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Weak Laws with Random Indices for Arrays of Random Elements in Rademacher Type p Banach Spaces

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For a sequence of constants $\{a_n, n \geq 1\}$, an array of rowwise independent and stochastically dominated random elements $\{V_{nj}, j \geq 1, n \geq 1\}$ in a real separable Rademacher type p $(1 \leq p \leq 2)$ Banach space, and a sequence of positive integer-valued random variables $\{T_n, n \geq 1\}$, a general weak law of large numbers of the form $\sum_{j=1}^{T_n} a_j(V_{nj} - c_{nj})/b_{[\alpha_n]} \xrightarrow{P} 0$ is established where $\{c_{nj}, j \geq 1, n \geq 1\}$, $\alpha_n \to \infty$, $b_n \to \infty$ are suitable sequences. Some related results are also presented. No assumption is made concerning the existence of expected values or absolute moments of the $\{V_{nj}, j \geq 1, n \geq 1\}$. Illustrative examples include one wherein the strong law of large numbers fails.

KEY WORDS: Rademacher type p Banach space; array of rowwise independent random elements; weighted sums; weak law of large numbers; random indices.

1. INTRODUCTION

In this paper, for an array $\{V_{nj}, j \geq 1, n \geq 1\}$ of rowwise independent Banach space valued random elements, general weak laws of large numbers (WLLNs) will be established for the *weighted sums* $\sum_{j=1}^{T_n} a_j V_{nj}$ where T_n is random. The general setting will now be described. Let $(\Omega, \mathscr{F}, P)$ be a probability space and let $\mathscr{X}$ be a real separable Banach space with

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norm $\|\cdot\|$. The *expected value* or *mean* of a random element V, denoted by EV, is defined to be the *Pettis integral* provided it exists. That is, V has expected value $EV \in \mathscr{X}$ if $f(EV) = E(f(V))$ for every $f \in \mathscr{X}^*$ where $\mathscr{X}^*$ is the (*dual*) space of all continuous linear functionals on $\mathscr{X}$.

Let $\{V_{nj}, j \geq 1, n \geq 1\}$ be an array of rowwise independent $\mathscr{X}$-valued random elements defined on $(\Omega, \mathscr{F}, P)$ and let $\{a_n \neq 0, n \geq 1\}$, $\{b_n, n \geq 1\}$, and $\{\alpha_n, n \geq 1\}$ be sequences of constants with $0 < b_n \to \infty$, $1 \leq \alpha_n \to \infty$. Let $\{T_n, n \geq 1\}$ be a sequence of positive integer-valued random variables and let $\{c_{nj}, j \geq 1, n \geq 1\}$ be a "centering" array consisting of (suitably selected) elements in $\mathscr{X}$. In this paper, general WLLNs of the forms

$$\frac{\sum_{j=1}^{T_n} a_j(V_{nj} - c_{nj})}{b_{[\alpha_n]}} \xrightarrow{P} 0 \tag{1.1}$$

and

$$\frac{\sum_{j=1}^{T_n} a_j(V_{nj} - c_{nj})}{b_{T_n}} \xrightarrow{P} 0 \tag{1.2}$$

will be established. The number of terms in the sums in (1.1) and (1.2) is random, and the $\{T_n, n \geq 1\}$ are referred to as *random indices.*

For normed weighted sums of the form $\sum_{j=1}^{n} a_j V_j / b_n$ where $\{V_n, n \geq 1\}$ is a sequence of Banach space valued random elements, the strong law of large numbers (SLLN) problem (wherein the convergence to 0 is almost certain (a.c.)) was studied by Mikosch and Norvaiša[17, 18] and by Adler *et al.*[4] and the WLLN problem was studied by Adler *et al.*[5] In Adler *et al.*,[5] the corresponding SLLN need not necessarily hold.

In the current work, the Banach space $\mathscr{X}$ is assumed to satisfy the geometric condition of being of Rademacher type p $(1 \leq p \leq 2)$. (Technical definitions such as this will be discussed in Section 2.) Conditions are placed on the growth behavior of the constants $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$. The random elements $\{V_{nj}, j \geq 1, n \geq 1\}$ are assumed to be stochastically dominated by a random element V in the sense that (2.2) holds. The tail $P\{\|V\| > t\}$ of the distribution of $\|V\|$ as $t \to \infty$ is controlled by (4.4). Moreover, conditions are imposed on the marginal distributions of the random indices $\{T_n, n \geq 1\}$. Examples are provided to illustrate various aspects of the results.

For convenience, technical definitions will be consolidated into Section 2. The lemmata which are needed to establish the WLLNs will be presented in Section 3. Finally, the symbol C denotes throughout a generic constant $(0 < C < \infty)$ which is not necessarily the same one in each appearance.

2. PRELIMINARY DEFINITIONS

Technical definitions relevant to the current work will be discussed in this section.

Let $\{\varepsilon_n, n \geqslant 1\}$ be a symmetric *Bernoulli sequence*, that is, $\{\varepsilon_n, n \geqslant 1\}$ are independent and identically distributed (i.i.d.) random variables with $P\{\varepsilon_1 = 1\} = P\{\varepsilon_1 = -1\} = 1/2$. Let $\mathscr{X}^\infty = \mathscr{X} \times \mathscr{X} \times \mathscr{X} \times \cdots$ and define $\mathscr{C}(\mathscr{X}) = \{(v_1, v_2, \ldots) \in \mathscr{X}^\infty \colon \sum_{n=1}^\infty \varepsilon_n v_n$ converges in probability$\}$. Let $1 \leqslant p \leqslant 2$. Then $\mathscr{X}$ is said to be of *Rademacher type p* if there exists a finite constant C such that $E \|\sum_{n=1}^\infty \varepsilon_n v_n\|^p \leqslant C \sum_{n=1}^\infty \|v_n\|^p$ for all $(v_1, v_2, \ldots) \in \mathscr{C}(\mathscr{X})$. Hoffmann-Jørgensen and Pisier[14] proved for $1 \leqslant p \leqslant 2$ that a real separable Banach space is of Rademacher type p if and only if there exists a finite constant C such that

$$E \left\| \sum_{j=1}^n V_j \right\|^p \leqslant C \sum_{j=1}^n E \|V_j\|^p \tag{2.1}$$

for every finite collection $\{V_1, \ldots, V_n\}$ of independent random elements with $EV_j = 0$, $1 \leqslant j \leqslant n$.

