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COMPLETE CONVERGENCE FOR ARRAYS OF ROWWISE INDEPENDENT RANDOM VARIABLES

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Abstract. Under some conditions on an array of rowwise independent random variables, Hu et al.(1998) obtained a complete convergence result for law of large numbers with rate $\{a_n, n \geq 1\}$ which is bounded away from zero. We investigate the general situation for rate $\{a_n, n \geq 1\}$ under similar conditions.

1. Introduction

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [5] as follows. A sequence $\{U_n, n \geq 1\}$ of random variables *converges completely* to the constant θ if

$$\sum_{n=1}^{\infty} P(|U_n - \theta| > \epsilon) < \infty$$

for all $\epsilon > 0$. We refer to [3] for a survey on results on complete convergence related to strong laws.

Recently, Hu et al. [6] and Hu and Volodin [8] proved the following complete convergence theorem for arrays of rowwise independent random variables.

Theorem 1. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent random variables and $\{a_n, n \geq 1\}$ a sequence of positive*

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constants bounded away from zero, that is, $\liminf_{n\to\infty} a_n > 0$. *Suppose that for every* $\epsilon > 0$ *and some* $\delta > 0$:

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(|X_{ni}| > \epsilon) < \infty$,

(ii) *there exists* $J \geq 2$ *such that*

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} EX_{ni}^2 I(|X_{ni}| \leq \delta) \right)^J < \infty,$$

(iii) $\sum_{i=1}^{k_n} EX_{ni} I(|X_{ni}| \leq \delta) \to 0$ *as* $n \to \infty$.

Then $\sum_{n=1}^{\infty} a_n P(|\sum_{i=1}^{k_n} X_{ni}| > \epsilon) < \infty$ *for all* $\epsilon > 0$.

This result was generalized on Banach space setting in [7].

The proof of Theorem 1 is based on the fact that

$$\sum_{i=1}^{k_n} X_{ni} \to 0 \text{ in probability} \tag{1}$$

as $n \to \infty$. We mention that (1) does not necessarily follow from the conditions of Theorem 1 if $\{a_n, n \geq 1\}$ is not bounded away from zero. To give such an example, we will need the following lemma.

Lemma 1. *If the random variable* X *is* $N(0,1)$, *then for every* $\epsilon > 0$

$$P(X > \epsilon) \leq e^{-\frac{\epsilon^2}{2}}.$$

Proof. For any $t > 0$,

$$P(X > \epsilon) = P(tX > t\epsilon) = P(e^{tX} > e^{t\epsilon}) \leq e^{-t\epsilon} E[e^{tX}] = e^{-t\epsilon + \frac{t^2}{2}},$$

since X has moment generating function $e^{\frac{t^2}{2}}$. The result follows by putting $t = \epsilon$. $\square$

Remark 1. It is well known that $P(X > \epsilon) \leq \frac{1}{\epsilon\sqrt{2\pi}} \exp(-\frac{\epsilon^2}{2})$(see [2], p. 175). Hence, the upper bound of $P(X > \epsilon)$ in Lemma 1 is good when $0 < \epsilon < \frac{1}{\sqrt{2\pi}}$.

EXAMPLE 1. Define a sequence $\{a_n, n \geq 1\}$ by

$$a_n = \begin{cases} 1/n^2, & \text{if } n \text{ is odd,} \\ 1/n, & \text{if } n \text{ is even.} \end{cases}$$

Let $X_1, X_2, \cdots$ be independent and identically distributed $N(0,1)$ random variables. Define an array $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ by

$$X_{ni} = \begin{cases} X_i/\sqrt{n}, & \text{if } n \text{ is odd and } 1 \leq i \leq n, \\ X_i/n, & \text{if } n \text{ is even and } 1 \leq i \leq n. \end{cases}$$

Then we have by Lemma 1 that

$$P(|X_{n1}| > \epsilon) \leq \begin{cases} 2\ \exp(-n\epsilon^2/2), & \text{if } n \text{ is odd,} \\ 2\ \exp(-n^2\epsilon^2/2), & \text{if } n \text{ is even.} \end{cases}$$

