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ON THE MARCINKIEWICZ WEAK LAWS OF LARGE NUMBERS IN BANACH SPACES

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ABSTRACT

Some general theorems concerning the weak law of large numbers are proved. Necessary and sufficient conditions for the validity of the weak law of large numbers and the stochastic boundedness of weighted sums (under some conditions on weights are found) of independent identically distributed random elements with zero means assuming values in an arbitrary Banach space. These results are applied to certain random elements.

INTRODUCTION

The interesting problem arising in the study of the laws of large numbers is to find necessary and sufficient conditions for the validity of the weak law and the stochastic boundedness of weighted sums (under certain conditions on the weights) of independent identically distributed random elements with zero means assuming values in an Banach space. Note that in applications of the law of large numbers its weak variant and stochastic boundedness are of the utmost importance.

Let us introduce some notations and formulate the conditions ensuring the validity of the main result of this section.

Let X be a random element with zero mean assuming values in a Banach space E, $(X_k)_1^\infty$ be independent copies of X,

$$T_n = T_n(X) = \sum_{k=1}^{n} a_k(n) X_k,$$

where $\mathbf{a} = (a_k(n),\ 1 \le k \le n,\ n \in \mathbf{N})$ is a triangular array of constants which is called a weight in what follows, and let

$$S_n = S_n(X) = \sum_{k=1}^{n} X_k.$$

Let any weight $\mathbf{a}$ under consideration satisfy the following assumptions.

For all $k \in \mathbf{N}$

(i) $\lim_{n\to\infty} |a_k(n)| = 0$, and there exists $B > 0$ such that for all $m > n$: $|a_k(m)| \le B|a_k(n)|$.

(ii) there exists $C > 1/2$ such that for all n: $|a_k(2n)| \ge C|a_k(n)|$.

Introduce the sequence $\varphi = (\varphi(n))$: $\varphi(n) = 1/\max_{1\le k\le n} |a_k(n)|$. Since we are interested in the law of large numbers, it is natural to use the following condition:

(iii) there exist $p < 2$ and $A > 0$ such that $\varphi(n) \ge An^{1/p}$ for all n.

Note that condition (iii) is necessary even for the random variables assuming values in $\mathbb{R}$.

Let us say a few words about condition (ii). We can assume that $C < 1$. To simplify the proofs of some technical results, we define the functions $a_k(t)$ and $\varphi(t)$, $t \ge 1$, $a_k(t) = a_k([t])$, $\varphi(t) = \varphi([t])$, where $[\cdot]$ denotes the integer part of a number. Then (ii) can be rewritten as

(ii′) $|a_k(tn)| \ge D\, t^q\, |a_k(n)|$ for all $n \in \mathbb{N}$ and $t \ge 1$, where $q = \log_2 C$ (< 0) and $D = B^2/C$.

In fact, if by $\langle t\rangle$ we denote the last integer number not smaller than t (≥ 1), then, by (i) and (ii), we have

$$|a_k(tn)| \ge B|a_k(\langle t\rangle n)| = B|a_k(2^{\log_2\langle t\rangle}n)| \ge B^2|a_k(2^{\langle\log_2\langle t\rangle\rangle}n)|$$
$$\ge B^2C^{\langle\log_2\langle t\rangle\rangle}|a_k(n)| \ge B^2C^{\log_2 t-1}|a_k(n)| \ge (B^2/C)t^q|a_k(n)|.$$

We say that a sequence of random elements (Y_n) is stochastically bounded, if $\lim_{t\to\infty}\sup_n \mathbf{P}\{\|Y_n\| > t\} = 0$.

Let us introduce the following notations:

$$\mathbf{SB}_a(E) = \{X \in L_0(E)\colon \mathbf{E}X = 0, \text{ and the sequence } (T_n(X)) \text{ is stochastically bounded}\},$$

$$\mathbf{WLLN}_a(E) = \{X \in L_0(E)\colon \mathbf{E}X = 0,\ T_n(X) \to 0 \text{ as } n\to\infty \text{ in probability}\},$$

$$\mathbf{WLLN}_p(E) = \{X \in L_0(E)\colon \mathbf{E}X = 0,\ S_n(X)/n^{1/p} \to 0 \text{ as } n\to\infty \text{ in probability}\},\quad 1 < p < 2.$$

In the sequel, we often use the following inequalities. In their formulations given below X_k are independent random elements and $t > 0$.

0.1. *Levy's inequality* (Vakhania et al., 1987, Proposition V.2.3)
If X_k are symmetric, then

$$\mathbf{P}\{\max_{1\le k\le n} \|X_k\| > t\} \le 2\mathbf{P}\{\|\sum_{k=1}^{n} X_k\| > t\}.$$

0.2. *Kolmogorov's inequality* (Kwapien and Woyczynski, 1987)
If X_k have zero means, then, whenever the right-hand side below is positive,

$$\mathbf{E}\|\sum_{k=1}^{n} X_k\| \leq 3(\mathbf{E}\max_{1\leq k\leq n}\|X_k\| + 8t)/(1 - 12\mathbf{P}\{\|\sum_{k=1}^{n} X_k\| > t\}).$$

0.3. *Kwapien's inequality* (Vakhania et. al., 1987, Lemma V.4.1 (a))
If X_k are symmetric and $|\beta_k| \geq |\alpha_k|$, then

$$\mathbf{P}\{\|\sum_{k=1}^{n} \alpha_k X_k\| > t\} \leq 2\mathbf{P}\{\|\sum_{k=1}^{n} \beta_k X_k\| > t\}.$$

0.4. *Hoffmann–Jorgencen's inequality* (Hoffmann–Jorgencen, 1974)
If X_k are symmetric, then

$$\mathbf{P}\{\|\sum_{k=1}^{n} X_k\| > 3t\} \leq 4\mathbf{P}^2\{\|\sum_{k=1}^{n} X_k\| > t\} + \mathbf{P}\{\max_{1\leq k\leq n}\|X_k\| > t\}.$$

0.5. *Contraction principle* (Vakhania et al., 1987, Lemma V.4.1 (c))
If $\alpha_k \in \mathbf{R}$, then

$$\mathbf{E}\|\sum_{k=1}^{n} \alpha_k X_k\| \leq 2\max_{1\leq k\leq n}|\alpha_k|\mathbf{E}\|\sum_{k=1}^{n} X_k\|.$$

0.6. *de Acosta's inequality* (Acosta, 1981)
There exists $C > 0$ such that

$$\mathbf{E}|\,\|\sum_{k=1}^{n} X_k\|^2 - \mathbf{E}\|\sum_{k=1}^{n} X_k\|^2| \leq C\sum_{k=1}^{n}\mathbf{E}\|X_k\|^2.$$

0.7. *Inequality for means* (Araujo and Gine, 1980, Lemma III.2.7)
If $\mathbf{E}X_1 = 0$, then $\mathbf{E}\|X_2\| \leq \mathbf{E}\|X_1 + X_2\|$.
0.8. *Elementary inequality*
For all $t \leq 0$

$$\mathbf{P}\{\|\sum_{k=1}^{n} X_k\| > t\} \leq \sum_{k=1}^{n}\mathbf{P}\{\|X_k\| > t\} + \mathbf{P}\{\|\sum_{k=1}^{n} X_k I\{\|X_k\| \leq t\}\| > t\}.$$