Random elements $\{V_{nj}, j \geqslant 1, n \geqslant 1\}$ are said to be *stochastically dominated* by a random element V if for some finite constant D,

$$P\{\|V_{nj}\| > t\} \leqslant DP\{\|DV\| > t\}, \qquad t \geqslant 0, \quad j \geqslant 1, \quad n \geqslant 1 \tag{2.2}$$

This condition is, of course, automatic with $V = V_{11}$ and $D = 1$ if the $\{V_{nj}, j \geqslant 1, n \geqslant 1\}$ are identically distributed. It follows from Lemma 5.2.2 of Taylor,[22] p. 123 (or Lemma 3 of Wei and Taylor[23]) that stochastic dominance can be accomplished by the array of random elements having a bounded absolute rth moment ($r > 0$). Specifically, if $\sup_{n \geqslant 1, j \geqslant 1} E \|V_{nj}\|^r < \infty$ for some $r > 0$, then there exists a random element V with $E \|V\|^p < \infty$ for all $0 < p < r$ such that (2.2) holds with $D = 1$. (The proviso that $r > 1$ in Lemma 5.2.2 of Taylor,[22] p. 123 (or Lemma 3 of Wei and Taylor[23]) is not needed as was pointed out by Adler *et al.*[6]).

3. PRELIMINARY LEMMATA

In this section, lemmata needed to establish the results in this paper will be presented. Some of them may be of independent interest. The first lemma, due to Etemadi,[11] provides a maximal inequality for a sum of independent random elements and will be used to give a proof of a Kolmogorov type maximal inequality (Lemma 2) for random elements in Rademacher type p Banach spaces. Lemma 1 may also be found in Billingsley,[8] p. 288. For some related results see Etemadi.[12]

Lemma 1 (Etemadi[11]). Let $\{V_j,\ 1 \leqslant j \leqslant n\}$ be independent random elements in a real separable Banach space. Then

$$P\left\{\max_{1\leqslant k\leqslant n}\left\|\sum_{j=1}^{k} V_j\right\| > t\right\} \leqslant 4 \max_{1\leqslant k\leqslant n} P\left\{\left\|\sum_{j=1}^{k} V_j\right\| > \frac{t}{4}\right\}, \qquad t>0$$

The next lemma is a Kolmogorov type maximal inequality for random elements in Rademacher type p $(1 \leqslant p \leqslant 2)$ Banach spaces. It was obtained by Jain[15] and Woyczyński[24] when $p=2$ and perhaps is also known when $1 \leqslant p < 2$ but the authors are not able to find it in the literature. Two proofs of it will be provided and the arguments are completely different from those of Jain[15] and Woyczyński.[24]

Lemma 2. Let $\{V_j,\ j \geqslant 1\}$ be independent mean 0 random elements in a real separable Rademacher type p $(1 \leqslant p \leqslant 2)$ Banach space. Then for all $n \geqslant 1$

$$P\left\{\max_{1\leqslant k\leqslant n}\left\|\sum_{j=1}^{k} V_j\right\| > t\right\} \leqslant \frac{C}{t^p} \sum_{j=1}^{n} E\,\|V_j\|^p, \qquad t>0$$

where C is a constant independent of n.

Proof No. 1. Let $\mathscr{F}_n = \sigma(V_1,..., V_n)$, $n \geqslant 1$. Now it is well known but appears to have first been observed by Scalora[20] that $\{\|\sum_{j=1}^{n} V_j\|,\ \mathscr{F}_n,\ n \geqslant 1\}$ is a real submartingale and hence so is (see e.g., Chow and Teicher,[9] p. 236) $\{\|\sum_{j=1}^{n} V_j\|^p,\ \mathscr{F}_n,\ n \geqslant 1\}$ via convexity and monotonicity of the function $\phi(x) = x^p$, $0 \leqslant x < \infty$. Then employing the Doob submartingale maximal inequality (see e.g., Doob,[10] p. 314; or Rao,[19] p. 173; or Shiryayev,[21] p. 464) and (2.1), it follows for all $n \geqslant 1$ and $t > 0$ that

$$\begin{aligned} P\left\{\max_{1\leqslant k\leqslant n}\left\|\sum_{j=1}^{k} V_j\right\| > t\right\} &= P\left\{\max_{1\leqslant k\leqslant n}\left\|\sum_{j=1}^{k} V_j\right\|^p > t^p\right\} \\ &\leqslant \frac{1}{t^p} E\left\|\sum_{j=1}^{n} V_j\right\|^p \\ &\leqslant \frac{C}{t^p} \sum_{j=1}^{n} E\,\|V_j\|^p \end{aligned}$$

□

Proof No. 2. Employing Lemma 1, the Markov inequality, and (2.1), it follows for all $n \geqslant 1$ and $t > 0$ that

$$P\left\{\max_{1\leqslant k\leqslant n}\left\|\sum_{j=1}^{k} V_j\right\|>t\right\}\leqslant 4\max_{1\leqslant k\leqslant n} P\left\{\left\|\sum_{j=1}^{k} V_j\right\|>\frac{t}{4}\right\}$$

$$\leqslant \frac{4^{p+1}}{t^p}\max_{1\leqslant k\leqslant n} E\left\|\sum_{j=1}^{k} V_j\right\|^p$$

$$\leqslant \frac{4^{p+1}C}{t^p}\max_{1\leqslant k\leqslant n}\sum_{j=1}^{k} E\|V_j\|^p$$

$$=\frac{4^{p+1}C}{t^p}\sum_{j=1}^{n} E\|V_j\|^p \qquad \square$$

4. MAINSTREAM

With the preliminaries accounted for, the main result of this paper, Theorem 1, may be established. Theorem 1 extends the WLLN of Adler *et al.*[5] in three directions, namely:

(i) Theorem 1 involves an array rather than a sequence of random elements.

(ii) Random indices $\{T_n, n\geqslant 1\}$ determine the number of terms in the row sums.

(iii) Less restrictive conditions are imposed on the $\{a_n, n\geqslant 1\}$ and $\{b_n, n\geqslant 1\}$.