It follows that

$$\sum_{n=1}^{\infty} a_n \sum_{i=1}^{n} P(|X_{ni}| > \epsilon) = \sum_{n \text{ is odd}} \frac{1}{n} P(|X_{n1}| > \epsilon) + \sum_{n \text{ is even}} P(|X_{n1}| > \epsilon)$$

$$\leq 2\left[\sum_{n \text{ is odd}} \exp(-\frac{n\epsilon^2}{2})/n + \sum_{n \text{ is even}} \exp(-\frac{n^2\epsilon^2}{2})\right] < \infty,$$

and so the condition (i) of Theorem 1 holds. Next, we claim that conditions (ii) and (iii) hold. Noting that

$$EX_{n1}^2 = \begin{cases} 1/n, & \text{if } n \text{ is odd,} \\ 1/n^2, & \text{if } n \text{ is even,} \end{cases}$$

we get

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{n} EX_{ni}^2 I(|X_{ni}| \leq \delta)\right)^J \leq \sum_{n=1}^{\infty} a_n (nEX_{n1}^2)^J$$

$$= \sum_{n \text{ is odd}} \frac{1}{n^2} + \sum_{n \text{ is even}} \frac{1}{n^{1+J}} < \infty,$$

which implies (ii). Since X_{ni} is symmetric, $EX_{ni}I(|X_{ni}| \leq \delta) = 0$. Thus (iii) holds. But, (1) does not hold, since for odd n

$$\sum_{i=1}^{n} X_{ni} = \frac{X_1 + \cdots + X_n}{\sqrt{n}} \sim N(0,1).$$

However, it is easy to see that $\sum_{n=1}^{\infty} a_n P(|\sum_{i=1}^{k_n} X_{ni}| > \epsilon) < \infty$ for all $\epsilon > 0$.

Remark 2. For a different example and a general discussion about Theorem 1 we refer to [8].

It is an interesting project to investigate whether Theorem 1 is true or not for general sequences. In this paper, we obtain a complete convergence result without assuming that $\{a_n, n \geq 1\}$ is bounded away from zero, but under slightly modified conditions of Theorem 1. The proof is different from that of Hu et al. [6] and it does not use symmetrization procedure.

2. Main result

To prove the main result, we will need the following lemma which is a version of Hoffmann-Jørgensen [4] inequality for independent, but not necessarily symmetric, random variables.

Lemma 2. *Let $X_1, \cdots, X_n$ be independent random variables. Let $S_i = \sum_{l=1}^{i} X_l, 1 \leq i \leq n$, and let $S_0 \equiv 0$. Then for every integer $j \geq 1$ and $t > 0$*

(2)

$$P(|S_n| > 6^j t) \leq C_j P(\max_{1\leq i\leq n} |X_i| > \frac{t}{4^{j-1}}) + D_j \max_{1\leq i\leq n} \left[P(|S_i| > \frac{t}{4^j}) \right]^{2^j}$$

for some positive constants C_j and D_j depending only on j.

Proof. From Lemma 1 and Lemma 2 in [1], it follows that

$$P(|S_n| > 6t) \leq P(\max_{1\leq i\leq n} |X_i| > t) + 64 \max_{1\leq i\leq n} \left[P(|S_i| > \frac{t}{4}) \right]^2. \tag{3}$$

Thus (2) holds for $j = 1$ with $C_1 = 1$ and $D_1 = 64$. Assume that (2) holds for some j for some positive constants C_j and D_j. Then using (3), we have

$$\begin{aligned} & P(|S_n| > 6^{j+1} t) \\ \leq\ & P(\max_{1\leq i\leq n} |X_i| > 6^j t) + 64 \max_{1\leq i\leq n} \left[P(|S_i| > \frac{6^j t}{4}) \right]^2 \\ \leq\ & P(\max_{1\leq i\leq n} |X_i| > 6^j t) \end{aligned}$$

$$
\begin{aligned}
&+\ 64 \max_{1\le i\le n}\Big[C_j P(\max_{1\le l\le i}|X_l| > \frac{t}{4^j}) + D_j \max_{1\le l\le i}\Big[P(|S_l| > \frac{t}{4^{j+1}})\Big]^{2^j}\Big]^2 \\
&= P(\max_{1\le i\le n}|X_i| > 6^j t) + 64\Big[C_j^2\Big[P(\max_{1\le i\le n}|X_i| > \frac{t}{4^j})\Big]^2 \\
&+\ 2C_jD_jP(\max_{1\le i\le n}|X_i| > \frac{t}{4^j})\max_{1\le i\le n}\Big[P(|S_i| > \frac{t}{4^{j+1}})\Big]^{2^j} \\
&+\ D_j^2 \max_{1\le i\le n}\Big[P(|S_i| > \frac{t}{4^{j+1}})\Big]^{2^{j+1}}\Big] \\
&\le P(\max_{1\le i\le n}|X_i| > 6^j t) \\
&+\ 64C_j^2 P(\max_{1\le i\le n}|X_i| > \frac{t}{4^j}) + 128C_jD_jP(\max_{1\le i\le n}|X_i| > \frac{t}{4^j}) \\
&+\ 64D_j^2 \max_{1\le i\le n}\Big[P(|S_i| > \frac{t}{4^{j+1}})\Big]^{2^{j+1}} \\
&\le (1 + 64C_j^2 + 128C_jD_j)P(\max_{1\le i\le n}|X_i| > \frac{t}{4^j}) \\
&+\ 64D_j^2 \max_{1\le i\le n}\Big[P(|S_i| > \frac{t}{4^{j+1}})\Big]^{2^{j+1}}.
\end{aligned}
$$