Note that one usually regards the relation

$$\mathbf{P}\{\|\sum_{k=1}^{n} \alpha_k X_k\| > t\} \leq 2\mathbf{P}\{\max_{1\leq k\leq n}|\alpha_k|\,\|\sum_{k=1}^{n} X_k\| > t\}$$

as Kwapien's inequality.
Our inequality follows easily from this one, because

$$\mathbf{P}\{\|\sum_{k=1}^{n} \alpha_k X_k\| > t\} \leq \mathbf{P}\{\|\sum_{k=1}^{n}(\alpha_k/\beta_k)\beta_k X_k\| > t\}$$

$$\leq 2\mathbf{P}\{\max_{1\leq k\leq n}|\alpha_k/\beta_k|\,\|\sum_{k=1}^{n} \beta_k X_k\| > t\} \leq 2\mathbf{P}\{\|\sum_{k=1}^{n} \beta_k X_k\| > t\}.$$

1. CONDITIONS OF STOCHASTIC BOUNDEDNESS

In this section we state the necessary and sufficient condition for stochastic boundedness of weighed sums of i.i.d. random elements zero means. Note that in this section we do not use condition (iii).

Let us formulate and prove some technical results with the aid of which we shall prove the main statement of this section.

For $s > 0$ and $t > 0$ set $Y_{k,s}(tn) = a_k(tn)X_k I\{\|a_k(tn)X_k\| \le s\}$.

LEMMA 1.1. *Let X be a symmetric random element. Consider the conditions*

a) $\lim_{s\to\infty} \sup_{t>0} t\mathbf{P}\{\|X\| > s\varphi(t)\} = 0$,

b) $\lim_{s\to\infty} \sup_{n} \left(\mathbf{E}\|\sum_{k=1}^{n} Y_{k,s}(n)\|/s\right) = 0$,

c) *there exists $u > 0$ such that*

$$\sup_{s>u}\sup_{n}\sup_{t\ge 1} \frac{\mathbf{E}\|\sum_{k=1}^{n} Y_{k,s}(tn)\|}{s} < \infty.$$

Conditions a) *and* b) *are sufficient, and conditions* a) *and* c) *are necessary for the inclusion $X \in \mathrm{SB}_a(E)$.*

Proof. Sufficiency. Let a) and b) hold true. Set

$$T_n' = \sum_{k=1}^{n} a_k(n)X_k I\{\|a_k(n)X_k\| \le s\} = \sum_{k=1}^{n} Y_{k,s}(n),$$

$$T_n'' = \sum_{k=1}^{n} a_k(n)X_k I\{\|a_k(n)X_k\| > s\}.$$

In this notation, $T_n = T_n' + T_n''$, and it suffices to show that $\sup_n \mathbf{P}\{\|T_n'\| > s\} \to 0$ and $\sup_n \mathbf{P}\{\|T_n''\| > s\} \to 0$ as $s \to \infty$.

By Chebyshev's inequality and condition b),

$$\sup_n \mathbf{P}\{\|T_n'\| > s\} \le \sup_n \frac{\mathbf{E}\|T_n'\|}{s} = \sup_n \frac{\mathbf{E}\|\sum_{k=1}^{n} Y_{k,s}(n)\|}{s} \to 0 \quad \text{as} \quad s \to \infty.$$

Furthermore, we have

$$\{\|T_n''\| > s\} \subset \{\|T_n''\| > 0\}$$
$$= \{\|\sum_{k=1}^{n} a_k(n)X_k I\{\|a_k(n)X_k\| > s\}\| > 0\} \subset \bigcup_{k=1}^{n}\{\|a_k(n)X_k\| > s\},$$

hence, by condition a),

$$\mathbf{P}\{\|T_n''\| > s\} \le \sum_{k=1}^{n} \mathbf{P}\{\|a_k(n)X_k\| > s\} \le \sum_{k=1}^{n} \mathbf{P}\{\|X_k\| > s\varphi(n)\}$$
$$= n\mathbf{P}\{\|X\| > s\varphi(n)\} \to 0 \quad \text{as} \quad s \to \infty.$$

Necessity. Now let $X \in \mathbf{SB}_a(E)$. Then $f(s) \equiv \sup_n \mathbf{P}\{\|T_n\| > s\} \to 0$ as $s \to \infty$. By Levy's inequality (0.1), for all n $f(s) \geq \mathbf{P}\{\max_{1\leq k\leq n} |a_k(n)|\,\|X_k\| > s\}/2$ or

$$1 - 2f(s) \leq \mathbf{P}\{\max_{1\leq k\leq n} \|X_k\| \leq s\varphi(n)\}$$

$$= \mathbf{P}\{\bigcap_{k=1}^{n}\{\|X_k\| \leq s\varphi(n)\}\} = \mathbf{P}^n\{\|X\| \leq s\varphi(n)\}.$$

Hence, $\mathbf{P}\{\|X\| > s\varphi(n)\} \leq 1 - (1 - f(s))^{1/n} \leq -\log(1 - 2f(s))/n$. So, $\lim_{s\to\infty} \sup_{t>0} t\mathbf{P}\{\|X\| > s\varphi(t)\} = 0.$

Furthermore, for all u, $0 < u \leq 2s$, we have

$$\mathbf{P} = \mathbf{P}\{\|\sum_{k=1}^{n} Y_{k,s}(tn)\| > u\}$$

$$= \mathbf{P}\{\|\sum_{k=1}^{n} a_k(tn)X_k - \sum_{k=1}^{n} a_k(tn)X_k I\{\|a_k(tn)X_k\| > s\}\| > u\}$$

$$\leq \mathbf{P}\{\|\sum_{k=1}^{n} a_k(tn)X_k\| > \frac{u}{2}\} + \mathbf{P}\{\|\sum_{k=1}^{n} a_k(tn)X_k I\{\|a_k(tn)X_k\| > s\}\| > \frac{u}{2}\}.$$

We estimate the first term in the right-hand side of the previous inequality by Kwapien's inequality (0.3) taking into account (i) and $t \geq 1$ and the second term by elementary inequality (0.8). As a result, we obtain

$$\mathbf{P} \leq 2\mathbf{P}\left\{\|T_n\| > \frac{u}{2B}\right\} + \sum_{k=1}^{n} \mathbf{P}\left\{\|a_k(tn)X_k I\{\|a_k(tn)X_k\| > s\}\| > \frac{u}{2}\right\}$$

$$+\mathbf{P}\left\{\|\sum_{k=1}^{n} a_k(tn)X_k I\{\|a_k(tn)X_k\| > s\} I\left\{\|a_k(tn)X_k\| \leq \frac{u}{2}\right\}\right\}.$$

By the choice of $u < 2s$, the latter event is impossible and

$$\mathbf{P} \leq 2\mathbf{P}\left\{\|T_n\| > \frac{u}{2B}\right\} + n\mathbf{P}\left\{\|X\| > \frac{u\varphi(n)}{2}\right\}.$$

Since $X \in \mathbf{SB}_a(E)$ and a) takes place, there exists $U > 0$ such that for all $s \geq U/2$

$$\sup_n \sup_{t\geq 1} \mathbf{P}\left\{\|\sum_{k=1}^{n} Y_{k,s}(tn)\| > U\right\} \leq \frac{1}{24}.$$

Then, by Kolmogorov's inequality (0.2),

$$\mathbf{E}\|\sum_{k=1}^{n} Y_{k,s}(tn)\| \leq \frac{3(\mathbf{E}\max_{1\leq k\leq n}\|Y_{k,s}(tn)\| + 8U)}{1 - 12\mathbf{P}\left\{\|\sum_{k=1}^{n} Y_{k,s}(tn)\| > U\right\}} \leq 6(s + 8U).$$

For any $s > 0$ and any random element X define $\rho_{n,s}(X) = \sup_{t\geq 0} t\mathbf{P}\{t^q\|T_n(X)\| > s\}$. Recall (see (ii)') that $q = \log_2 C$ and C is given by (ii).