For a sequence of i.i.d. random variables, Adler and Rosalsky[2] proved a very special case of Theorem 1 in that substantially stronger restrictions were placed on the $\{a_n, n\geqslant 1\}$ and $\{b_n, n\geqslant 1\}$.

It should be noted that in Theorem 1, it is not being assumed that the $\{V_{nj}, j\geqslant 1, n\geqslant 1\}$ have expected values or absolute moments. Moreover, the first condition of (4.1) ensures that $b_n\to\infty$. However, it is not assumed that $\{b_n, n\geqslant 1\}$ is monotone. In addition, no assumptions are made regarding the joint distributions of the random indices $\{T_n, n\geqslant 1\}$ whose marginal distributions are constrained solely by (4.2). Nor is it assumed that the stochastic processes $\{T_n, n\geqslant 1\}$ and $\{V_{nj}, j\geqslant 1, n\geqslant 1\}$ are independent of each other. It should be noted that the condition (4.2) is considerably weaker than $T_n/\alpha_n \xrightarrow{P} c$ for some constant $c\in[0,\infty)$. Finally, observe that the condition (4.4) is of the spirit of the condition $nP\{|X_1|>n\}=o(1)$ of the classical WLLN with random indices for i.i.d. random variables (see e.g., Chow and Teicher,[9] p. 131), that the condition

(4.3) is automatic (with $\kappa_n = [\alpha_n]$, $n \geqslant 1$) if $0 < \lambda \leqslant 1$, and that if $\lambda > 1$ and either

$$\sum_{j=1}^{[\lambda\alpha_n]} |a_j|^p = \mathcal{O}\left(\sum_{j=1}^{[\alpha_n]} |a_j|^p\right) \qquad \text{or} \qquad b_{[\lambda\alpha_n]} = \mathcal{O}(b_{[\alpha_n]})$$

then (4.3) holds with $\kappa_n = [\alpha_n]$, $n \geqslant 1$, or $\kappa_n = [\lambda\alpha_n]$, $n \geqslant 1$, respectively.

Theorem 1. Let $\{V_{nj}, j \geqslant 1, n \geqslant 1\}$ be an array of rowwise independent random elements in a real separable Rademacher type p $(1 \leqslant p \leqslant 2)$ Banach space, and suppose that $\{V_{nj}, j \geqslant 1, n \geqslant 1\}$ is stochastically dominated by a random element V. Let $\{a_n, n \geqslant 1\}$ and $\{b_n, n \geqslant 1\}$ be constants with $a_n \neq 0$, $b_n > 0$, $n \geqslant 1$, and suppose that $b_n/|a_n| \uparrow$ and

$$\sum_{j=1}^{n} |a_j|^p = o(b_n^p), \qquad \sum_{j=1}^{n} |a_j|^p = \mathcal{O}(n\,|a_n|^p)$$

and

$$\sum_{j=1}^{n} \frac{b_j^p}{j^2\,|a_j|^p} = \mathcal{O}\left(\frac{b_n^p}{\sum_{j=1}^{n} |a_j|^p}\right) \tag{4.1}$$

Let $\{T_n, n \geqslant 1\}$ be positive integer-valued random variables and $1 \leqslant \alpha_n \to \infty$ be constants such that

$$P\left\{\frac{T_n}{\alpha_n} > \lambda\right\} = o(1) \qquad \text{for some constant} \quad 0 < \lambda < \infty \tag{4.2}$$

Suppose that there exists a sequence of integers $\{\kappa_n, n \geqslant 1\}$ such that

$$\kappa_n \geqslant [\alpha_n],\ n \geqslant 1, \qquad \text{and} \qquad b_{[\alpha_n]}^{-p} \sum_{j=1}^{[\lambda\alpha_n]} |a_j|^p = \mathcal{O}\left(b_{\kappa_n}^{-p} \sum_{j=1}^{\kappa_n} |a_j|^p\right) \tag{4.3}$$

Then if

$$nP\left\{\|DV\| > \frac{b_n}{|a_n|}\right\} = o(1) \tag{4.4}$$

where D is as in (2.2), the WLLN

$$\frac{\sum_{j=1}^{T_n} a_j(V_{nj} - EV_{nj}I(\|V_{nj}\| \leqslant (b_{[\alpha_n]}/|a_{[\alpha_n]}|))}{b_{[\alpha_n]}} \xrightarrow{P} 0 \tag{4.5}$$

obtains.

Proof. Let the three conditions of (4.1) be referred to as (4.1a), (4.1b), and (4.1c), respectively, and set

$$c_0 = 0,\ c_n = \frac{b_n}{|a_n|}, \qquad U_{nj} = V_{nj} I(\| V_{nj} \| \leqslant c_{[\alpha_n]}), \qquad j \geqslant 1, \quad n \geqslant 1$$

Note at the outset that $E \| U_{nj} \| < \infty$, $j \geqslant 1$, $n \geqslant 1$ and so (see e.g., Taylor,[22] p. 40) the $\{ U_{nj}, j \geqslant 1, n \geqslant 1 \}$ all have expected values.

Firstly, it will be verified that

$$\frac{\sum_{j=1}^{T_n} a_j (V_{nj} - U_{nj})}{b_{[\alpha_n]}} \xrightarrow{P} 0 \tag{4.6}$$

For arbitrary $\varepsilon > 0$ and all large n,

$$\begin{aligned}
&P \left\{ \frac{\| \sum_{j=1}^{T_n} a_j (V_{nj} - U_{nj}) \|}{b_{[\alpha_n]}} > \varepsilon \right\} \\
&\leqslant P \left\{ \sum_{j=1}^{T_n} a_j V_{nj} \neq \sum_{j=1}^{T_n} a_j U_{nj} \right\} \\
&\leqslant P \left\{ \left[\sum_{j=1}^{T_n} a_j V_{nj} \neq \sum_{j=1}^{T_n} a_j U_{nj} \right] [T_n \leqslant \lambda \alpha_n] \right\} + P\{ T_n > \lambda \alpha_n \} \\
&\leqslant P \left\{ \bigcup_{j=1}^{[\lambda \alpha_n]} [\| V_{nj} \| > c_{[\alpha_n]}] \right\} + o(1) \qquad \text{(by (4.2))} \\
&\leqslant \sum_{j=1}^{[\lambda \alpha_n]} P\{ \| V_{nj} \| > c_{[\alpha_n]} \} + o(1) \\
&\leqslant D[\lambda \alpha_n]\, P\{ \|DV\| > c_{[\alpha_n]} \} + o(1) \qquad \text{(by (2.2))} \\
&= (1 + o(1))\, D\lambda [\alpha_n]\, P\{ \|DV\| > c_{[\alpha_n]} \} + o(1) \\
&= o(1) \qquad \text{(by (4.4))}
\end{aligned}$$

thereby establishing (4.6).

