STOCHASTIC ANALYSIS AND APPLICATIONS
Vol. 21, No. 5, pp. 1169–1187, 2003

Mean Convergence Theorems with or without Random Indices for Randomly Weighted Sums of Random Elements in Rademacher Type p Banach Spaces

Andrew Rosalsky,[1,*] M. Sreehari,[2] and Andrei I. Volodin[3]

[1]Department of Statistics, University of Florida, Gainesville, Florida, USA
[2]Department of Statistics, Faculty of Science, Maharaja Sayajirao University of Baroda, Vadodara, India
[3]Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada

ABSTRACT

Some mean convergence theorems are established for randomly weighted sums of the form $\sum_{j=1}^{k_n} A_{nj} V_{nj}$ and $\sum_{j=1}^{T_n} A_{nj} V_{nj}$ where $\{A_{nj}, j \geq 1, n \geq 1\}$ is an array of random variables, $\{V_{nj}, j \geq 1, n \geq 1\}$ is an array of mean 0 random elements in a separable real Rademacher type p $(1 \leq p \leq 2)$ Banach space, and $\{k_n, n \geq 1\}$ and $\{T_n, n \geq 1\}$ are sequences of positive integers and positive integer-valued random variables, respectively. The results take the form $\| \sum_{j=1}^{k_n} A_{nj} V_{nj} \| \xrightarrow{\mathcal{L}_r} 0$ or $\| \sum_{j=1}^{T_n} A_{nj} V_{nj} \| \xrightarrow{\mathcal{L}_r} 0$

*Correspondence: Andrew Rosalsky, Department of Statistics, University of Florida, Gainesville, FL 32611, USA; Fax: (352) 392-5175; E-mail: rosalsky@stat.ufl.edu.

DOI: 10.1081/SAP-120024708 0736-2994 (Print); 1532-9356 (Online)

www.dekker.com

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

where $1 \leq r \leq p$. It is assumed that the array $\{A_{nj}V_{nj}, j \geq 1, n \geq 1\}$ is comprised of rowwise independent random elements and that for all $n \geq 1$, A_{nj} and V_{nj} are independent for all $j \geq 1$ and T_n and $\{A_{nj}V_{nj}, j \geq 1\}$ are independent. No conditions are imposed on the joint distributions of the random indices $\{T_n, n \geq 1\}$. The sharpness of the results is illustrated by examples.

Key Words: Separable real Rademacher type p Banach space; Array of rowwise independent random elements; Weighted sums; Random weights; Random indices; Mean convergence.

1. INTRODUCTION

Consider an array $\{V_{nj}, j \geq 1, n \geq 1\}$ of mean 0 random elements defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a separable real Banach space $\mathcal{X}$ with norm $\|\cdot\|$. Let $\{A_{nj}, j \geq 1, n \geq 1\}$ be an array of (real-valued) random variables (called *random weights*), let $\{k_n, n \geq 1\}$ be a sequence of positive integers, and let $\{T_n, n \geq 1\}$ be a sequence of positive integer-valued random variables (called *random indices*). In the current work, mean convergence theorems will be established for the *weighted sums* $\sum_{j=1}^{k_n} A_{nj}V_{nj}$ and $\sum_{j=1}^{T_n} A_{nj}V_{nj}$. These results take the form

$$\left\| \sum_{j=1}^{k_n} A_{nj}V_{nj} \right\| \xrightarrow{\mathcal{L}_r} 0 \tag{1}$$

or

$$\left\| \sum_{j=1}^{T_n} A_{nj}V_{nj} \right\| \xrightarrow{\mathcal{L}_r} 0 \tag{2}$$

where $1 \leq r \leq 2$. It should be noted that each $A_{nj}V_{nj}$ is automatically a random element (see, e.g., Lemma 2.1.5 of Ref.[1], p. 24). Of course, it follows from the Markov inequality that (1) implies the general weak law of large numbers (WLLN)

$$\sum_{j=1}^{k_n} A_{nj}V_{nj} \xrightarrow{P} 0$$

and a similar observation can be made concerning (2). It is assumed that the array $\{A_{nj}V_{nj}, j \geq 1, n \geq 1\}$ is comprised of rowwise independent random

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

elements and that A_{nj} and V_{nj} are independent for all $n \geq 1$ and $j \geq 1$. These assumptions are of course automatic if $\{A_{nj}, j \geq 1\}$ and $\{V_{nj}, j \geq 1\}$ are independent sequences of independent random variables and random elements, respectively, for all $n \geq 1$. It is not assumed that the sequences $\{A_{nj}V_{nj}, j \geq 1\}$ and $\{A_{n'j}V_{n'j}, j \geq 1\}$ are independent for $n \neq n'$. Moreover, it is assumed that T_n and $\{A_{nj}V_{nj}, j \geq 1\}$ are independent for all $n \geq 1$. No conditions are imposed on the joint distributions of the $\{T_n, n \geq 1\}$ whose marginal distributions are constrained solely by (9). The Banach space $\mathcal{X}$ is assumed to be of *Rademacher type p* ($1 \leq p \leq 2$). (Technical definitions such as this will be discussed in Sec. 2.)

Except for some recent work on the WLLN in Banach spaces by Hu et al.,[2] Rosalsky and Sreehari,[3] and Rosalsky et al.[4] and some recent work on mean convergence in martingale type p Banach spaces by Rosalsky and Sreehari,[5] we are unaware of any literature of investigation on the limiting behavior of *both* randomly weighted and randomly indexed sums $\sum_{j=1}^{T_n} A_{nj}V_{nj}$ even when the V_{nj} are random variables. The only mean convergence results that the authors are aware of for randomly weighted sums are those of Rosalsky and Sreehari.[5] In the current work, we establish Rademacher type p Banach space versions of the main results of Rosalsky and Sreehari.[5] The current work owes much to this earlier work.

