples. The first two examples illustrate Theorem 3.1. According to Proposition 2.1, for each example there exists a sequence of PNQD random variables which are not independent and which has the specified marginal distributions.

Example 5.1. Let $\lambda \geq 1$ and let $\{\alpha_n,\ n \geq 1\}$ and $\{\beta_n,\ n \geq 1\}$ be sequences of positive constants such that $\alpha_n \beta_n = \mathcal{O}\left(n^{\lambda-1}\right)$ and $\beta_n = \mathcal{O}\left(n^{\lambda}/(\log n)^{1+\varepsilon}\right)$ for some $\varepsilon > 0$. Let $\{X_n,\ n \geq 1\}$ be a sequence of PNQD random variables where X_n has probability density function

$$f_n(x) = \frac{|x|^{\alpha_n - 1} e^{-|x|/\beta_n}}{2\beta_n^{\alpha_n} \Gamma(\alpha_n)}, \quad -\infty < x < \infty, \quad n \geq 1$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt,\ \alpha > 0$. Let $B_n = n^\lambda, \quad n \geq 1$. Note that

$$E|X_n| = \alpha_n \beta_n, \quad EX_n^2 = \alpha_n^2 \beta_n^2 + \alpha_n \beta_n^2, \quad n \geq 1.$$

Thus the series of (1.3) is majorized by

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{EX_n^2}{B_n^2} &\leq \sum_{n=1}^{\infty}\frac{Cn^{2\lambda-2}+Cn^{\lambda-1}\beta_n}{n^{2\lambda}} \\ &\leq C+\sum_{n=2}^{\infty}\left(\frac{C}{n^2}+\frac{C}{n(\log n)^{1+\varepsilon}}\right)<\infty\end{aligned}$$

and so (3.1) holds by either Remark 3.1(i) or Remark 3.1(ii). Now (3.2) also holds whence

$$\frac{\sum_{j=1}^{n} X_j}{n^{\lambda}} \longrightarrow 0 \text{ a.c.}$$

follows from Theorem 3.1(i) noting that $E\left(X_n I(|X_n| \leq n^{\lambda})\right)=0,\ n\geq 1$.

The random variables $\{X_n,\ n\geq 1\}$ in the next example are in $\mathcal{L}_p$ for all $0<p<1$ but are not in $\mathcal{L}_1$.

Example 5.2. Let $\{X_n,\ n\geq 1\}$ be a sequence of PNQD i.d. random variables where X_1 has probability density function $f(x)=x^{-2}I_{[1,\infty)}(x),\ -\infty<x<\infty$. Let $B_n=n(\log n)^{1+\varepsilon},\ n\geq 2$ where $\varepsilon>0$ and let $B_1=B_2/2$. Now $P\{|X_1|>x\}=x^{-1},\ x\geq 1$ and so

$$\begin{aligned}\sum_{n=3}^{\infty} B_n^{-2}\int_0^{B_n} xP\left\{|X_n|>x\right\}dx &= \sum_{n=3}^{\infty} B_n^{-2}\left(\int_0^1 xdx+\int_1^{B_n} 1dx\right) \\ &\leq \sum_{n=3}^{\infty} B_n^{-1}<\infty.\end{aligned}$$

Thus (3.1) holds. Now

$$E\left(|X_n|I(|X_n|\leq B_n)\right)=E\left(X_1 I(X_1\leq B_n)\right)\sim \log n$$

and so (3.28) holds. Thus by Corollary 3.1(ii),

$$\frac{\sum_{j=1}^{n} X_j}{n(\log n)^{1+\varepsilon}} \longrightarrow 0 \text{ a.c.} \tag{5.1}$$

Remark 5.1. In Theorem 2 of Martikainen and Petrov [18], a SLLN was proved for i.d. random variables irrespective of their joint distributions. But the conclusion (5.1) cannot be obtained from it since their condition

$$B_n\sum_{j=n}^{\infty} B_j^{-1}=\mathcal{O}(n)$$

fails for the norming constants $\{B_n,\ n\geq 1\}$ in Example 5.2. However, Theorem 2 of Martikainen and Petrov [18] will establish for the random variables $\{X_n,\ n\geq 1\}$ of Example 5.2 that for all $\varepsilon>0$

$$\frac{\sum_{j=1}^{n} X_j}{n^{1+\varepsilon}} \longrightarrow 0 \text{ a.c.}$$

and this is weaker than (5.1).