Hence, we can take $C_{j+1} = 1 + 64C_j^2 + 128C_jD_j$ and $D_{j+1} = 64D_j^2$. □

Now, let $\{a_n, n \ge 1\}$ be a sequence of positive constants without the assumption that it is of bounded away from zero. We state and prove our main result.

THEOREM 2. *Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be as in Theorem 1 except that* (ii) *and* (iii) *are replaced by* (ii′) *and* (iii′), *respectively:*

(ii′) *there exists $J \ge 2$ such that*

$$\sum_{n=1}^{\infty} a_n \Big(\sum_{i=1}^{k_n} Var(X_{ni}I(|X_{ni}| \le \delta))\Big)^J < \infty,$$

(iii′) $\max_{1\le i\le k_n} |\sum_{l=1}^{i} EX_{nl}I(|X_{nl}| \le \delta)| \to 0$ *as* $n \to \infty$.

Then $\sum_{n=1}^{\infty} a_n P(|\sum_{i=1}^{k_n} X_{ni}| > \epsilon) < \infty$ *for all* $\epsilon > 0$.

Proof. Let $X'_{ni} = X_{ni}I(|X_{ni}| \le \delta), X''_{ni} = X_{ni}I(|X_{ni}| > \delta)$ for $1 \le i \le k_n, n \ge 1$. Then

$$P(|\sum_{i=1}^{k_n} X_{ni}| > \epsilon) \le P(|\sum_{i=1}^{k_n} X'_{ni}| > \frac{\epsilon}{2}) + P(|\sum_{i=1}^{k_n} X''_{ni}| > \frac{\epsilon}{2})$$
$$\le P(|\sum_{i=1}^{k_n} X'_{ni}| > \frac{\epsilon}{2}) + \sum_{i=1}^{k_n} P(|X_{ni}| > \delta).$$

By (i), it suffices to estimate $P(|\sum_{i=1}^{k_n} X'_{ni}| > \frac{\epsilon}{2})$. Take j such that $2^j \ge J$. Then we have by Lemma 2 that

$$P(|\sum_{i=1}^{k_n} X'_{ni}| > \frac{\epsilon}{2})$$
$$\le C_j P(\max_{1\le i\le k_n} |X'_{ni}| > \frac{2\epsilon}{24^j}) + D_j \max_{1\le i\le k_n} P(|\sum_{l=1}^{i} X'_{nl}| > \frac{\epsilon}{2\cdot 24^j})^{2^j}$$
$$\le C_j \sum_{i=1}^{k_n} P(|X_{ni}| > \frac{2\epsilon}{24^j}) + D_j \max_{1\le i\le k_n} P(|\sum_{l=1}^{i} X'_{nl}| > \frac{\epsilon}{2\cdot 24^j})^{J}.$$

Hence by (i) it suffices to estimate $\max_{1\le i\le k_n} P(|\sum_{l=1}^{i} X'_{nl}| > \frac{\epsilon}{2\cdot 24^j})^J$. On the other hand, condition (iii′) implies that there exists an integer N such that

$$\max_{1\le i\le k_n} |\sum_{l=1}^{i} EX'_{nl}| < \frac{\epsilon}{4\cdot 24^j} \text{ if } n \ge N.$$