The proof will thus be completed if it can be demonstrated that

$$\frac{\sum_{j=1}^{T_n} a_j (U_{nj} - EU_{nj})}{b_{[\alpha_n]}} \xrightarrow{P} 0 \tag{4.7}$$

To this end, for arbitrary $\varepsilon > 0$ and all large n,

$$P\left\{\frac{\|\sum_{j=1}^{T_n} a_j(U_{nj}-EU_{nj})\|}{b_{[\alpha_n]}} > \varepsilon\right\}$$

$$\leqslant P\left\{\left[\frac{\|\sum_{j=1}^{T_n} a_j(U_{nj}-EU_{nj})\|}{b_{[\alpha_n]}} > \varepsilon\right][T_n \leqslant \lambda\alpha_n]\right\} + P\{T_n > \lambda\alpha_n\}$$

$$\leqslant P\left\{\bigcup_{k=1}^{[\lambda\alpha_n]}\left[\left\|\sum_{j=1}^{k} a_j(U_{nj}-EU_{nj})\right\| > \varepsilon b_{[\alpha_n]}\right]\right\} + o(1) \qquad \text{(by (4.2))}$$

$$= P\left\{\max_{1\leqslant k\leqslant[\lambda\alpha_n]}\left\|\sum_{j=1}^{k} a_j(U_{nj}-EU_{nj})\right\| > \varepsilon b_{[\alpha_n]}\right\} + o(1)$$

$$\leqslant \frac{C}{\varepsilon^p b_{[\alpha_n]}^p}\sum_{j=1}^{[\lambda\alpha_n]} |a_j|^p E\,\|U_{nj}-EU_{nj}\|^p + o(1) \qquad \text{(by Lemma 2)} \qquad (4.8)$$

$$\leqslant \frac{C2^p}{\varepsilon^p b_{[\alpha_n]}^p}\sum_{j=1}^{[\lambda\alpha_n]} |a_j|^p E\,\|U_{nj}\|^p + o(1) \qquad (4.9)$$

$$\leqslant \frac{C2^p}{\varepsilon^p b_{[\alpha_n]}^p}\sum_{j=1}^{[\lambda\alpha_n]} |a_j|^p (E\,\|V_{nj}\|^p I(\|V_{nj}\| \leqslant c_{[\alpha_n]})$$

$$+ c_{[\alpha_n]}^p P\{\|V_{nj}\| > c_{[\alpha_n]}\}) + o(1)$$

$$= \frac{C2^p}{\varepsilon^p b_{[\alpha_n]}^p}\sum_{j=1}^{[\lambda\alpha_n]} |a_j|^p \int_0^{c_{[\alpha_n]}} pt^{p-1} P\{\|V_{nj}\| > t\}\,dt$$

$$+ o(1) \qquad \text{(by integration by parts)}$$

$$\leqslant \frac{CD2^p}{\varepsilon^p b_{[\alpha_n]}^p}\sum_{j=1}^{[\lambda\alpha_n]} |a_j|^p \sum_{k=1}^{[\alpha_n]}\int_{c_{k-1}}^{c_k} pt^{p-1} P\{\|DV\| > t\}\,dt + o(1) \qquad \text{(by (2.2))}$$

$$\leqslant \frac{C}{b_{\kappa_n}^p}\sum_{j=1}^{\kappa_n} |a_j|^p \sum_{k=1}^{\kappa_n}\int_{c_{k-1}}^{c_k} pt^{p-1} P\{\|DV\| > t\}\,dt + o(1) \qquad \text{(by (4.3))}$$

$$\leqslant \frac{C}{b_{\kappa_n}^p}\sum_{j=1}^{\kappa_n} |a_j|^p \sum_{k=1}^{\kappa_n} \frac{c_k^p - c_{k-1}^p}{k} kP\{\|DV\| > c_{k-1}\} + o(1) \qquad (4.10)$$

Now for $n \geqslant 2$,

$$\begin{aligned}
&\left(\frac{1}{b_n^p}\sum_{j=1}^{n}|a_j|^p\right)\sum_{k=1}^{n}\frac{c_k^p-c_{k-1}^p}{k}\\
&\quad=\left(\frac{1}{b_n^p}\sum_{j=1}^{n}|a_j|^p\right)\left(\frac{c_n^p}{n}+\sum_{k=1}^{n-1}\frac{c_k^p}{k(k+1)}\right)\\
&\quad\leqslant\left(\frac{1}{b_n^p}\sum_{j=1}^{n}|a_j|^p\right)\left(\frac{c_n^p}{n}+\sum_{k=1}^{n}\frac{c_k^p}{k^2}\right)\\
&\quad=\mathcal{O}(1)+\mathcal{O}(1)\qquad\text{(by (4.1b) and (4.1c))}\\
&\quad=\mathcal{O}(1)
\end{aligned}$$

Moreover, (4.1a) ensures that

$$\left(\frac{1}{b_n^p}\sum_{j=1}^{n}|a_j|^p\right)\left(\frac{c_k^p-c_{k-1}^p}{k}\right)\to 0 \qquad \text{as} \quad n\to\infty \text{ for all fixed } k\geqslant 1$$

Since for $n \geqslant 2$

$$nP\{\|DV\|>c_{n-1}\}=(1+o(1))(n-1)\,P\{\|DV\|>c_{n-1}\}=o(1)\qquad\text{(by (4.4))}$$

it follows from the Toeplitz lemma (see e.g., Loève,[16] p. 250) that

$$\left(\frac{1}{b_n^p}\sum_{j=1}^{n}|a_j|^p\right)\sum_{k=1}^{n}\frac{c_k^p-c_{k-1}^p}{k}kP\{\|DV\|>c_{k-1}\}=o(1)$$

implying (4.7) via (4.10) and $\kappa_n \geqslant [\alpha_n] \to \infty$. □

Remarks 1.