In the random variable case, there is a small literature of mean convergence results for randomly indexed sums; see Refs.[6–8]. The work of Gut[6] generalizes a famous result of Pyke and Root[9] to the case of randomly indexed partial sums.

Let θ_{nj} be a generic symbol for a (suitably selected) conditional expectation, $j \geq 1$, $n \geq 1$. Assuming $\mathcal{X}$ is of martingale type p, Adler et al.[10] proved under a Cesàro type condition of Hong and Oh[11] (which is weaker than Cesàro uniform integrability as introduced by Chandra[12]) the WLLN

$$\sum_{j=1}^{k_n} a_{nj}(V_{nj} - \theta_{nj}) \xrightarrow{P} 0$$

where $\{a_{nj}, 1 \leq j \leq k_n < \infty, n \geq 1\}$ is an array of constants with $k_n \to \infty$. In a martingale type p Banach space setting, Hong et al.[13,14] gave conditions for the WLLN (with random indices)

$$\sum_{j=1}^{T_n} a_{nj}(V_{nj} - \theta_{nj}) \xrightarrow{P} 0$$

to hold. Also in a martingale type p Banach space setting, Adler et al.[10] proved the $\mathcal{L}_r$ convergence result

$$\left\| \sum_{j=1}^{k_n} a_{nj}(V_{nj} - \theta_{nj}) \right\| \xrightarrow{\mathcal{L}_r} 0$$

where $1 \leq r \leq p$ under a uniform integrability type condition introduced by Ordóñez Cabrera[15] which contains Cesàro uniform integrability as a special case.

When $\mathcal{X}$ is of Rademacher type p $(1 \leq p \leq 2)$ and the array $\{V_{nj}, j \geq 1, n \geq 1\}$ is comprised of rowwise independent random elements, Adler et al.[16] established a WLLN (with random indices) of the form

$$\frac{\sum_{j=1}^{T_n} a_j(V_{nj} - \mu_{nj})}{b_n} \xrightarrow{P} 0$$

where $\{\mu_{nj}, j \geq 1, n \geq 1\}$ is an array of suitable elements of $\mathcal{X}$ and $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ are sequences of constants with $0 < b_n \to \infty$.

Taylor and Padget,[17] Wei and Taylor,[18,19] Taylor and Calhoun,[20] Taylor et al.,[21] Ordóñez Cabrera,[22] and Adler et al.[23] studied either the weak or the almost sure (a.s.) limiting behavior of randomly weighted partial sums of Banach space valued random elements. However, in all of those articles, the number of terms in the partial sums is deterministic.

The plan of the paper is as follows. Some technical definitions will be given in Sec. 2. The main results will be stated and proved in Sec. 3. In Sec. 4, examples will be presented which illustrate the sharpness of the results.

Finally, the symbol C denotes a generic constant $(0 < C < \infty)$ which is not necessarily the same one in each appearance.

2. PRELIMINARIES

Technical definitions relevant to the current work will be discussed in this section. The *expected value* or *mean* of a random element V, denoted by EV, is defined to be the *Pettis integral* provided it exists. That is, V has expected value $EV \in \mathcal{X}$ if $f(EV) = E[f(V)]$ for every $f \in \mathcal{X}^*$ where $\mathcal{X}^*$ denotes the (*dual*) space of all continuous linear functionals on $\mathcal{X}$. A sufficient condition for EV to exist is that $E\|V\| < \infty$ (see, e.g., Ref.[1], p. 40).

Let $\{Y_n, n \geq 1\}$ be a symmetric *Bernoulli sequence*; that is, $\{Y_n, n \geq 1\}$ is a sequence of independent and identically distributed random variables with $P\{Y_1 = 1\} = P\{Y_1 = -1\} = 1/2$. Let $\mathcal{X}^\infty = \mathcal{X} \times \mathcal{X} \times \mathcal{X} \times \cdots$ and define $\mathcal{C}(\mathcal{X}) = \{(v_1, v_2, \dots) \in \mathcal{X}^\infty : \sum_{n=1}^\infty Y_n v_n$ converges in probability$\}$. Let $1 \le r \le 2$. Then the separable real Banach space $\mathcal{X}$ is said to be of *Rademacher type* r if there exists a constant $0 < C < \infty$ such that $E\| \sum_{n=1}^\infty Y_n v_n \|^r \le C \sum_{n=1}^\infty \|v_n\|^r$ for all $(v_1, v_2, \dots) \in \mathcal{C}(\mathcal{X})$. Hoffmann-Jørgensen and Pisier[24] proved for $1 \le r \le 2$ that a separable real Banach space is of Rademacher type r if and only if there exists a constant $0 < C < \infty$ such that

$$E\left\| \sum_{j=1}^n V_j \right\|^r \le C \sum_{j=1}^n E\|V_j\|^r \tag{3}$$

for every finite collection $\{V_1, \dots, V_n\}$ of independent mean 0 random elements.

If a separable real Banach space is of Rademacher type p for some $1 < p \le 2$, then it is of Rademacher type r for all $1 \le r < p$. Every separable real Banach space is of Rademacher type (at least) 1 while the $\mathcal{L}_p$-spaces and ℓ_p-spaces are of Rademacher type $2 \wedge p$ for $p \ge 1$. Every separable real Hilbert space and separable real finite-dimensional Banach space is of Rademacher type 2.