LEMMA 1.2. *There exists $H > 0$ such that $\mathbf{E}\|T_n(X)\| \leq Hs(\rho_{n,s}(X) + \rho_{n,s}^{2+1/q}(X))$ for all symmetric random elements X and $s > 0$.*

Proof. For the sake of bravity, we put $\rho = \rho_{n,s}(X)$. We have

$$\mathbf{E}\|T_n(X)\| = \int_0^\infty \mathbf{P}\{\|T_n(X)\| > t\}dt$$

$$= s\int_0^\infty \mathbf{P}\{\|T_n\| > st\}dt = s\Big(\int_0^\rho \mathbf{P}\{\|T_n\| > st\}dt\Big) + \int_\rho^\infty \mathbf{P}\left\{\|\frac{T_n}{t}\| > s\right\} dt$$

$$\leq s\Big(\rho + \int_\rho^\infty t^{-1/q}\mathbf{P}\left\{(t^{-1/q})^q\|T_n\| > s\right\} t^{1/q}dt\Big) \leq s\Big(\rho + \rho\int_\rho^\infty t^{1/q}dt\Big).$$

Since we have $C > 1/2$ in condition (ii), it follows that $q > -1$ and the last integral converges. The straightforward computation completes the proof.
On the ground of these lemmas, we prove

THEOREM 1.1. *Let X be a symmetric random element. The inclusion $X \in \mathbf{SB}_a(E)$ holds, if and only if $\lim_{s\to\infty}\sup_n \rho_{n,s}(X) = 0$.*

Proof. Necessity. Let $X \in \mathbf{SB}_a(E)$, $L = 3^{1/q}$, and $0 < \varepsilon < L/2$. We shall prove that there exists an $s_0 = s_0(\varepsilon)$ such that $\rho_{n,s}(X) \leq \varepsilon$ for all $s > s_0$ and $n \in \mathbf{N}$. Note that $\rho_{n,s}(X) = \sup_{t\geq 0} t\mathbf{P}\{t^q\|T_n(X)\| > s\} = \sup_{t\geq 0} D(t) = \max(\sup_{0<t\leq\varepsilon} D(t), \sup_{\varepsilon<t\leq 1} D(t), \sup_{t>1} D(t)) \leq \max(\varepsilon, \sup_{\varepsilon<t\leq 1} D(t), \sup_{t>1} D(t))$. Moreover, $\sup_{\varepsilon<t\leq 1} D(t) \leq \mathbf{P}\{\varepsilon^q\|T_n\| > s\}$ and, since $X \in \mathbf{SB}_a(E)$, there exists $s_1 = s_1(\varepsilon)$ such that $\sup_{\varepsilon<t\leq 1} D(t) \leq \varepsilon$ for all $s > s_1$.
Furthermore, there exists $s_2 = s_2(\varepsilon)$ such that for all $s > s_2$ and $n \in \mathbf{N}$ we have

$$\mathbf{P}\{\|T_n(X)\| > s\} < \frac{\varepsilon}{4} < \frac{L}{8} \tag{1}$$

(the last inequality follows from the choice of ε). By assertion (1) of Lemma 1.1, there exists $s_3 = s_3(\varepsilon)$ such that

$$\sup_{t>0} t\mathbf{P}\{\|X\| > s\varphi(t)\} < \frac{\varepsilon}{4} \tag{2}$$

for all $s > s_3$.

Let $s > s_0(\varepsilon) = \max(s_1, s_2, s_3)$ and $A > L$. Then, according to inequality (1),

$$I(A) \equiv \sup_{1<t<A} t\mathbf{P}\{t^q\|T_n\| > s\} = \sup_{1\le t\le L} D(t) + \sup_{L<t<A} D(t)$$

$$\le L\mathbf{P}\{\|T_n\| > s\} + \sup_{L<t<A} D(t) \le \frac{\varepsilon}{4} + \sup_{L<t<A} t\mathbf{P}\{3t^q\|T_n(X)\| > 3s\}.$$

Furthermore, by the choice of L,

$$I(A) \le \frac{\varepsilon}{4} + \sup_{1<u<A/L} \frac{u}{L}\mathbf{P}\{\|u^q T_n\| > 3s\}.$$

Applying Hoffmann–Jorgensen inequality (0.4), we obtain

$$I(A) \le \frac{\varepsilon}{4} + \frac{1}{L}\Big(4 \sup_{1<u<A/L} u\mathbf{P}^2\{\|u^q T_n\| > s\} +$$

$$\sup_{1<u<A/L} u\mathbf{P}\{\max_{1\le k\le n} \|u^q a_k(n)X_k\| > s\}\Big)$$

$$\le \frac{\varepsilon}{4} + \frac{4}{L}\Big(\mathbf{P}\{\|T_n\| > s\} \sup_{1<u<A/L} u\mathbf{P}\{\|u^q T_n\| > s\} +$$

$$\sup_{n>1} u \sum_{k=1}^{n} \mathbf{P}\{\|u^q a_k(n)X_k\| > s\}\Big).$$

Then, by condition (ii)′, in view of the inequality $L > 1$, and by virtue of (1) and (2), we have

$$I(A) \le \frac{\varepsilon}{4} + 4\mathbf{P}\{\|T_n\| > s\}D \sup_{1<u<A/L} \frac{u}{L}\mathbf{P}\{\|u^q T_n\| > s\}$$

$$+ \sup_{n>1} un\mathbf{P}\{\|X\| > s\varphi(un)\} \le \frac{\varepsilon}{4} + \frac{I(A)}{2} + \frac{\varepsilon}{4}.$$

Hence, $I(A) < \varepsilon$. Thus, $\rho_{n,s}(X) \le \varepsilon$ for $s > s_0$.