(i) The authors had originally proved Theorem 1 using a condition slightly stronger than (4.3) and with condition (4.1) replaced by the assumption that either

$$\frac{b_n}{n\,|a_n|}\downarrow,\qquad \sum_{j=1}^{n}|a_j|^p=o(b_n^p),\qquad\text{and}\qquad \sum_{j=1}^{n}\frac{b_j^p}{j^2\,|a_j|^p}=\mathcal{O}\left(\frac{b_n^p}{\sum_{j=1}^{n}|a_j|^p}\right)\tag{4.11}$$

or

$$\frac{b_n}{n\,|a_n|}\to\infty,\qquad \sum_{j=1}^{n}|a_j|^p=\mathcal{O}(n\,|a_n|^p)$$

and

$$\sum_{j=1}^{n} \frac{b_j^p}{j^2 |a_j|^p} = \mathcal{O}\left(\frac{b_n^p}{\sum_{j=1}^{n} |a_j|^p}\right) \tag{4.12}$$

or

$$p > 1, \qquad \frac{b_n}{n |a_n|} \uparrow, \qquad \text{and} \qquad \sum_{j=1}^{n} |a_j|^p = \mathcal{O}(n |a_n|^p) \tag{4.13}$$

or

$$\frac{b_n}{n |a_n|} \uparrow \qquad \text{and} \qquad \sum_{j=1}^{n} |a_j| = \mathcal{O}\left(\frac{n |a_n|}{\log n}\right) \tag{4.14}$$

hold. The referee so kindly pointed out to the authors the unified set of conditions (4.1) and (4.3) in Theorem 1 and indeed the referee supplied the modifications needed to adapt the proof of the initial version of Theorem 1 to the new improved conditions. The conditions (4.11)–(4.14) will now be compared with this new condition (4.1). As in the proof of Theorem 1, let the three conditions of (4.1) be referred to as (4.1a)–(4.1c), respectively, and set $c_n = b_n/|a_n|$, $n \geqslant 1$. Note that if $\sum_{n=1}^{\infty} c_n^p/n^2 = \infty$ (*a fortiori*, c_n/n is bounded from 0) and (4.1c) holds, then (4.1a) holds. If $c_n^p/n^2 \downarrow$ (*a fortiori*, $c_n/n \downarrow$) and (4.1c) holds, then

$$\frac{b_n^p}{n |a_n|^p} = \frac{c_n^p}{n} \leqslant \sum_{j=1}^{n} \frac{c_j^p}{j^2} \leqslant \frac{C b_n^p}{\sum_{j=1}^{n} |a_j|^p}$$

implying (4.1b). If $c_n^p/n^\alpha \uparrow$ for some $\alpha > 1$ (*a fortiori*, $p > 1$ and $c_n/n \uparrow$) and (4.1b) holds, then $\sum_{n=1}^{\infty} c_n^p/n^2 = \infty$ and

$$\sum_{j=1}^{n} \frac{c_j^p}{j^2} \leqslant \frac{c_n^p}{n^\alpha} \sum_{j=1}^{n} j^{\alpha-2} \leqslant \frac{C c_n^p}{n} \leqslant \frac{C b_n^p}{\sum_{j=1}^{n} |a_j|^p}$$

implying (4.1a) as noted above. Similarly, if $c_n^p/n \uparrow$ and $\sum_{j=1}^{n} |a_j|^p = \mathcal{O}(n |a_n|^p/\log n)$, then (4.1b) holds, $\sum_{n=1}^{\infty} c_n^p/n^2 = \infty$, and

$$\sum_{j=1}^{n} \frac{c_j^p}{j^2} \leqslant \frac{c_n^p}{n} \sum_{j=1}^{n} \frac{1}{j} \leqslant \frac{C c_n^p \log n}{n} \leqslant \frac{C b_n^p}{\sum_{j=1}^{n} |a_j|^p}$$

again implying (4.1a). It follows in particular from these observations that the conditions (4.11)–(4.13) each imply (4.1). Moreover, if (4.14) holds, then without any loss of generality p may be taken to be 1 and so it also follows from the above that (4.1) (with $p=1$) holds. The sequences $a_n \equiv 1$ and $b_n = n$ (n odd), $b_n = n + \frac{1}{2}$ (n even) satisfy (4.1) for any $1 < p \leqslant 2$ but fail to satisfy (4.11)–(4.14).

(ii) If $p=1$, the hypothesis of independence is not needed. To see this, observe that (4.8) with $C=1$ follows immediately from the Markov inequality when $p=1$ without independence and without invoking Lemma 2. A perusal of the argument reveals that independence was not used anywhere else.

(iii) Moreover, if the hypotheses of Theorem 1 obtain with $p=1$, then (as will be shown later)

$$\frac{\sum_{j=1}^{T_n} a_j E V_{nj} I(\|V_{nj}\| \leqslant c_{[\alpha_n]})}{b_{[\alpha_n]}} \xrightarrow{P} 0 \tag{4.15}$$

and, consequently,

$$\frac{\sum_{j=1}^{T_n} a_j V_{nj}}{b_{[\alpha_n]}} \xrightarrow{P} 0$$

To prove (4.15), observe that

$$\begin{aligned}
&\frac{\|\sum_{j=1}^{T_n} a_j E V_{nj} I(\|V_{nj}\| \leqslant c_{[\alpha_n]})\|}{b_{[\alpha_n]}} \\
&\leqslant \frac{(\sum_{j=1}^{T_n} |a_j|\, E\,\|V_{nj}\|\, I(\|V_{nj}\| \leqslant c_{[\alpha_n]}))\, I(T_n \leqslant \lambda\alpha_n)}{b_{[\alpha_n]}} \\
&\quad + \frac{(\sum_{j=1}^{T_n} |a_j|\, E\,\|V_{nj}\|\, I(\|V_{nj}\| \leqslant c_{[\alpha_n]}))\, I(T_n > \lambda\alpha_n)}{b_{[\alpha_n]}} \\
&\leqslant \frac{\sum_{j=1}^{[\lambda\alpha_n]} |a_j|\, E\,\|V_{nj}\|\, I(\|V_{nj}\| \leqslant c_{[\alpha_n]})}{b_{[\alpha_n]}} + o_P(1) \qquad \text{(by (4.2))} \\
&= o(1) + o_P(1) = o_P(1)
\end{aligned}$$

noting that it was shown in the *proof* of (4.7) that the expression in (4.9) is $o(1)$. □

(iv) Since p may be taken to be 1 under (4.14) without any loss of generality, it follows in view of the previous two remarks that

$$\frac{\sum_{j=1}^{T_n} a_j V_{nj}}{b_{[\alpha_n]}} \xrightarrow{P} 0$$

obtains under (4.14) without assuming independence provided the other hypotheses of Theorem 1 remain in force.