3. THE MAIN RESULTS

With the preliminaries accounted for, the main results may now be established.

Theorem 1

Let $1 \le r \le p \le 2$ and let $\{V_{nj}, 1 \le j \le k_n < \infty, n \ge 1\}$ be an array of mean 0 random elements in a separable real Rademacher type p Banach space $\mathcal{X}$. Let $\{A_{nj}, 1 \le j \le k_n, n \ge 1\}$ be an array of random variables and suppose that the array $\{A_{nj} V_{nj}, 1 \le j \le k_n, n \ge 1\}$ is comprised of rowwise independent random elements and that A_{nj} and V_{nj} are independent for all $n \ge 1$ and $1 \le j \le k_n$. Let $\{c_{nj}, 1 \le j \le k_n, n \ge 1\}$ be an array of positive constants such that

$$\sum_{j=1}^{k_n} c_{nj}^r E|A_{nj}|^r = o(1) \tag{4}$$

and

$$\sum_{j=1}^{k_n} E\|A_{nj}V_{nj}I(\|V_{nj}\| > c_{nj})\|^r = o(1). \tag{5}$$

Then

$$\left\| \sum_{j=1}^{k_n} A_{nj}V_{nj} \right\| \xrightarrow{\mathcal{L}_r} 0 \tag{6}$$

and, *a fortiori*, the WLLN

$$\sum_{j=1}^{k_n} A_{nj}V_{nj} \xrightarrow{P} 0 \tag{7}$$

obtains.

Proof.
Note that for all $n \geq 1$ and $1 \leq j \leq k_n$,

$$\begin{aligned} E\|A_{nj}V_{nj}\| &= E\|A_{nj}V_{nj}I(\|V_{nj}\| \leq c_{nj})\| + E\|A_{nj}V_{nj}I(\|V_{nj}\| > c_{nj})\| \\ &\leq c_{nj}E|A_{nj}| + E\|A_{nj}V_{nj}I(\|V_{nj}\| > c_{nj})\| \\ &< \infty \text{ [by (4) and (5)]} \end{aligned}$$

implying that $E(A_{nj}V_{nj})$ exists. Then since A_{nj} and V_{nj} are independent, $j \geq 1$, $n \geq 1$, we have

$$E(A_{nj}V_{nj}) = (EA_{nj})(EV_{nj}) = 0, \quad j \geq 1, \ n \geq 1.$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Then for all $n \geq 1$, since $\mathcal{X}$ is of Rademacher type r, it follows from (3) with V_j replaced by $A_{nj}V_{nj}$ that

$$\begin{aligned}
E\left\|\sum_{j=1}^{k_n} A_{nj}V_{nj}\right\|^r &\leq C\sum_{j=1}^{k_n} E\|A_{nj}V_{nj}\|^r \\
&= C\sum_{j=1}^{k_n} E\|A_{nj}V_{nj}I(\|V_{nj}\| \leq c_{nj})\|^r \\
&\quad + C\sum_{j=1}^{k_n} E\|A_{nj}V_{nj}I(\|V_{nj}\| > c_{nj})\|^r \\
&\leq C\sum_{j=1}^{k_n} c_{nj}^r E|A_{nj}|^r + C\sum_{j=1}^{k_n} E\|A_{nj}V_{nj}I(\|V_{nj}\| > c_{nj})\|^r \\
&= o(1) \text{ [by (4) and (5)].}
\end{aligned}$$

■

The next result is a random indices version of Theorem 1. It should be noted that if $T_n/k_n \xrightarrow{P} c$ for some $c \in [0, 1)$, then the condition (9) of Theorem 2 holds. An example wherein the converse fails is given by a positive integer sequence $k_n \to \infty$ with k_n divisible by 4, $n \geq 1$ and a sequence of random variables $\{T_n, n \geq 1\}$ with $P\{T_n = (1/2)k_n\} = P\{T_n = (1/4)k_n\} = 1/2$, $n \geq 1$. The condition (9) is not comparable with the condition $ET_n/k_n \to c \in (0, \infty)$ assumed by Gut.[6] Finally, note in Theorem 2 that the slower $k_n \to \infty$ can be taken to satisfy (9), the conditions (10) and (11) become less stringent.

Theorem 2

Let $1 \leq r \leq p \leq 2$ and let $\{V_{nj}, j \geq 1, n \geq 1\}$ be an array of mean 0 random elements in a separable real Rademacher type p Banach space. Let $\{A_{nj}, j \geq 1, n \geq 1\}$ be an array of random variables and suppose that the array $\{A_{nj}V_{nj}, j \geq 1, n \geq 1\}$ is comprised of rowwise independent random elements and that A_{nj} and V_{nj} are independent for all $n \geq 1$ and $j \geq 1$. Let $\{T_n, n \geq 1\}$ be a sequence of positive integer-valued random variables such that

$$T_n \text{ is independent of } \{A_{nj}V_{nj}, j \geq 1\}, \quad n \geq 1 \tag{8}$$

and

$$P\{T_n > k_n\} = o(1) \tag{9}$$

MARCEL DEKKER, INC.
270 Madison Avenue, New York, New York 10016

where $\{k_n, n \geq 1\}$ is a sequence of positive integers. Let $\{c_{nj}, j \geq 1, n \geq 1\}$ be an array of positive constants such that

$$\sum_{j=1}^{k_n} c_{nj}^r E|A_{nj}|^r = o(1), \tag{10}$$

$$\sum_{j=1}^{k_n} E\|A_{nj} V_{nj} I(\|V_{nj}\| > c_{nj})\|^r = o(1), \tag{11}$$

and for some positive integer n_0

$$\sup_{n \geq n_0} \sum_{j=1}^{\infty} c_{nj}^r E|A_{nj}|^r < \infty \tag{12}$$

and

$$\sup_{n \geq n_0} \sum_{j=1}^{\infty} E\|A_{nj} V_{nj} I(\|V_{nj}\| > c_{nj})\|^r < \infty. \tag{13}$$