Sufficiency. By Chebyshev's inequality and Lemma 1.2,

$$\sup_n \mathbf{P}\{\|T_n\| > s\} \le \sup_n \frac{\mathbf{E}\|T_n\|}{s} \le H(\rho_{n,s}(X) + \rho_{n,s}^{2+1/q}(X)) \to 0 \quad \text{as} \quad s \to \infty.$$

Put

$$\lambda_{\mathbf{a}}(X) = \sup_n \mathbf{E}\|T_n(X)\|$$

for a random element X and introduce the space

$$\mathbf{SB}_{\mathbf{a}}^{\infty}(E) = \{X \in L_0(E)\colon \mathbf{E}X = 0 \ \text{ and } \ \lambda_{\mathbf{a}}(X) < \infty\}.$$

This is a linear space, and $\lambda_a(\cdot)$ is a norm. Let us prove, for example, the triangular inequality. Let X, $Y \in \mathbf{SB}_a^\infty(E)$. If (X_k, Y_k) is a sequence of independent copies of the vector (X, Y), then

$$\lambda_a(X+Y) = \sup_n \mathbf{E}\|T_n(X+Y)\| = \sup_n \mathbf{E}\|\sum_{k=1}^{n} a_k(n)(X_k+Y_k)\|$$

$$= \sup_n \mathbf{E}\|T_n(X) + T_n(Y)\| \le \lambda_a(X) + \lambda_a(Y).$$

It is easy to show that $\mathbf{SB}_a^\infty(E)$ is a Banach space.

THEOREM 1.2. $\mathbf{SB}_a(E) = \mathbf{SB}_a^\infty(E)$.

Proof. If $X \in \mathbf{SB}_a^\infty(E)$, then $X^s \in \mathbf{SB}_a(E)$, where X^s is the symmetrization of X. By inequality (0.7) for means and by Lemma 1.2, we have

$$\lambda_a(X) = \sup_n \mathbf{E}\|T_n(X)\| \le \sup_n \mathbf{E}\|T_n(X^s)\| \le sH(\rho_{n,s}(X) + \rho_{n,s}^{2+1/q}(X)) < \infty$$

for any $s > 0$ (wherein the last inequality follows from Theorem 1.1).
The reverse is evident by Chebyshev's inequality.

2. WEAK LAWS OF LARGE NUMBERS FOR THE WEIGHTED SUM

In this section we state necessary and sufficient conditions for the weak law of large numbers to hold for weighted sums of independent identically distributed random elements with zero means. We shall prove that the set of all such elements is the closure of the set of step random elements (i.e., taking on a finite number of values only) by the norm $\lambda_a(\cdot)$ introduced at the end of the previous section. This result is an analogue of the well-known results of G. Pisier for the cases of the central limit theorem and the law of the iterated logarithm (Pisier, 1975); it is also an extension of the result of R. Norvaisa to the case of the Marcinkiewicz law of large numbers (Norvaisa, 1984).
Let us formulate and prove some technical results which enable us to obtain the main results of this section. Recall the notation

$$Y_{k,s}(tn) = a_k(tn)X_k I\|a_k(nt)X_k\| \le s\}.$$

LEMMA 2.1. *Let X be a symmetric random element. Consider the conditions*
a) $\lim_{t\to\infty} t\mathrm{P}\{\|X\| > s\varphi(t)\} = 0$,

b) $\lim_{n\to\infty} \sup_{t\ge 1} \mathbf{E}\|\sum_{k=1}^{n} Y_{k,s}(tn)\| = 0$.
In order for the inclusion $X \in \mathbf{WLLN}_a(E)$ to hold, it is sufficient that a) *and* b) *are satisfied for at least one $s > 0$ and it is necessary that* a) *and* b) *are satisfied for all $s > 0$.*

Proof. Sufficiency. Let a) and b) be satisfied for some $s > 0$. Define

$$T'_n = \sum_{k=1}^{n} a_k(n) X_k I\{\|a_k(n)X_k\| \le s\} = \sum_{k=1}^{n} Y_{k,s}(n),$$

$$T''_n = \sum_{k=1}^{n} a_k(n) X_k I\{\|a_k(n)X_k\| > s\}.$$

In this notation, $T_n = T'_n + T''_n$, and it is sufficient to show that $\mathbf{P}\{\|T'_n\| > \varepsilon\} \to 0$ and $\mathbf{P}\{\|T''_n\| > \varepsilon\} \to 0$ as $n \to \infty$ for all $\varepsilon > 0$.
By Chebyshev's inequality and condition b),

$$\mathbf{P}\{\|T'_n\| > \varepsilon\} \le \frac{1}{\varepsilon}\mathbf{E}\|T'_n\| = \frac{1}{\varepsilon}\mathbf{E}\|\sum_{k=1}^{n} Y_{k,s}(n)\| \to 0 \quad \text{as} \quad n \to \infty.$$

Furthermore, we have

$$\{\|T''_n\| > \varepsilon\} = \{\|T''_n\| > 0\}$$

$$\stackrel{=}{=} \{\|\sum_{k=1}^{n} a_k(n) X_k I\{\|a_k(n)X_k\| > s\}\| > 0\} \subset \bigcup_{k=1}^{n} \{\|a_k(n)X_k\| > s\},$$

hence, by condition a),

$$\mathbf{P}\{\|T''_n\| > \varepsilon\} \le \sum_{k=1}^{n} \mathbf{P}\{\|a_k(n)X_k\| > s\} \le \sum_{k=1}^{n} \mathbf{P}\{\|X_k\| > s\varphi(n)\}$$

$$= n\mathbf{P}\{\|X\| > s\varphi(n)\} \to 0 \quad \text{as} \quad n \to \infty.$$

Necessity. Let $X \in \mathbf{WLLN_a}(E)$. Fix some $s > 0$. Then $f(n) = \mathbf{P}\{\|T_n(X)\| > s\} \to 0$ as $n \to \infty$. By Levy's inequality (0.1), for all n we have

$$f(n) \ge \frac{1}{2}\mathbf{P}\{\max_{1\le k\le n} |a_k(n)|\,\|X_k\| > s\}$$

or

$$1 - 2f(n) \le \mathbf{P}\{\max_{1\le k\le n} \|X_k\| \le s\varphi(n)\}$$

$$= \mathbf{P}\{\bigcap_{k=1}^{n} \{\|X_k\| \le s\varphi(n)\}\} = \mathbf{P}^n\{\|X\| \le s\varphi(n)\}.$$

Hence, $\mathbf{P}\{\|X\| > s\varphi(n)\} \le 1 - (1 - f(n))^{1/n}$. So, $n\mathbf{P}\{\|X\| > s\varphi(n)\} \to 0$ as $n \to \infty$.
Furthermore, for all ε such that $0 < \varepsilon \le 2s$ and for all $t \ge 1$ we obtain

$$\mathbf{P} = \mathbf{P}\{\|\sum_{k=1}^{n} Y_{k,s}(tn)\| > \varepsilon\}$$

$$= \mathbf{P}\{\|\sum_{k=1}^{n} a_k(tn)X_k - \sum_{k=1}^{n} a_k(tn)X_k I\{\|a_k(tn)X_k\| > s\}\| > \varepsilon\}$$

$$\leq \mathbf{P}\left\{\|\sum_{k=1}^{n} a_k(tn)X_k\| > \frac{\varepsilon}{2}\right\} + \mathbf{P}\left\{\|\sum_{k=1}^{n} a_k(tn)X_kI\{\|a_k(tn)X_k\| > s\}\| > \frac{\varepsilon}{2}\right\}.$$