For $n \ge N$, we get by the Markov's inequality that

$$\max_{1\le i\le k_n} P(|\sum_{l=1}^{i} X'_{nl}| > \frac{\epsilon}{2\cdot 24^j})^J$$
$$\le \max_{1\le i\le k_n} P(|\sum_{l=1}^{i}(X'_{nl} - EX'_{nl})| + |\sum_{l=1}^{i} EX'_{nl}| > \frac{\epsilon}{2\cdot 24^j})^J$$
$$\le \max_{1\le i\le k_n} P(|\sum_{l=1}^{i}(X'_{nl} - EX'_{nl})| > \frac{\epsilon}{4\cdot 24^j})^J$$

$$\leq (\frac{4\cdot 24^j}{\epsilon})^{2J} \max_{1\leq i\leq k_n} \left(Var(\sum_{l=1}^{i} X'_{nl}) \right)^J$$
$$= (\frac{4\cdot 24^j}{\epsilon})^{2J} \left(\sum_{i=1}^{k_n} Var(X'_{ni}) \right)^J.$$

In view of (ii′), the proof is complete. □

REMARK 3. Condition (ii′) in Theorem 2 is a slight modification of condition (ii) in Theorem 1. Although condition (iii′) in Theorem 2 is stronger than condition (iii) in Theorem 1, Corollary 1 and Corollary 2 in [6] can be proved by Theorem 2.

Theorem 2 can be generalized to Banach space setting. Recall that a real separable Banach space $(B, \|\ \|)$ is said to be of (Rademacher) *type* $p, 1 \leq p \leq 2$, if there exists a positive constant C such that

$$E\|\sum_{i=1}^{n} X_i\|^p \leq C\sum_{i=1}^{n} E\|X_i\|^p$$

for all independent mean zero and finite p-th moment random elements $X_1, \cdots, X_n$ with values in B. For discussion of this notion and some equivalent definitions, see [10].

Let us mention that a version of Hoffmann-Jørgensen [4] inequality (Lemma 2) is still valid for independent, but not necessarily symmetric, random elements with values in B. For a random element X with expected value and $p > 0$ denote $\sigma_p(X) = E\|X - EX\|^p$.

THEOREM 3. *Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent random elements taking values in a real separable Banach space $(B, \|\ \|)$ of type $p, 1 \leq p \leq 2$, and $\{a_n, n \geq 1\}$ a sequence of positive constants. Suppose that for every $\epsilon > 0$ and some $\delta > 0$:*

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \epsilon) < \infty$,

(ii) *there exists $J \geq 2$ such that*

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{k_n} \sigma_p(X_{ni}I(\|X_{ni}\| \leq \delta)) \right)^J < \infty,$$

(iii) $\max_{1\leq i\leq k_n} \|\sum_{l=1}^{i} EX_{nl}I(\|X_{nl}\| \leq \delta)\| \to 0$ *as* $n \to \infty$.

Then $\sum_{n=1}^{\infty} a_n P(\|\sum_{i=1}^{k_n} X_{ni}\| > \epsilon) < \infty$ *for all* $\epsilon > 0$.

Proof. Let $X'_{ni} = X_{ni} I(\|X_{ni}\| \le \delta)$ for $1 \le i \le k_n, n \ge 1$. If $k_n = \infty$ we have to prove that the series $\sum_{i=1}^{\infty} X_{ni}$ converges a.s. By Corollary 2.2.1 in [9] it is sufficient to prove that for some $\delta > 0$:

(a) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{k_n} P(\|X_{ni}\| > \delta) < \infty$,
(b) $\sum_{i=1}^{\infty} X'_{ni}$ converges a.s.

Condition (a) is satisfied by (i). Since the Banach space is of type p, for any positive integer m we have $\sigma_p(\sum_{i=1}^{m} X'_{ni}) \le C \sum_{i=1}^{m} \sigma_p(X'_{ni})$. By (ii) $\sum_{i=1}^{\infty} \sigma_p(X'_{ni}) < \infty$. This implies that $\sum_{i=1}^{\infty}(X'_{ni} - EX'_{ni})$ converges a.s. Hence (b) is satisfied by (iii). The rest of the proof is the same as that in Theorem 2 except that

$$\max_{1\le i\le k_n} P(\|\sum_{l=1}^{i}(X'_{nl} - EX'_{nl})\| > \frac{\epsilon}{4\cdot 24^j})^J$$
$$\le (\frac{4\cdot 24^j}{\epsilon})^{pJ} \max_{1\le i\le k_n}\left(\sigma_p(\sum_{l=1}^{i} X'_{nl})\right)^J$$
$$\le C(\frac{4\cdot 24^j}{\epsilon})^{pJ}\left(\sum_{i=1}^{k_n}\sigma_p(X'_{ni})\right)^J,$$

since B is of type p. □

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