Then

$$\left\| \sum_{j=1}^{T_n} A_{nj} V_{nj} \right\| \xrightarrow{\mathcal{L}_r} 0 \tag{14}$$

and, *a fortiori*, the WLLN

$$\sum_{j=1}^{T_n} A_{nj} V_{nj} \xrightarrow{P} 0 \tag{15}$$

obtains.

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Proof.
Set $p_{nk} = P\{T_n = k\}$, $k \geq 1$, $n \geq 1$. Then

$$\begin{aligned}
E\left\|\sum_{j=1}^{T_n} A_{nj}V_{nj}\right\|^r &= E\left(\sum_{k=1}^{\infty}\left\|\sum_{j=1}^{k} A_{nj}V_{nj}\right\|^r I(T_n = k)\right) \\
&= \sum_{k=1}^{\infty} E\left\|\sum_{j=1}^{k} A_{nj}V_{nj}\right\|^r p_{nk} \text{ [by the Lebesgue monotone} \\
&\quad \text{convergence theorem and (8)]} \\
&\leq C\sum_{k=1}^{\infty}\sum_{j=1}^{k} c_{nj}^r E|A_{nj}|^r p_{nk} \\
&\quad + C\sum_{k=1}^{\infty}\sum_{j=1}^{k} E\|A_{nj}V_{nj}I(\|V_{nj}\| > c_{nj})\|^r p_{nk} \text{ (by the} \\
&\quad \text{the same argument as in Theorem 1).} \qquad (16)
\end{aligned}$$

Now for all large n

$$\begin{aligned}
\sum_{k=1}^{\infty}\sum_{j=1}^{k} c_{nj}^r E|A_{nj}|^r p_{nk} &= \sum_{k=1}^{k_n} p_{nk}\sum_{j=1}^{k} c_{nj}^r E|A_{nj}|^r + \sum_{k=k_n+1}^{\infty} p_{nk}\sum_{j=1}^{k} c_{nj}^r E|A_{nj}|^r \\
&\leq \sum_{j=1}^{k_n} c_{nj}^r E|A_{nj}|^r + C\sum_{k=k_n+1}^{\infty} p_{nk} \text{ [by (12)]} \\
&= \sum_{j=1}^{k_n} c_{nj}^r E|A_{nj}|^r + CP\{T_n > k_n\} \\
&= o(1) \quad \text{[by (10) and (9)].} \qquad (17)
\end{aligned}$$

A similar argument employing (13), (11), and (9) yields

$$\sum_{k=1}^{\infty}\sum_{j=1}^{k} E\|A_{nj}V_{nj}I(\|V_{nj}\| > c_{nj})\|^r p_{nk} = o(1). \qquad (18)$$

The conclusion (14) follows immediately from (16)–(18). ∎

Remark 1

Rosalsky and Sreehari[5] obtained versions of Theorems 1 and 2 for the case $0 < r \leq 1$ when the underlying Banach space is not necessarily of Rademacher type p for some $p \in (1, 2]$. All of the independence assumptions from Theorem 1 and some of the independence assumptions from Theorem 2 are eliminated in those earlier versions.

4. SOME INTERESTING EXAMPLES

Eight illustrative examples will now be presented. For $1 \leq q \leq 2$, let ℓ_q denote the Banach space of absolute pth power summable real sequences $v = \{v_i, i \geq 1\}$ with norm $\|v\| = (\sum_{i=1}^{\infty} |v_i|^q)^{1/q}$. The element having 1 in its jth position and 0 elsewhere will be denoted by $v^{(j)}, j \geq 1$. Define a sequence $\{V_j, j \geq 1\}$ of independent random elements in ℓ_1 by requiring the $\{V_j, j \geq 1\}$ to be independent with

$$P\{V_j = v^{(j)}\} = P\{V_j = -v^{(j)}\} = \frac{1}{2}, \quad j \geq 1.$$

Set $V_{nj} = V_j, j \geq 1, n \geq 1$. Hence for all $1 \leq q \leq 2$, $\{V_{nj}, j \geq 1, n \geq 1\}$ is an array of mean 0 random elements in ℓ_q.

Let $\{A_{nj}, j \geq 1, n \geq 1\}$ be an array of rowwise independent random variables such that the sequences $\{A_{nj}, j \geq 1\}$ and $\{V_j, j \geq 1\}$ are independent for all $n \geq 1$. Then the array $\{A_{nj}V_{nj}, j \geq 1, n \geq 1\}$ is comprised of rowwise independent random elements and A_{nj} and V_{nj} are independent for all $n \geq 1$ and $j \geq 1$.

The random elements $\{V_{nj}, j \geq 1, n \geq 1\}$ and random variables $\{A_{nj}, j \geq 1, n \geq 1\}$ will be used in the examples. The (marginal) distribution of each A_{nj} will be indicated in each example. Let $\{k_n, n \geq 1\}$ be a sequence of positive integers with $k_n \to \infty$.

The first example illustrates the essential role that condition (4) plays in Theorem 1.