We estimate the first term in the right-hand side of the previous inequality by Kwapien's inequality (0.3) taking into account (i) and $t \geq 1$ and the second term — by elementary inequality (0.8). As a result, we have

$$\mathbf{P} \leq 2\mathbf{P}\left\{\|T_n\| > \frac{\varepsilon}{2B}\right\} + \sum_{k=1}^{n} \mathbf{P}\left\{\|a_k(tn)X_kI\{\|a_k(tn)X_k\| > s\}\| > \frac{\varepsilon}{2}\right\}$$

$$+\mathbf{P}\left\{\|\sum_{k=1}^{n} a_k(tn)X_kI\{\|a_k(tn)X_k\| > s\}I\left\{\|a_k(tn)X_k\| \leq \frac{\varepsilon}{2}\right\}\| > \frac{\varepsilon}{2}\right\}.$$

By the choice of $u < 2s$, the last event is impossible and

$$\mathbf{P} \leq 2\mathbf{P}\left\{\|T_n\| > \frac{\varepsilon}{2B}\right\} + n\mathbf{P}\left\{\|X\| > \frac{\varepsilon\varphi(n)}{2}\right\}.$$

Since $X \in \mathbf{WLLN_a}(E)$ and a) takes place, there exists $N = N(\varepsilon)$ such that

$$\sup_{n \geq N}\sup_{t \geq 1} \mathbf{P}\left\{\|\sum_{k=1}^{n} Y_{k,s}(tn)\| > \varepsilon\right\} \leq \frac{1}{24}.$$

Applying Hoffmann–Jorgencen's inequality (0.4) to the integral

$$I(A) = \int_0^A \mathbf{P}\left\{\|\sum_{k=1}^{n} Y_{k,s}(tn)\| > u\right\} du = \int_0^{A/3} \mathbf{P}\left\{\|\sum_{k=1}^{n} Y_{k,s}(tn)\| > 3v\right\} dv,$$

we obtain

$$I(A) \leq 3(4 \int_0^{A/3} \mathbf{P}^2\left\{\|\sum_{k=1}^{n} Y_{k,s}(tn)\| > v\right\} dv$$

$$+ \int_0^{A/3} \mathbf{P}\{\max_{1\leq k\leq n} |a_k(tn)|\,\|X_kI\{\|a_k(tn)X_k\| \leq s\}\| > v\}dv$$

$$\leq 12 \int_0^{\varepsilon} \mathbf{P}^2\{\|\sum_{k=1}^{n} Y_{k,s}(tn)\| > v\}dv$$

$$+12 \int_{\varepsilon}^{A/3} \mathbf{P}\{\|\sum_{k=1}^{n} Y_{k,s}(tn)\| > v\}\mathbf{P}\{\|\sum_{k=1}^{n} Y_{k,s}(tn)\| > v\}dv$$

$$+3\int_0^s \mathbf{P}\{\max_{1\le k\le n}|a_k(tn)|\,\|X_k\|>v\}dv.$$

By the previous reasonings, the first multiplier in the second integral is not greater than 1/24. Applying Levy's inequality (0.1) to the last integral, we have

$$I(A)\le 12\varepsilon+\frac{1}{2}\int_0^A \mathbf{P}\{\|\sum_{k=1}^n Y_{k,s}(tn)\|>v\}dv+6\int_0^s \mathbf{P}\{\|\sum_{k=1}^n a_k(tn)X_k\|>v\}dv$$

$$\le 12\varepsilon+\frac{I(A)}{2}+12B\int_0^s \mathbf{P}\{\|T_n\|>v\}dv$$

by condition (i). So, $I(A)\le 24\varepsilon+24B\int_0^s \mathbf{P}\{\|T_n\|>v\}dv$. The last integral converges to zero as $n\to\infty$ by the Lebesgue dominated convergence theorem. Hence,

$$\lim_{n\to\infty}\sup_{t\ge1}\mathbf{E}\|\sum_{k=1}^n Y_{k,s}(tn)\|=\lim_{n\to\infty}I(\infty)=0.$$

Lemma 2.2. *Let η be a positive random variable and $b(t)$ be a positive function such that $\lim_{t\to\infty} t\mathbf{P}\{\eta>b(t)\}=0$ for any $s>0$, then $t\mathbf{E}\eta I\{\eta\le sb(t)\}/b(t)\to 0$ as $t\to\infty$.*

Proof. Applying the formula of integration by parts and using the substitution $v=ub(t)$, we find

$$\frac{t}{b(t)}\mathbf{E}\eta I\{\eta\le sb(t)\}=-\frac{t}{b(t)}\int_0^{b(t)} v\,d\mathbf{P}\{\eta>v\}$$

$$=-t\mathbf{P}\{\eta>b(t)\}+\frac{t}{b(t)}\int_0^{b(t)}\mathbf{P}\{\eta>v\}dv$$

$$=-t\mathbf{P}\{\eta>b(t)\}+\int_0^1 t\mathbf{P}\{\eta>ub(t)\}\,du\to 0$$

as $t\to\infty$ by the Lebesgue dominated convergence theorem.

Lemma 2.3. *If $X\in \mathbf{WLLN}_a(E)$, then for all $k\in\mathbf{N}$ and $s>0$*

$$\lim_{n\to\infty}\sup_{t\ge1} nt\mathbf{E}\|Y_{k,s}(tn)\|^2=0.$$

Proof. We have

$$d_n = \sup_{t\geq 1} nt\mathbf{E}\|Y_{k,s}(tn)\|^2 = \sup_{u\geq n} u a_k^2(u)\mathbf{E}\|X_k I\|a_k(u)X_k\| \leq s\}\|^2.$$

Let $b(u) = s^2/a_k^2(u)$ and $\eta = \|X_k\|^2$. In this notation, by Lemma 2.2, we have $d_n = \sup_{u\geq n}(u\mathbf{E}\eta I\{\eta \leq b(u)\}/b(u)) \to 0$ as $n \to \infty$, since $\mathbf{P}\{\eta I\{\eta \leq b(u)\} = u\mathbf{P}\{\|a_k(u)X_k\| > s\} \leq u\mathbf{P}\{\eta > b(u)\}$ as $n \to \infty$. With the aid of these lemmas, we prove the following theorem. Recall the notation $\rho_{n,s}(X) = \sup_{t\geq 0} t\mathbf{P}\{t^q\|T_n(X)\| > s\}$, where $q = \log_2 C$ and the constant C is taken from condition (ii).

THEOREM 2.1. *Let X be a symmetric random element. Next statements are equivalent:*
a) $X \in \mathbf{WLLN_a}(E)$,
b) $\rho_{n,s}(X) \to 0$ *as* $n \to \infty$ *for all* $s > 0$,
c) *there exists* $s > 0$ *such that* $\rho_{n,s}(X) \to 0$ *as* $n \to \infty$.