(v) Apropos of (4.13) and (4.14), the example of Beck[7] considered by Adler *et al.*[5] shows that Theorem 1 can fail for well-behaved sequences $\{a_n, n \geqslant 1\}$ and $\{b_n, n \geqslant 1\}$ when

$$\frac{b_n}{n\,|a_n|}\uparrow \qquad \text{and} \qquad \sum_{j=1}^{n} |a_j| = \mathcal{O}(n\,|a_n|)$$

Take $T_n = n$, $\alpha_n = n$, $n \geqslant 1$, and $\lambda = 1$ (hence (4.2) holds) and refer to the discussion given in Adler *et al.*[5] Since the sequences $\{a_n, n \geqslant 1\}$ and $\{b_n, n \geqslant 1\}$ in this example satisfy (4.1) for all p in $(1, 2]$, this example also shows that Theorem 1 can fail if the Rademacher type p hypothesis is dispensed with.

(vi) It is natural to ask whether Theorem 1 holds if the norming sequence is replaced by $\{b_{T_n}, n \geqslant 1\}$. The answer is negative in view of the following example of Adler and Rosalsky.[2] Let $\{V_n, n \geqslant 1\}$ be i.i.d. random variables with V_1 having probability density function

$$f(v) = \frac{c}{v^2 \log v} I_{[e, \infty)}(v), \qquad -\infty < v < \infty$$

where c is a constant and let $V_{nj} = V_j$, $j \geqslant 1$, $n \geqslant 1$. Set $a_n = 1$, $b_n = n$, $n \geqslant 1$. The condition (4.1) holds with $p = 2$. Let $T_n = [n^{1/2}]$, $\alpha_n = n$, $n \geqslant 1$ and let $\lambda = 1$. Now for all $n \geqslant 3$,

$$nP\{|V_1| > n\} = nc \int_n^{\infty} \frac{1}{v^2 \log v}\, dv \leqslant \frac{c}{\log n} = o(1)$$

Thus by Theorem 1 (with $D = 1$ and $V = V_1$),

$$\frac{\sum_{j=1}^{[n^{1/2}]} (V_j - EV_1 I(|V_1| \leqslant n))}{n} = \frac{\sum_{j=1}^{T_n} (V_{nj} - EV_{nj} I(|V_{nj}| \leqslant n))}{b_{[\alpha_n]}} \xrightarrow{P} 0$$

However, Adler and Rosalsky[2] showed that

$$\frac{\sum_{j=1}^{[n^{1/2}]} (V_j - EV_1 I(|V_1| \leqslant n))}{[n^{1/2}]} \not\xrightarrow{P} 0$$

whence

$$\frac{\sum_{j=1}^{T_n} (V_{nj} - EV_{nj} I(|V_{nj}| \leqslant n))}{b_{T_n}} \xrightarrow{P} 0$$

fails.

(vii) The ensuing theorem shows that if the hypotheses to Theorem 1 are suitably strengthened, then the conclusion holds with the norming sequence replaced by $\{b_{T_n}, n \geq 1\}$. The pair of conditions (4.2) and (4.17) is equivalent to the single condition

$$P\left\{\lambda' \leq \frac{T_n}{\alpha_n} \leq \lambda\right\} \to 1 \qquad \text{for some constants} \quad 0 < \lambda' \leq \lambda < \infty \quad (4.16)$$

which is substantially weaker than $T_n/\alpha_n \xrightarrow{P} c$ for some constant $0 < c < \infty$. Clearly the sequence $\{T_n, n \geq 1\}$ in the example in Remark 1(v) does not satisfy (4.17).

Theorem 2. Let $\{V, V_{nj}, j \geq 1, n \geq 1\}$, $\{a_n, n \geq 1\}$, $\{b_n, n \geq 1\}$, $\{\alpha_n, n \geq 1\}$, and $\{T_n, n \geq 1\}$ satisfy the hypotheses of Theorem 1 and suppose, additionally, that $b_n \uparrow$ and for some constant $0 < \lambda' < \infty$ that

$$P\left\{\frac{T_n}{\alpha_n} < \lambda'\right\} = o(1) \tag{4.17}$$

and $b_{[\alpha_n]} = \mathcal{O}(b_{[\lambda'\alpha_n]})$. Then the WLLN

$$\frac{\sum_{j=1}^{T_n} a_j(V_{nj} - EV_{nj}I(\|V_{nj}\| \leq b_{[\alpha_n]}/|a_{[\alpha_n]}|))}{b_{T_n}} \xrightarrow{P} 0 \tag{4.18}$$

obtains.

Proof. In view of Theorem 1, it suffices to show that $b_{[\alpha_n]}/b_{T_n} = \mathcal{O}_P(1)$ and this follows as in Corollary 4 of Adler and Rosalsky.[2] □