Example 1

Let $1 \leq r \leq 2$, and consider the Rademacher type $p = r$ Banach space ℓ_r and the array $\{V_{nj}, 1 \leq j \leq k_n, n \geq 1\}$ of random elements in ℓ_r. Suppose that

$$P\{A_{nj} = k_n^{-\alpha}\} = P\{A_{nj} = -k_n^{-\alpha}\} = \frac{1}{2}, \quad 1 \leq j \leq k_n, \ n \geq 1 \tag{19}$$

where $\alpha > 0$. Let $c_{nj} = 1, 1 \leq j \leq k_n, n \geq 1$. Note that $I(\|V_{nj}\| > c_{nj}) = 0$ a.s., $1 \leq j \leq k_n, n \geq 1$ and so (5) holds.

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Now if $\alpha > r^{-1}$, then

$$\sum_{j=1}^{k_n} c_{nj}^r E|A_{nj}|^r = \frac{k_n}{k_n^{\alpha r}} = o(1)$$

whence (4) holds. Thus by Theorem 1

$$\left\| \sum_{j=1}^{k_n} A_{nj} V_{nj} \right\| \xrightarrow{\mathcal{L}_r} 0 \quad \text{and} \quad \sum_{j=1}^{k_n} A_{nj} V_{nj} \xrightarrow{P} 0.$$

On the other hand, if $\alpha \leq r^{-1}$, then

$$\sum_{j=1}^{k_n} c_{nj}^r E|A_{nj}|^r = \frac{k_n}{k_n^{\alpha r}} \neq o(1)$$

whence (4) fails. Moreover, if $\alpha \leq r^{-1}$, then with probability 1

$$\left\| \sum_{j=1}^{k_n} A_{nj} V_{nj} \right\| = \frac{k_n^{r^{-1}}}{k_n^{\alpha}} \geq 1, \quad n \geq 1$$

and so the conclusions (6) and (7) of Theorem 1 also fail.

The next example shows that in Theorem 1 the condition (5) cannot be dispensed with.

Example 2

Let $1 \leq r \leq p \leq 2$, and consider the Rademacher type p Banach space ℓ_p and the array of random elements $\{V_{nj}, 1 \leq j \leq k_n, n \geq 1\}$ in ℓ_p. Let the (marginal) distributions of the $\{A_{nj}, 1 \leq j \leq k_n, n \geq 1\}$ be as in (19) where $\alpha \in (0, p^{-1}]$. Let $c_{nj} = 2^{-j}$, $1 \leq j \leq k_n$, $n \geq 1$. Now (4) holds since

$$\sum_{j=1}^{k_n} c_{nj}^r E|A_{nj}|^r = \frac{1}{k_n^{\alpha r}} \sum_{j=1}^{k_n} \frac{1}{2^{jr}} \leq \frac{1}{k_n^{\alpha r}} = o(1).$$

Now $I(\|V_{nj}\| > c_{nj}) = 1$ a.s., $1 \le j \le k_n$, $n \ge 1$. Thus

$$\sum_{j=1}^{k_n} E\|A_{nj}V_{nj}I(\|V_{nj} > c_{nj})\|^r = \sum_{j=1}^{k_n} E\|A_{nj}V_{nj}\|^r$$
$$= \frac{k_n}{k_n^{\alpha r}} \ge \frac{k_n}{k_n^{\alpha p}} \ge 1, \quad n \ge 1$$

and so (5) fails. Moreover, with probability 1

$$\left\| \sum_{j=1}^{k_n} A_{nj}V_{nj} \right\| = \frac{k_n^{p^{-1}}}{k_n^{\alpha}} \ge 1, \quad n \ge 1$$

whence the conclusions (6) and (7) of Theorem 1 also fail.

In the next example, we will show apropos of Theorem 1 that the Rademacher type p hypothesis cannot be dispensed with.

Example 3

Let $1 \le q < r \le p \le 2$, and consider the Banach space ℓ_q and the array of random elements $\{V_{nj}, 1 \le j \le k_n, n \ge 1\}$ in ℓ_q. It is well known that ℓ_q is not of Rademacher type p. Let the (marginal) distributions of the $\{A_{nj}, 1 \le j \le k_n, n \ge 1\}$ be as in (19) where $\alpha \in (r^{-1}, q^{-1}]$. Let $c_{nj} = 1$, $1 \le j \le k_n$, $n \ge 1$. All of the hypotheses of Theorem 1 are satisfied except for the underlying Banach space being of Rademacher type p. But the conclusions (6) and (7) of Theorem 1 fail since with probability 1

$$\left\| \sum_{j=1}^{k_n} A_{nj}V_{nj} \right\| = \frac{k_n^{q^{-1}}}{k_n^{\alpha}} \ge 1, \quad n \ge 1.$$

It might be conjectured in view of (10) [resp. (12)] that Theorem 2 will hold without (12) [resp. (10)]. The next two examples demonstrate the falsity of such conjectures.