Proof. a)⇒b). Let us show that for all ε such that $0 < \varepsilon < 1$ and for all $s > 0$ there exists $N = N(\varepsilon, s)$ such that $\rho_{n,s}(X) < \varepsilon$ for all $n > N$. We have:

$$\rho_{n,s}(X) = \sup_{t>0} t\mathbf{P}\{t^q\|T_n(X)\| > s\} = \sup_{t>0} D(t)$$

$$= \max(\sup_{0<t\leq\varepsilon} D(t), \sup_{\varepsilon<t\leq 1} D(t), \sup_{t>1} D(t)) \leq \max(\varepsilon, \sup_{\varepsilon<t\leq 1} D(t), \sup_{t>1} D(t)).$$

Moreover, $\sup_{\varepsilon<t\leq 1} D(t) \leq \mathbf{P}\{\varepsilon^q\|T_n\| > s\}$ and, since $X \in \mathbf{WLLN_a}(E)$, there exists $N_1 = N_1(\varepsilon)$ such that for all $n \geq N_1$

$$\mathbf{P}\{\varepsilon^q\|T_n(X)\| > s\} < \frac{\varepsilon}{4}.$$

It remains to estimate the last supremum, that is, to prove that $A = \sup_{t>1} D(t) < \varepsilon$ for sufficiently large n.
Note that, by Lemmas 2.1 and 2.3, there exists $N_2 = N_2(\varepsilon, s)$ such that for all $n \geq N_2$ and $k \leq n$

$$\sup_{u\geq n} u\mathbf{P}\{\|X\| > s\varphi(u)\} < \frac{\varepsilon}{4}, \tag{3}$$

$$\sup_{t\geq 1} \mathbf{E}\|Y_{k,s}(tn)\| \leq \frac{s}{2}, \tag{4}$$

$$\sup_{t\geq 1} t\mathbf{E}\|Y_{k,s}(tn)\|^2 \leq s^2\frac{\varepsilon}{16Cn}, \tag{5}$$

where the constant C is taken from de Acosta's inequality (0.6).
By condition (ii)′, $A \leq \sup_{t\geq 1} Dt\mathbf{P}\{\|\sum_{k=1}^n a_k(tn)X_k\| > s\}$. Now, applying elementary inequality (0.8), we get

$$A \le 2(\sup_{t\ge 1}\sum_{k=1}^{n} t\mathbf{P}\{\|a_k(tn)X_k\| > s\}$$

$$+\sup_{t\ge1} t\mathbf{P}\{\|\sum_{k=1}^{n} a_k(tn)X_k I\{\|a_k(tn)X_k\| \le s\}\| > s\}).$$

Let us estimate each term separately. Using the substitution $u = tn$ and taking into account (3), we obtain the following estimate $I_1 \le \sup_{u\ge n} u\mathbf{P}\{\|X\| > s\varphi(u)\} < (\varepsilon/4)$ for the first term. For the second term, we have, by (4),

$$I_2 = \sup_{t\ge1} t\mathbf{P}\{\|\sum_{k=1}^{n} Y_{k,s}(tn)\| > s\} \le \sup_{t\ge1} t\mathbf{P}\{|\|\sum_{k=1}^{n} Y_{k,s}(tn)\| - \mathbf{E}\|\sum_{k=1}^{n} Y_{k,s}(tn)\|| > \frac{s}{2}\}.$$

Applying Chebyshev's inequality, de Acosta's inequality (0.6), and (5), we obtain

$$I_2 \le 4s^{-2}\sup_{t\ge1} t\mathbf{E}(\|\sum_{k=1}^{n} Y_{k,s}(tn)\| - \mathbf{E}\|\sum_{k=1}^{n} Y_{k,s}(tn)\|)^2$$

$$\le 4s^{-2}C\sup_{t\ge1}\sum_{k=1}^{n} t\mathbf{E}\|Y_{k,s}(tn)\|^2 \le \frac{\varepsilon}{4}.$$

So, there exists $N = N(\varepsilon, s) = \max(N_1, N_2)$ such that $\rho_{n,s}(X) < \varepsilon$ for all $n > N$. Implication b)$\Rightarrow$c) is trivial, and c)$\Rightarrow$a) follows from Chebyshev's inequality and Lemma 1.2: for any $\varepsilon > 0$,

$$\mathbf{P}\{\|T_n(X)\| > \varepsilon\} \le \frac{\mathbf{E}\|T_n(X)\|}{\varepsilon} \le \frac{Hs}{\varepsilon}(\rho_{n,s}(X) + \rho_{n,s}^{2+1/q}(X)) \to 0 \quad \text{as} \quad n \to \infty.$$

COROLLARY. *The inclusion $X \in \mathbf{WLLN_a}(E)$ holds for a random element X with the zero mean, if and only if $\mathbf{E}\|T_n(X)\| \to 0$ as $n \to \infty$.*

Proof. If X satisfies $X \in \mathbf{WLLN_a}(E)$, then its symmetrization X^s satisfies the law of large numbers also. By inequality for means (0.7), Lemma 1.2, and condition (5) of the previous theorem, $\mathbf{E}\|T_n(X)\| \le Hs(\rho_{n,s}(X) + \rho_{n,s}^{2+1/q}(X)) \to 0$ as $n \mapsto \infty$. The convergence follows easily from Chebyshev's inequality.
From the Corollary and Theorem 1.2 we conclude that

$$\mathbf{WLLN_a}(E) \subset \mathbf{SB_a}(E) = \mathbf{SB_a}^{\infty}(E).$$

THEOREM 2.2. *The linear set $\mathbf{WLLN_a}(E)$ is closed in the Banach space $(\mathbf{SB_a}(E), \lambda_a(\cdot))$ or, equivalently, $X \in \mathbf{WLLN_a}(E)$, if and only if for any $\varepsilon > 0$ there exists $Y \in \mathbf{WLLN_a}(E)$ such that $\lambda_a(X - Y) < \varepsilon$.*

Proof. Take any $X \in \overline{\mathbf{WLLN_a}(E)}^{\lambda_a(E)}$. Then for any $\varepsilon > 0$ and $\delta > 0$ there exists $Y \in \mathbf{WLLN_a}(E)$ such that $\lambda_a(X - Y) \le \varepsilon\delta/2$. By the corollary to Theorem 2.1, there exists $N = N(\varepsilon, \delta)$ such that $\mathbf{E}\|T_n(Y)\| \le \varepsilon\delta/2$. By Chebyshev's inequality,

$$\mathbf{P}\{\|T_n(X)\| > \varepsilon\} \le \frac{1}{\varepsilon}\mathbf{E}\|T_n(X)\| \le \frac{1}{\varepsilon}(\mathbf{E}\|T_n(X-Y)\| + \mathbf{E}\|T_n(Y)\|)$$
$$\le \frac{1}{\varepsilon}(\lambda_a(X-Y) + \mathbf{E}\|T_n(Y)\|) \le \delta\cdot$$

and, since ε and δ are arbitrary, $X \in \mathbf{WLLN}_a(E)$.
In the next section we shall need the following result which is originally due to R. Norvaisa (Norvaisa, 1984, Lemma 3.6).
We use the notation

$$\mathbf{WLLN}_p(E) = \{X \in L_0(E)\colon \mathbf{E}X = 0 \text{ and } S_n(X)/n^{1/p} \to 0$$
$$\text{as } n \to \infty \text{ in probability}\}, \quad 1 < p < 2.$$

COROLLARY. *The space* $\mathbf{WLLN}_p(E)$ *is closed in the sense of the norm* $\lambda_p(X) = \sup_n \mathbf{E}\|S_n(X)\|$ *or, equivalently,* $X \in \mathbf{WLLN}_p(E)$, *if and only if for any* $\varepsilon > 0$ *there exists* $Y \in \mathbf{WLLN}_p(E)$ *such that* $\lambda_p(X - Y) < \varepsilon$.