Example 1. For $1 \leq p < \infty$, consider the real separable Banach space ℓ_p of absolute pth power summable real sequences $v = \{v_k, k \geq 1\}$ with norm $\|v\| = (\sum_{k=1}^{\infty} |v_k|^p)^{1/p}$. Let $\{K_n, n \geq 1\}$ be i.i.d. positive integer-valued random variables and let $\{Y_n, n \geq 1\}$ be i.i.d. random variables with the generalized St. Petersburg distribution

$$P\{Y_1 = q_o^{-y}\} = p_o q_o^{y-1}, \qquad y = 1, 2, \ldots$$

where $0 < p_o = 1 - q_o < 1$. Furthermore, suppose that $\{K_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are independent stochastic processes. Let $V_n = \{Y_n I(K_n = k), k \geq 1\}$, $n \geq 1$. Then $\{V_n, n \geq 1\}$ are i.i.d. random elements in ℓ_p for each $1 \leq p < \infty$ and $E\|V_1\| = EY_1 = \infty$. Let $V_{nj} = V_j$, $j \geq 1$, $n \geq 1$. Let $a_n = n^{\alpha}$, $n \geq 1$, $b_1 = \text{Log}\, 2$, $b_n = n^{\alpha+1} \text{Log}\, n$, $n \geq 2$ where $\alpha > -1$ and Log denotes the logarithm to the base q_o^{-1}. Let $\{T_n, n \geq 1\}$ be positive integer-valued random variables with $T_n/n \xrightarrow{P} 1$ and let $\alpha_n = n$, $n \geqslant 1$. Let $1 < p \leqslant 2$ be such that $\alpha > -1/p$. Now ℓ_p is of Rademacher type p, (2.2) holds with $D = 1$ and $V = V_1$, and (4.13) also holds since $\alpha p > -1$. It was shown by Adler and Rosalsky[1] that $P(Y_1 > a) < (q_o a)^{-1}$, $a > 0$, whence

$$nP\left\{\|DV_1\| > \frac{b_n}{|a_n|}\right\} = nP\{Y_1 > n \operatorname{Log} n\} = o(1)$$

The other hypotheses of Theorem 2 are easily seen to be satisfied. Thus by Theorem 2

$$\frac{\sum_{j=1}^{T_n} j^\alpha (V_{nj} - EV_{nj} I(\|V_{nj}\| \leqslant n \operatorname{Log} n))}{T_n^{\alpha+1} \operatorname{Log} T_n} \xrightarrow{P} 0 \tag{4.19}$$

Now $\alpha > -1$ ensures that

$$\frac{\sum_{j=1}^{n} j^\alpha}{n^{\alpha+1}} \to \frac{1}{\alpha+1}$$

and so it follows from $T_n/n \xrightarrow{P} 1$ and Lemma 3.3.2 of Chow and Teicher,[9] (p. 67) that

$$\frac{\sum_{j=1}^{T_n} j^\alpha}{T_n^{\alpha+1}} \xrightarrow{P} \frac{1}{\alpha+1} \tag{4.20}$$

Similarly,

$$\frac{\operatorname{Log} T_n}{\operatorname{Log} n} \xrightarrow{P} 1 \tag{4.21}$$

It was shown by Adler and Rosalsky[1] that

$$p_o q_o^{-1}((\operatorname{Log} a) - 1) \leqslant EY_1 I(Y_1 \leqslant a) \leqslant p_o q_o^{-1} \operatorname{Log} a, \; a \geqslant q_o^{-1}$$

and so

$$EY_1 I(Y_1 \leqslant n \operatorname{Log} n) \sim p_o q_o^{-1} \operatorname{Log} n \tag{4.22}$$

Set $v_o = \{P\{K_1 = k\}, k \geqslant 1\}$. Then by (4.20)–(4.22)

$$\frac{\sum_{j=1}^{T_n} j^\alpha EV_{nj} I(\|V_{nj}\| \leqslant n \operatorname{Log} n)}{T_n^{\alpha+1} \operatorname{Log} T_n} = \frac{\sum_{j=1}^{T_n} j^\alpha (EY_1 I(Y_1 \leqslant n \operatorname{Log} n))\, v_o}{T_n^{\alpha+1} \operatorname{Log} T_n}$$

$$\xrightarrow{P} \frac{p_o q_o^{-1}}{\alpha+1} v_o \tag{4.23}$$

Adding (4.19) and (4.23) yields

$$\frac{\sum_{j=1}^{T_n} j^\alpha V_{nj}}{T_n^{\alpha+1} \operatorname{Log} T_n} \xrightarrow{P} \frac{p_o q_o^{-1}}{\alpha+1} v_o \tag{4.24}$$

Next, suppose that $T_n \equiv n$, $n \geqslant 1$. Then

$$\frac{\sum_{j=1}^{n} j^\alpha V_{nj}}{n^{\alpha+1} \operatorname{Log} n} \xrightarrow{P} \frac{p_o q_o^{-1}}{\alpha+1} v_o$$

However, according to Theorem 2 of Adler and Rosalsky,[3] either

$$\liminf_{n\to\infty} \frac{\|\sum_{j=1}^{n} j^\alpha V_{nj}\|}{n^{\alpha+1} \operatorname{Log} n} = 0 \quad \text{a.c.} \qquad \text{or} \qquad \limsup_{n\to\infty} \frac{\|\sum_{j=1}^{n} j^\alpha V_{nj}\|}{n^{\alpha+1} \operatorname{Log} n} = \infty \quad \text{a.c.}$$

and so

$$\frac{\sum_{j=1}^{n} j^\alpha V_{nj}}{n^{\alpha+1} \operatorname{Log} n} \to \frac{p_o q_o^{-1}}{\alpha+1} v_o \quad \text{a.c.}$$

fails.

Remark 2. The conclusion (4.24) is stronger the closer p is chosen to 1 in view of the Pringsheim-Jensen inequality (see e.g., Hardy, Littlewood, and Pólya,[13] p. 28) which asserts that

$$\left(\sum_{k=1}^{\infty} |v_k|^p\right)^{1/p} \geqslant \left(\sum_{k=1}^{\infty} |v_k|^{p'}\right)^{1/p'} \qquad \text{for} \quad 0<p<p'<\infty \tag{4.25}$$

For any $\alpha > -1$, the choice of $p \in (1, 2]$ satisfying $\alpha > -1/p$ can always be made to be arbitrarily close to 1. Hence, in view of (4.25), the conclusion (4.24) must then hold for any $p \in (1, \infty)$.

Remark 3. Suppose that $\{V, V_{nj}, j \geqslant 1, n \geqslant 1\}$, $\{a_n, n \geqslant 1\}$, $\{b_n, n \geqslant 1\}$, $\{T_n, n \geqslant 1\}$, and $\{\alpha_n, n \geqslant 1\}$ satisfy the hypotheses of Theorem 2 with $T_n/\alpha_n \xrightarrow{P} c$ where $0<c<\infty$. It is natural to ask whether

$$\frac{\sum_{j=1}^{T_n} a_j(V_{nj} - EV_{nj} I(\|V_{nj}\| \leqslant b_{[T_n/c]}/|a_{[T_n/c]}|))}{b_{T_n}} \xrightarrow{P} 0 \tag{4.26}$$

necessarily holds. The following example shows that the answer is negative.