Example 4

Let $p = r = 1$, and consider the Rademacher type 1 Banach space ℓ_1 and the array $\{V_{nj}, j \ge 1, n \ge 1\}$ of random elements in ℓ_1. Suppose that

$$P\{A_{nj} = b_n^{-1}\} = P\{A_{nj} = -b_n^{-1}\} = \frac{1}{2}, \qquad j \ge 1, \quad n \ge 1$$

where $\{b_n, n \geq 1\}$ is a sequence of positive constants with $2^n = o(b_n)$. Let $c_{nj} = 1, j \geq 1, n \geq 1$. Let $\{T_n, n \geq 1\}$ be a sequence of identically distributed random variables satisfying (8) and with the distribution of T_1 given by

$$P\{T_1 = 2^j\} = 2^{-j}, \quad j \geq 1. \tag{20}$$

Suppose that $k_n = 2^n$, $n \geq 1$. Then (9) holds since

$$P\{T_n > k_n\} = \sum_{j=n+1}^{\infty} 2^{-j} = o(1).$$

Now (11) and (13) are immediate since $I(\|V_{nj}\| > c_{nj}) = 0$ a.s., $j \geq 1$, $n \geq 1$. Moreover, (10) holds since

$$\sum_{j=1}^{k_n} c_{nj} E|A_{nj}| = \frac{2^n}{b_n} = o(1).$$

On the other hand, (12) fails since for all $n \geq 1$

$$\sum_{j=1}^{\infty} c_{nj} E|A_{nj}| = \sum_{j=1}^{\infty} b_n^{-1} = \infty.$$

Finally, (14) fails since for all $n \geq 1$

$$E\left\|\sum_{j=1}^{T_n} A_{nj} V_{nj}\right\| = \frac{ET_n}{b_n} = \infty.$$

We remark, however, that (15) does hold since for arbitrary $\varepsilon > 0$,

$$P\left\{\left\|\sum_{j=1}^{T_n} A_{nj} V_{nj}\right\| > \varepsilon\right\} = P\left\{\frac{T_n}{b_n} > \varepsilon\right\} = P\{T_1 > \varepsilon b_n\} \to 0.$$

Example 5

Let $1 \leq r \leq p \leq 2$, and consider the Rademacher type p Banach space ℓ_p and the array $\{V_{nj}, j \geq 1, n \geq 1\}$ of random elements in ℓ_p. Suppose that

$$P\{A_{nj} = j^{-\alpha/r}\} = P\{A_{nj} = -j^{-\alpha/r}\} = \frac{1}{2}, \quad j \geq 1, \; n \geq 1$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

where $\alpha > 1$. Let $c_{nj} = 1, j \geq 1, n \geq 1$ and let $\{T_n, n \geq 1\}$ be a sequence of positive integer-valued random variables satisfying (8) and (9). Now (11) and (13) are immediate since $I(\|V_{nj}\| > c_{nj}) = 0$ a.s., $j \geq 1$, $n \geq 1$. Moreover, since $\alpha > 1$

$$\sup_{n\geq 1} \sum_{j=1}^{\infty} c_{nj}^r E|A_{nj}|^r = \sum_{j=1}^{\infty} j^{-\alpha} < \infty$$

and so (12) holds. On the other hand, (10) fails since

$$\sum_{j=1}^{k_n} c_{nj}^r E|A_{nj}|^r = \sum_{j=1}^{k_n} j^{-\alpha} \geq 1, \quad n \geq 1.$$

The conclusions of (14) and (15) of Theorem 2 also fail since with probability 1

$$\left\| \sum_{j=1}^{T_n} A_{nj} V_{nj} \right\| = \left(\sum_{j=1}^{T_n} j^{-\alpha p/r} \right)^{1/p} \geq 1, \quad n \geq 1.$$

It might be conjectured in view of (11) [resp. (13)] that Theorem 2 will hold without (13) [resp. (11)]. The next two examples demonstrate the falsity of such conjectures.

Example 6

Let $p = r = 1$, and consider the Rademacher type 1 Banach space ℓ_1 and the array $\{V_{nj}, j \geq 1, n \geq 1\}$ of random elements in ℓ_1. Suppose that

$$P\{A_{nj} = n^{-\alpha}\} = P\{A_{nj} = -n^{-\alpha}\} = \frac{1}{2}, \qquad j \geq 1, \quad n \geq 1$$

where $\alpha \in (0, 1]$. Let $\{T_n, n \geq 1\}$ be a sequence of identically distributed random variables satisfying (8) and with the distribution of T_1 given by (20). Let $\delta \in (0, \alpha)$ and set $k_n = [n^\delta]$, $n \geq 1$. Then

$$P\{T_n > k_n\} = P\{T_1 > k_n\} = o(1)$$

and so (9) holds. Let $\theta > 1$ and set

$$c_{n1} = \frac{1}{2}, \quad c_{nj} = j^{-\theta}, \qquad j \geq 2, \quad n \geq 1.$$

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Now

$$\sum_{j=1}^{k_n} c_{nj} E|A_{nj}| \leq \frac{\sum_{j=1}^{k_n} j^{-\theta}}{n^\alpha} \leq \frac{\sum_{j=1}^{\infty} j^{-\theta}}{n^\alpha} = o(1)$$

since $\theta > 1$ and so (10) holds. Also, for every positive integer n_0, $\theta > 1$ ensures that

$$\sup_{n \geq n_0} \sum_{j=1}^{\infty} c_{nj} E|A_{nj}| \leq \sup_{n \geq n_0} \frac{\sum_{j=1}^{\infty} j^{-\theta}}{n^\alpha} = \frac{\sum_{j=1}^{\infty} j^{-\theta}}{n_0^\alpha} < \infty$$

and so (12) holds. Now $I(\|V_{nj}\| > c_{nj}) = 1$ a.s., $j \geq 1$, $n \geq 1$. Thus

$$\begin{aligned}\sum_{j=1}^{k_n} E\|A_{nj} V_{nj} I(\|V_{nj}\| > c_{nj})\| &= \sum_{j=1}^{k_n} E\|A_{nj} V_{nj}\| \\ &= \frac{k_n}{n^\alpha} = [1 + o(1)]\frac{n^\delta}{n^\alpha} = o(1)\end{aligned}$$