Let $\mathbf{E}(E)$ be the set of all step random elements (i.e., those assuming only a finite number of values) with zero means. It is easy to note that condition (iii) implies the inclusion $\mathbf{E}(E) \subset \mathbf{WLLN}_a(E)$. In fact, an element from $\mathbf{E}(E)$ is supported on a finite-dimensional and bounded space, hence, it satisfies the law of large numbers (Marcus and Woyczynski, 1987, Theorem 3.1).

THEOREM 2.3. *The space* $\mathbf{WLLN}_a(E)$ *is the closure of the set* $\mathbf{E}(E)$ *in* $(\mathbf{SB}_a(E), \lambda_a(\cdot))$.

Proof. Since the space $\mathbf{WLLN}_a(E)$ is closed, it is sufficient to find for any $X \in \mathbf{WLLN}_a(E)$ a sequence from $\mathbf{E}(E)$ which converges to X in the sense of the norm $\lambda_a(\cdot)$.
Let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots$ be finite σ-algebras such that the σ-algebra generated by the random element X is contained in the σ-algebra generated by $\bigcup_{k=1}^{n} \mathcal{F}_k$. Consider the martingale $M_k = \mathbf{E}(X|\mathcal{F}_k) \in \mathbf{E}(E)$ and show that $\lambda_a(X - M_k) \to 0$ as $k \to \infty$.
We have

$$T_n(X) - T_n(M_k) = (T_n(X) - T_n(M_1)) - (T_n(M_k) - T_n(M_1))$$
$$= (T_n(X) - T_n(M_1)) - \mathbf{E}(T_n(X) - T_n(M_1)|\mathcal{F}_k).$$

Hence, $\mathbf{E}\|T_n(X - M_k)\| = \mathbf{E}\|T_n(X) - T_n(M_k)\| \le 2\mathbf{E}\|T_n(X) - T_n(M_1)\|$.
By the corollary to Theorem 2.1, for any $\varepsilon > 0$ there exists $N = N(\varepsilon)$ such that $\mathbf{E}\|T_n(X) - T_n(M_1)\| < \varepsilon/2$ for all $n \ge N$, i.e., for all $k \in \mathbf{N}$ and $n \ge N$

$$\mathbf{E}\|T_n(X - M_k)\| \le \varepsilon. \tag{6}$$

Moreover, since $M_k \to X$ as $k \to \infty$ in $L_1(E)$, (Vakhania et al., 1987, Theorem II.4.1) there exists $K = K(\varepsilon)$ such that for all $n\colon 1 \le n \le N$ and $k \ge K\colon \mathbf{E}\|T_n(X - M_k)\| \le \varepsilon$.
Combining this inequality and (6), we obtain

$$\lambda_{\mathbf{a}}(X - M_k) = \max\Big(\sup_{1\le k\le n} \mathbf{E}\|T_n(X - M_k)\|, \sup_{n\ge N} \mathbf{E}\|T_n(X - M_k)\|\Big) \le \varepsilon$$

for $k \ge K$.

Note that the case of the classical law of large numbers ($a_k(n) = 1/n$ for all $k \in \mathbf{N}$) is not covered by our results. This is caused by the fact that $C > 1/2$ in condition (ii) while in the classical case $C = 1/2$. It is evident that the existence of $\mathbf{E}X$ is not necessary for the proof of the classical weak law of large numbers (even the strong law takes place).

3. AN APPLICATION

Using Pisier's inequality (see below) and the results of the previous section, we examine the weak law of large numbers for certain random elements with values in an arbitrary Banach space.

Let $1 < s < \infty$ and let $(b_k)_{k\ge 1}$ be a sequence of real numbers. Set $\|(b_k)_{k\le n}\|_{s,\infty} = \max_{k\le n} k^{1/s} b_k^*$, where $(b_k^*)_{k\le n}$ is the nonincreasing permutation of $(|b_k|)_{k\le n}$. Define $\Lambda_s(\xi) \triangleq (\sup_{t>0} t^s \mathbf{P}\{|\xi| > t\})^{1/s}$ for a random variable ξ.

To facilitate the formulation of the theorem, we introduce the random element $X = \sum_{i=1}^{\infty} \eta_i x_i$ (with the series converging almost surely), where (x_i) is a nonrandom sequence of elements in E and (η_i) is a sequence of independent random variables with zero means. Note that we do not require the sequence (η_i) to be identically distributed. It should be mentioned that I. K. Matsak (Matsak, 1988) proved the Central Limit Theorem for such random elements.

By β_q denote the symmetric random variable whose absolute value has the Weibull distribution with a parameter $q > 0$, i.e., $\mathbf{P}\{|\beta_q| > t\} = \exp\{-t^q\}$, and let $(\beta_{q,i})_{i\ge 1}$ be independent copies of β_q.

THEOREM 3.1. *Let $X = \sum_{i=1}^{\infty} \eta_i x_i$ and let the weight satisfy condition (iii) only. If there exists s: $2 > s > p$ such that*

a) *$W = \sum_{i=1}^{\infty} \beta_{q,i} x_i$ converges a.s. for $\frac{1}{p} + \frac{1}{q} = 1$,*

b) *$\Lambda_s(\sup_{i\ge 1} |\eta_i|) < \infty$,*

then $X \in \mathbf{WLLN_a}(E)$.

In order to prove this theorem, we need some lemmas. In the sequel, we shall use the same notation as in the formulation of the theorem.

LEMMA 3.1. *If $W = \sum_{i=1}^{\infty} \beta_{q,i} x_i$ converges a.s., then it converges in $\mathbf{L}_p(E)$.*

Proof. It is sufficient to show that $W \in \mathbf{L}_p(E)$ (Vakhania et al., 1987, corollary to Theorem V.3.2). It suffices to prove the following inequality for all $t > 0$ (Vakhania et al., 1987, Theorem V.5.1):

$$\int_t^{\infty} \mathbf{P}\{|\beta_q| > u\} u^{p-1} du \le C t^p \mathbf{P}\{|\beta_q| > t\},$$

where the constant C does not depend on t. In the particular case β_q of a Weibull random variable this inequality can be rewritten as

$$\int_t^\infty \exp\{-u^q\}u^{p-1}du \le Ct^p \exp\{-t^q\}.$$

In fact, even a more precise estimation holds:

$$\int_t^\infty \exp\{-u^q\}u^{p-1}du \le Ct^{p-q} \exp\{-t^q\}, \tag{7}$$

since

$$t^{p-1}\exp\{-t^q\} \le t^{p-1}\exp\{-t^q\}\left(q + \frac{q-p}{p}t^{-q}\right) \tag{8}$$

and $p < 2 < q$. The terms of this inequality are the derivatives of the corresponding terms of (7) taken with the opposite sign, hence, (7) follows from (8) after integration of (8) on the half-line (t, ∞).

Pisier's inequality (Pisier, 1986, Lemma 2.7). Suppose $1 < s < 2$, q is the conjugate of s: $(1/q) + (1/s) = 1$, $(b_k)_{k\le n} \subset \mathbf{R}$, and let $(\varepsilon_k)_{k\le n}$ be independent Bernoulli random variables (i.e., $\mathbf{P}\{\varepsilon_1 = \pm 1\} = 1/2$). Then, for $k_s = q(q-2)/2$

$$\mathbf{P}\{|\sum_{k=1}^n b_k\varepsilon_k| > t\} \le 2\exp\left\{-\frac{t^q}{(k_s\|(b_k)_{k\le n}\|_{s,\infty})^q}\right\}.$$

The following lemma is just a reformulation of Pisier's inequality.