Example 2. Let $\{V_n, n \geqslant 1\}$ be nonnegative i.i.d. nonintegrable random variables with $nP\{V_1 > n\} = o(1)$ and let $V_{nj} = V_j, j \geqslant 1, n \geqslant 1$. Set $a_n = 1$ and $b_n = \alpha_n = n$, $n \geqslant 1$. Since V_1 is nonintegrable, there exists an integer sequence $\{A_n, n \geqslant 1\}$ with $EV_1 I(n < V_1 \leqslant A_n) \to \infty$. Let $\{T_n, n \geqslant 1\}$ be a sequence of random variables independent of $\{V_n, n \geqslant 1\}$ with $P\{T_n = A_n\} = p_n = 1 - P\{T_n = n\}$, $n \geqslant 1$ where $p_n = o(1)$. Then $T_n/n \xrightarrow{P} c = 1$. Now (4.1) holds for $1 < p \leqslant 2$ and so all of the hypotheses of Theorem 2 are satisfied with $1 < p \leqslant 2$, $D = 1$, and $V = V_1$. Then subtracting the expression in (4.26) from that in (4.18) (which is $o_P(1)$ by Theorem 2) yields

$$
\begin{aligned}
&EV_1 I(V_1 \leqslant T_n) - EV_1 I(V_1 \leqslant n) \\
&\quad = EV_1 I(n < V_1 \leqslant T_n) \\
&\quad = E\{V_1 I(n < V_1 \leqslant T_n)\, I(T_n = n)\} + E\{V_1 I(n < V_1 \leqslant T_n)\, I(T_n = A_n)\} \\
&\quad = 0 + E\{V_1 I(n < V_1 \leqslant A_n)\, I(T_n = A_n)\} \\
&\quad = (EV_1 I(n < V_1 \leqslant A_n))\, p_n
\end{aligned}
$$

Thus, (4.26) prevails if and only if $p_n = o((EV_1 I(n < V_1 \leqslant A_n))^{-1})$.

The next theorem provides sufficient conditions for (4.26) to hold with $c = 1$. Note that the condition (4.28) is considerably stronger than (4.16) and that the condition (4.27) is automatic if $|a_n| \uparrow$.

Theorem 3. Let $\{V, V_{nj}, j \geqslant 1, n \geqslant 1\}$, $\{a_n, n \geqslant 1\}$, and $\{b_n, n \geqslant 1\}$ satisfy the hypotheses of Theorem 1 with $b_n \uparrow$ and

$$\sum_{j=1}^{n} |a_j| = \mathcal{O}(n\, |a_n|) \tag{4.27}$$

Let $\{T_n, n \geqslant 1\}$ be positive integer-valued random variables and $1 \leqslant \alpha_n \to \infty$ be constants such that

$$\lambda' \leqslant \frac{T_n}{\alpha_n} \leqslant \lambda \text{ a.c.} \qquad \text{for some constants} \quad 0 < \lambda' \leqslant \lambda < \infty \tag{4.28}$$

Moreover, suppose that

$$b_{[\lambda^* \alpha_n]} = \mathcal{O}(b_{[\lambda_* \alpha_n]}) \tag{4.29}$$

where $\lambda^* = \lambda \vee 1$ and $\lambda_* = \lambda' \wedge 1$. Then the WLLN

$$\frac{\sum_{j=1}^{T_n} a_j(V_{nj} - EV_{nj} I(\|V_{nj}\| \leqslant b_{T_n}/|a_{T_n}|))}{b_{T_n}} \xrightarrow{P} 0 \tag{4.30}$$

obtains.

Proof. Set $c_n = b_n/|a_n|$, $n \geqslant 1$. Note that with probability 1,

$$\left\| \frac{\sum_{j=1}^{T_n} a_j(V_{nj} - EV_{nj} I(\|V_{nj}\| \leqslant c_{T_n}))}{b_{T_n}} - \frac{\sum_{j=1}^{T_n} a_j(V_{nj} - EV_{nj} I(\|V_{nj}\| \leqslant c_{[\alpha_n]}))}{b_{T_n}} \right\|$$

$$\leqslant \frac{1}{b_{T_n}} \sum_{j=1}^{T_n} |a_j| \, E\{ \|V_{nj}\| \, |I(\|V_{nj}\| \leqslant c_{T_n}) - I(\|V_{nj}\| \leqslant c_{[\alpha_n]})| \}$$

$$\leqslant \frac{1}{b_{[\lambda_* \alpha_n]}} \sum_{j=1}^{[\lambda^* \alpha_n]} |a_j| \, E \, \|V_{nj}\| \, I(c_{[\lambda_* \alpha_n]} < \|V_{nj}\| \leqslant c_{[\lambda^* \alpha_n]}) \qquad \text{(by (4.28))}$$

$$\leqslant \frac{1}{b_{[\lambda_* \alpha_n]}} \sum_{j=1}^{[\lambda^* \alpha_n]} |a_j| \, c_{[\lambda^* \alpha_n]} P\{ \|V_{nj}\| > c_{[\lambda_* \alpha_n]} \}$$

$$\leqslant \frac{D}{b_{[\lambda_* \alpha_n]}} \sum_{j=1}^{[\lambda^* \alpha_n]} |a_j| \, c_{[\lambda^* \alpha_n]} P\{ \|DV\| > c_{[\lambda_* \alpha_n]} \} \qquad \text{(by (2.2))}$$

$$\leqslant \frac{C}{|a_{[\lambda^* \alpha_n]}|} \sum_{j=1}^{[\lambda^* \alpha_n]} |a_j| \, P\{ \|DV\| > c_{[\lambda_* \alpha_n]} \} \qquad \text{(by (4.29))}$$

$$\leqslant C[\lambda^* \alpha_n] \, P\{ \|DV\| > c_{[\lambda_* \alpha_n]} \} \qquad \text{(by (4.27))}$$

$$\leqslant C[\lambda_* \alpha_n] \, P\{ \|DV\| > c_{[\lambda_* \alpha_n]} \}$$

$$= o(1) \qquad \text{(by (4.4))}$$

Then since (4.18) holds by Theorem 2, the conclusion (4.30) obtains. □

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