since $\delta < \alpha$. Thus, (11) holds. However, for every positive integer n_0

$$\begin{aligned}\sup_{n \geq n_0} \sum_{j=1}^{\infty} E\|A_{nj} V_{nj} I(\|V_{nj}\| > c_{nj})\| &= \sup_{n \geq n_0} \sum_{j=1}^{\infty} E\|A_{nj} V_{nj}\| \\ &= \sup_{n \geq n_0} \sum_{j=1}^{\infty} \frac{1}{n^\alpha} = \infty\end{aligned}$$

and hence (13) fails. Finally, for all $n \geq 1$

$$E\left\|\sum_{j=1}^{T_n} A_{nj} V_{nj}\right\| = \frac{ET_n}{n^\alpha} = \infty$$

and hence the conclusion (14) fails. However, (15) does hold for this example by an argument similar to that given at the end of Example 4.

Example 7

Let $p = r = 1$, and consider the Rademacher type 1 Banach space ℓ_1 and the array $\{V_{nj}, j \geq 1, n \geq 1\}$ of random elements in ℓ_1. Suppose that

$$P\{A_{nj} = j^{-\theta}\} = P\{A_{nj} = -j^{-\theta}\} = \frac{1}{2}, \qquad j \geq 1, \quad n \geq 1$$

where $\theta > 1$. Let $\{T_n, n \geq 1\}$ be a sequence of identically distributed positive integer-valued random variables satisfying (8). Then

$$P\{T_n > k_n\} = P\{T_1 > k_n\} = o(1)$$

and so (9) holds. Set

$$c_{n1} = \frac{1}{2}, \quad c_{nj} = \frac{1}{n}, \qquad j \geq 2, \quad n \geq 1.$$

Now

$$\sum_{j=1}^{k_n} c_{nj} E|A_{nj}| \leq \frac{\sum_{j=1}^{k_n} j^{-\theta}}{n} \leq \frac{\sum_{j=1}^{\infty} j^{-\theta}}{n} = o(1)$$

since $\theta > 1$ and so (10) holds. Also, for every positive integer n_0, $\theta > 1$ ensures that

$$\sup_{n \geq n_0} \sum_{j=1}^{\infty} c_{nj} E|A_{nj}| \leq \sup_{n \geq n_0} \frac{\sum_{j=1}^{\infty} j^{-\theta}}{n} = \frac{\sum_{j=1}^{\infty} j^{-\theta}}{n_0} < \infty$$

and so (12) holds. Now $I(\|V_{nj}\| > c_{nj}) = 1$ a.s., $j \geq 1$, $n \geq 1$. Thus for every positive integer n_0

$$\sup_{n \geq n_0} \sum_{j=1}^{\infty} E\|A_{nj} V_{nj} I(\|V_{nj}\| > c_{nj})\| = \sup_{n \geq n_0} \sum_{j=1}^{\infty} E\|A_{nj} V_{nj}\| = \sum_{j=1}^{\infty} \frac{1}{j^{\theta}} < \infty$$

since $\theta > 1$. Thus, (13) holds. However,

$$\sum_{j=1}^{k_n} E\|A_{nj} V_{nj} I(\|V_{nj}\| > c_{nj})\| = \sum_{j=1}^{k_n} E\|A_{nj} V_{nj}\| = \sum_{j=1}^{k_n} \frac{1}{j^{\theta}} \to \sum_{j=1}^{\infty} \frac{1}{j^{\theta}} > 0$$

and hence (11) fails. Finally, for all $n \geq 1$

$$E\left\|\sum_{j=1}^{T_n} A_{nj} V_{nj}\right\| = E\left(\sum_{j=1}^{T_n} \frac{1}{j^{\theta}}\right) \geq 1$$

and hence the conclusion (14) fails. It is easy to see that (15) also fails.

Marcel Dekker, Inc.
270 Madison Avenue, New York, New York 10016

Remark 2

A slight modification of Example 3 concerning the Banach space ℓ_q reveals that the Rademacher type p hypothesis cannot be dispensed with in Theorem 2. (Take $T_n = k_n$, $n \geq 1$ and take $A_{nj} = 0, j > k_n$, $n \geq 1$.) The details are left to the reader.

In the last example, the hypotheses of Theorem 1 are satisfied but not those of Theorem 2.

Example 8

Let $p = r = 1$, and consider the Rademacher type 1 Banach space ℓ_1 and the array $\{V_{nj}, j \geq 1, n \geq 1\}$ of random elements in ℓ_1. Suppose that

$$P\{A_{nj} = b_n^{-1}\} = P\{A_{nj} = -b_n^{-1}\} = \frac{1}{2}, \qquad j \geq 1, \quad n \geq 1$$

where $\{b_n, n \geq 1\}$ is a sequence of positive constants with $k_n = o(b_n)$. Let $c_{nj} = 1, j \geq 1$, $n \geq 1$ and $T_n = k_n$, $n \geq 1$. Now (4) holds since

$$\sum_{j=1}^{k_n} c_{nj} E|A_{nj}| = \frac{k_n}{b_n} = o(1).$$

Moreover, $I(\|V_{nj}\| > c_{nj}) = 0$ a.s., $j \geq 1$, $n \geq 1$ and so (5) holds. Thus, all of the hypotheses of Theorem 1 are satisfied and so

$$\left\| \sum_{j=1}^{k_n} A_{nj} V_{nj} \right\| \xrightarrow{\mathcal{L}_1} 0.$$

However, Theorem 2 is not applicable because (12) fails as in Example 4. It may be noted that all of the other hypotheses of Theorem 2 are indeed satisfied.