LEMMA 3.2. *Let* $\tau = \sum_{k=1}^n b_k\varepsilon_k$ *and* $\nu = \|(b_k)_{k\le n}\|_{s,\infty}\beta_q$, *where* $(b_k)_{k\le n} \subset \mathbf{R}$, *and let* (ε_k) *be a sequence of independent Bernoulli random variables. Then, for any* $t > 0$

$$\mathbf{P}\{|\tau| > t\} \le 2\mathbf{P}\{k_s|\nu| > t\}.$$

Let $(\eta_{k,i})_{k\ge 1}$ be independent copies of the sequence (η_i). We introduce the random variables $\mu(i,s,n) = \|(\eta_{k,i})_{k\le n}\|_{s,\infty}$.

LEMMA 3.3. *Let* η_i *be a sequence of symmetric random variables,* $X_k = \sum_{i=1}^\infty \eta_{k,i}x_i$, *and* $Y = \sum_{i=1}^\infty \beta_{q,i}\mu(i,s,n)x_i$. *Then the inequality* $\mathbf{E}\|\sum_{k=1}^n X_k\|^p \le C\mathbf{E}\|Y\|^p$ *holds, where* $C = 2^{1+p}k_s$.

Proof. First consider the case of $\eta_{k,i} = \varepsilon_{k,i}b_{k,i}$, where $b_{k,i} \in \mathbf{R}$ and $\varepsilon_{k,i}$ are independent Bernoulli random variables. Define

$$\tau_i = \sum_{k=1}^n b_{k,i}\varepsilon_{k,i} \quad \text{and} \quad \nu_i = \beta_{q,i}\|(b_{k,i})_{k\le n}\|_{s,\infty}.$$

It follows from Lemma 3.2 that $\mathbf{P}\{|\tau_i| > t\} \le 2\mathbf{P}\{k_s|\nu_i| > t\}$ for all $i \in \mathbf{N}$. Utilizing Theorem V.4.5 (Vakhania et al., 1987), we conclude that

$$\mathbf{E}\|\sum_{k=1}^{n} X_k\|^p = \mathbf{E}\|\sum_{i=1}^{\infty} \tau_i x_i\|^p \le 2^{1+p} k_s^p \mathbf{E}\|\sum_{i=1}^{\infty} \tau_i x_i\|^p = C\mathbf{E}\|Y\|^p.$$

In the general case, the desired result follows from this relation by a standard procedure. Replace the symmetric random variables η_i by the random variables $\eta_i \varepsilon_i$ which have the same distributions. Here ε_i are independent Bernoulli random variables which are independent of η_i. The left-hand side of the inequality from the statement of Lemma 3.3 is averaged over ε_i for a fixed η_i, and, finally, Fubini's theorem is applied (see, for example, the proof of Lemma V.2.1 (Vakhania et al., 1987)).

Proof of Theorem 3.1. First consider the case of a symmetric η_i. Let $X_k = \sum_{i=1}^{\infty} \eta_{k,i} x_i$ be independent copies of X. Applying Kwapien's inequality (0.3), we see that

$$\mathbf{P}\{\|T_n(X)\| > \varepsilon\} \le 2\mathbf{P}\left\{\frac{\|S_n(X)\|}{\max_{k\le n}|a_k(n)|} > \varepsilon\right\} \le 2\mathbf{P}\left\{\frac{\|S_n(X)\|}{n^{1/p}} > A\varepsilon\right\}$$

by condition (iii). Hence, it is sufficient to prove that $S_n(X)/n^{1/p} \to 0$ in probability as $n \to \infty$. Let $Z_m = \sum_{i=1}^{m} \eta_i x_i$ and let $Z_{k,m} = \sum_{i=1}^{m} \eta_{k,i} x_i$ be independent copies of Z_m. By Lemma 3.3 and inequality from (Vakhania et al., 1987, V.4, exercise 1(a)):

$$\mathbf{E}\left\|\frac{S_n(X - Z_m)}{n^{1/p}}\right\|^p \le C\mathbf{E}\left\|\frac{\sum_{i>m} \beta_{q,i}\mu(i,s,n)}{n}\right\|^p$$
$$\le \frac{C2^p}{n}\mathbf{E}(\sup_{i\ge 1}\mu(i,s,n))^p \mathbf{E}\|\sum_{i>m}\beta_{q,i}x_i\|^p.$$

Note that, by Lemma 3.1, $\mathbf{E}\|\sum_{i>m}\beta_{q,i}x_i\|^p \to 0$ as $m \to \infty$. Furthermore,

$$\mathbf{E}(\sup_{i\ge 1}\mu(i,s,n))^p \le (1 + \int_1^{\infty} \mathbf{P}\{\sup_{i\ge 1}\|(\eta_{k,i})_{k\le n}\|_{s,\infty} > t\}dt^p)$$
$$\le 1 + 2en\int_1^{\infty} \frac{\Lambda_s(\sup_{i\ge 1}|\eta_i|)}{t^s}dt^p$$

by Marcus–Pisier's result (Pisier, 1986, Lemma 4.11). Since $s > p$, we have

$$\frac{1}{n}\mathbf{E}(\sup_{i\ge 1}\mu(i,s,n))^p \le C\Lambda_s(\sup_{i\ge 1}|\eta_i|) + 1,$$

whence $\sup_n \mathbf{E}\|S_n(X - Z_m)/n^{1/p}\|^p \to 0$ as $m \to \infty$.

Note that the random vector Z_m takes on its values in the finite-dimensional space Span $(x_1, \ldots, x_m)$ and $\Lambda_p(\|Z_m\|) < \infty$, since $\Lambda_s(\sup_{i\geq 1}|\eta_i|) < \infty$ and $s > p$. Then $S_n(Z_m)/n^{1/p} \to 0$ in probability as $n \to \infty$ (Marcus and Woyczynski, 1979, Theorem 3.1). By the corollary to Theorem 2.2, we conclude that $S_n(X)/n^{1/p} \to 0$ in probability as $n \to \infty$.

If X is an arbitrary not necessarily symmetric centered random element satisfying the hypotheses of Theorem 1, then its symmetrization X^s also satisfies the same hypotheses and inequality for means (0.7):

$$\sup_n \mathbf{E}\left\|\frac{T_n(X - Z_m)}{n^{1/p}}\right\| \leq \sup_n \mathbf{E}\left\|\frac{T_n(X^s - Z_m)}{n^{1/p}}\right\| \to 0$$

as $m \to \infty$. It remains to apply the corollary to Theorem 2.2.

Mention should be made of an important particular case, namely, the case of random variables (η_i) which are nondegenerate, i.e., $\inf_i \mathbf{E}|\eta_i| > 0$. Then condition a) of Theorem 1 is not necessary, since the convergence of the series $\sum_{i=1}^{\infty} \eta_i x_i$ follows from the convergence of the series $\sum_{i=1}^{\infty} \beta_{q,i} x_i$ (Vakhania et al., Lemma V.5.1 and Proposition V.5.5).

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