The last two examples show, respectively, that parts (i) and (ii) of Theorem 3.1 can fail without the PNQD assumption.

Example 5.3. Let X be a symmetric random variable with $0 < E|X| < \infty$ and let $B_n = n,\ n \geq 1$. Set $X_n = X,\ n \geq 1$. Then $\{X_n,\ n \geq 1\}$ is *not* PNQD. Now the choice of $\{B_n,\ n \geq 1\}$ and $E|X| < \infty$ ensure, respectively, that

$$\left(\max_{1\leq j\leq n} B_j^2\right)\sum_{j=n}^{\infty} B_j^{-2} = \mathcal{O}(n) \quad \text{and} \quad \sum_{n=1}^{\infty} P\left\{|X_n| > B_n\right\} < \infty$$

and so by Lemma 2.3

$$\sum_{n=1}^{\infty} B_n^{-2} E\left(X_n^2 I\left(|X_n| \leq B_n\right)\right) < \infty.$$

Thus (3.1) holds via the equivalence between (3.6) and (3.7). Moreover, (3.2) clearly holds. But for $n \geq 1$,

$$\frac{\sum_{j=1}^{n}\left(X_j - E\left(X_j I(|X_j| \leq B_j)\right)\right)}{B_n} = \frac{nX}{n} = X$$

and so (3.3) fails.

The next example is taken from Counterexample 2 of Rosalsky [23] which had served an entirely different purpose from that of Example 5.4.

Example 5.4. Let $\{Y_n,\ n \geq 1\}$ be a sequence of i.i.d. random variables with

$$P\left\{Y_1 > y\right\} = \frac{e}{y\log y}, \quad y \geq e.$$

Let $n_0 = 0,\ n_k = 2^k,\ k \geq 1$ and define

$$X_j = Y_k, \quad n_{k-1} + 1 \leq j \leq n_k,\ k \geq 1.$$

Then $\{X_n,\ n \geq 1\}$ is *not* PNQD. Let $B_1 = 1$ and $B_n = n\log n,\ n \geq 2$. We first verify (3.1). Note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} B_n^{-2}\int_0^{B_n} xP\left\{|X_n| > x\right\}dx \\
&= C + \sum_{n=5}^{\infty} B_n^{-2}\left(C + \int_{e^2}^{B_n} xP\left\{|X_1| > x\right\}dx\right) \\
&= C + \sum_{n=5}^{\infty} B_n^{-2}\int_{e^2}^{B_n} \frac{e}{\log x}dx \\
&\leq C + \sum_{n=5}^{\infty} 2eB_n^{-2}\int_{e^2}^{B_n} \frac{\log x - 1}{(\log x)^2}dx \\
&= C + \sum_{n=5}^{\infty} 2eB_n^{-2}\left.\frac{x}{\log x}\right|_{e^2}^{B_n} \\
&\leq C + C\sum_{n=5}^{\infty} \frac{1}{B_n\log B_n} \\
&\leq C + C\sum_{n=5}^{\infty} \frac{1}{n(\log n)^2} \\
&< \infty
\end{aligned}$$

establishing (3.1).

Next, we verify (3.4). Note that for $n \geq 3$

$$
\begin{aligned}
E\left(|X_n| I\left(|X_n| \leq B_n\right)\right) &= \int_0^\infty P\left\{|X_1| I(|X_1| \leq B_n) > x\right\} dx \\
&= \int_0^{B_n} P\left\{|X_1| I(|X_1| \leq B_n) > x\right\} dx \\
&\leq \int_0^{B_n} P\left\{|X_1| > x\right\} dx \\
&= C + \int_e^{B_n} \frac{e}{x \log x} dx \\
&\leq C \log\log B_n \\
&\leq C \log\log n
\end{aligned}
$$

and so for $n \geq 3$

$$
\begin{aligned}
\frac{\sum_{j=1}^n E\left(|X_j| I(|X_j| \leq B_j)\right)}{B_n} &\leq o(1) + \frac{C \sum_{j=3}^n \log\log j}{B_n} \\
&\leq o(1) + \frac{Cn \log\log n}{B_n} \\
&= o(1) + \frac{C \log\log n}{\log n} \\
&= o(1)
\end{aligned}
$$

verifing (3.4).

Finally, it was shown by Rosalsky [23] that

$$
\limsup_{n \to \infty} \frac{\sum_{j=1}^n X_j}{B_n} = \infty \quad \text{a.c.}
$$

whence (3.5) fails.

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