Remark 3

Nevertheless, Theorem 1 can indeed be derived from Theorem 2 in the following manner: Suppose that all of the hypotheses of Theorem 1 are satisfied. Let $T_n = k_n$, $n \geq 1$. Let $V_{nj} = 0$, $c_{nj} = 0$, and $A_{nj} = 0, j > k_n$, $n \geq 1$. Now (12) and (13) follow from (4) [or (10)] and (5) [or (11)], respectively. The other assumptions to Theorem 2 either coincide with those of Theorem 1 or obviously hold. Thus, by Theorem 2, the conclusion (6) of Theorem 1 holds.

REFERENCES

1. Taylor, R.L. Stochastic convergence of weighted sums of random elements in linear spaces. *Lecture Notes in Mathematics*. Springer-Verlag: Berlin, 1978; Vol. 672.
2. Hu, T.-C.; Ordóñez Cabrera, M.; Volodin, A.I. Convergence of randomly weighted sums of Banach space valued random elements and uniform integrability concerning the random weights. Statist. Probab. Lett. **2001**, *51*, 155–164.
3. Rosalsky, A.; Sreehari, M. A weak law with random indices for randomly weighted sums of random elements in martingale type p Banach spaces. Nonlinear Anal. Theor. Meth. & Appl. **2001**, *47*, 1257–1270.
4. Rosalsky, A.; Sreehari, M.; Volodin, A.I. A weak law with random indices for randomly weighted sums of rowwise independent random elements in Rademacher type p Banach spaces. Calcutta Statist. Assoc. Bull. **2002**, *52*, 85–98.
5. Rosalsky, A.; Sreehari, M. $\mathcal{L}_r$ convergence theorems with or without random indices for randomly weighted sums of random elements in martingale type p Banach spaces. Stochastic Modelling Appl. **2001**, *4*, 1–15.
6. Gut, A. On convergence in r-mean of some first passage times and randomly indexed partial sums. Ann. Probab. **1974**, *2*, 321–323.
7. Krajka, A.; Rychlik, Z. On weak and moments convergence of randomly indexed sums. Math. Nachr. **1992**, *157*, 249–261.
8. Hong, D.H. On convergence in p-mean of randomly indexed partial sums and some first passage times for random variables which are dependent or non-identically distributed. J. Korean Statist. Assoc. **1996**, *25*, 175–183.
9. Pyke, R.; Root, D. On convergence in r-mean and normalized partial sums. Ann. Math. Statist. **1968**, *39*, 378–381.
10. Adler, A.; Rosalsky, A.; Volodin, A.I. A mean convergence theorem and weak law for arrays of random elements in martingale type p Banach spaces. Statist. Probab. Lett. **1997**, *32*, 167–174.
11. Hong, D.H.; Oh, K.S. On the weak law of large numbers for arrays. Statist. Probab. Lett. **1995**, *22*, 55–57.
12. Chandra, T.K. Uniform integrability in the Cesàro sense and the weak law of large numbers. Sankhyā Ser. A **1989**, *51*, 309–317.
13. Hong, D.H.; Ordóñez Cabrera, M.; Sung, S.H.; Volodin, A.I. Again on the weak law in martingale type p Banach spaces. Extracta Math. **1999**, *14*, 45–50.
14. Hong, D.H.; Ordóñez Cabrera, M.; Sung, S.H.; Volodin, A.I. On the weak law for randomly indexed partial sums for arrays of random elements in martingale type p Banach spaces. Statist. Probab. Lett. **2000**, *46*, 177–185.

15. Ordóñez Cabrera, M. Convergence of weighted sums of random variables and uniform integrability concerning the weights. Collect. Math. **1994**, *45*, 121–132.
16. Adler, A.; Rosalsky, A.; Volodin, A.I. Weak laws with random indices for arrays of random elements in Rademacher type *p* Banach spaces. J. Theor. Probab. **1997**, *10*, 605–623.
17. Taylor, R.L.; Padgett, W.J. Some laws of large numbers for normed linear spaces. Sankhyā Ser. A **1974**, *36*, 359–368.
18. Wei, D.; Taylor, R.L. Geometrical consideration of weighted sums convergence and random weighting. Bull. Inst. Math. Acad. Sinica **1978**, *6*, 49–59.
19. Wei, D.; Taylor, R.L. Convergence of weighted sums of tight random elements. J. Multiv. Anal. **1978**, *8*, 282–294.
20. Taylor, R.L.; Calhoun, C.A. On the almost sure convergence of randomly weighted sums of random elements. Ann. Probab. **1983**, *11*, 795–797.
21. Taylor, R.L.; Raina, C.C.; Daffer, P.Z. Stochastic convergence of randomly weighted sums of random elements. Stochastic Anal. Appl. **1984**, *2*, 299–321.
22. Ordóñez Cabrera, M. Limit theorems for randomly weighted sums of random elements in normed linear spaces. J. Multiv. Anal. **1988**, *25*, 139–145.
23. Adler, A.; Rosalsky, A.; Taylor, R.L. Some strong laws of large numbers for sums of random elements. Bull. Inst. Math. Acad. Sinica **1992**, *20*, 335–357.
24. Hoffmann-Jørgensen, J.; Pisier, G. The law of large numbers and the central limit theorem in Banach spaces. Ann. Probab. **1976**, *4*, 587–599.