Available online at www.sciencedirect.com
SCIENCE DIRECT
J. Math. Anal. Appl. 305 (2005) 644–658
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Mean convergence theorems and weak laws of large numbers for weighted sums of random variables under a condition of weighted integrability

Manuel Ordóñez Cabrera [a,*,1], Andrei I. Volodin [b,2]

[a] *Department of Mathematical Analysis, University of Sevilla, 41080 Sevilla, Spain*
[b] *Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada S4S 0A2*

Received 15 September 2004

Available online 11 January 2005

Submitted by J. Glaz

Abstract

From the classical notion of uniform integrability of a sequence of random variables, a new concept of integrability (called h-integrability) is introduced for an array of random variables, concerning an array of constants. We prove that this concept is weaker than other previous related notions of integrability, such as Cesàro uniform integrability [Chandra, Sankhyā Ser. A 51 (1989) 309–317], uniform integrability concerning the weights [Ordóñez Cabrera, Collect. Math. 45 (1994) 121–132] and Cesàro α-integrability [Chandra and Goswami, J. Theoret. Probab. 16 (2003) 655–669].

Under this condition of integrability and appropriate conditions on the array of weights, mean convergence theorems and weak laws of large numbers for weighted sums of an array of random variables are obtained when the random variables are subject to some special kinds of dependence: (a) rowwise pairwise negative dependence, (b) rowwise pairwise non-positive correlation, (c) when the sequence of random variables in every row is φ-mixing. Finally, we consider the general weak

* Corresponding author.
E-mail address: cabrera@us.es (M. Ordóñez Cabrera).

1 The research has been partially supported by DGICYT grant BFM2000-0344-C02-01 and Junta de Andalucia FQM 127.

2 The research has been partially supported by the National Science and Engineering Research Council of Canada.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2004.12.025

law of large numbers in the sense of Gut [Statist. Probab. Lett. 14 (1992) 49–52] under this new condition of integrability for a Banach space setting.

Keywords: Uniform integrability; Weighted sums; Integrability concerning the weights; Negative dependence; Non-positive correlation; φ-Mixing sequence; Random elements; Martingale type Banach space

1. Introduction

Laws of large numbers for sequences of random variables play a central role in the area of limit theorems in Probability Theory. Conditions of independence and identical distribution of random variables are basic in historic results due to Bernoulli, Borel or Kolmogorov. Since then, serious attempts have been made to relax these strong conditions. Hence, for example, independence has been relaxed to pairwise independence or pairwise non-positive correlation or, even replaced by conditions of dependence such as mixing or martingale models. In order to relax the identical distribution, several other conditions have been considered, such as stochastic domination by an integrable random variable or, in the case of the weak law, uniform integrability. It is in this condition in which we are interested.

Landers and Rogge [9] prove that the uniform integrability condition is sufficient in order that a sequence of pairwise independent random variables verifies the weak law of large numbers.

Chandra [3] obtains the weak law of large numbers under a new condition which is weaker than uniform integrability: the condition of Cesàro uniform integrability.

Ordóñez Cabrera [10], by studying the weak convergence for weighted sums of random variables, introduces the condition of uniform integrability concerning the weights, which is weaker than uniform integrability, and leads to Cesàro uniform integrability as a particular case. Under this condition, a weak law of large numbers for weighted sums of pairwise independent random variables is obtained; this condition of pairwise independence can be even dropped, at the price of slightly strengthening the conditions on the weights.

Chandra and Goswami [5] introduce the condition of Cesàro α-integrability ($\alpha > 0$), and show that Cesàro α-integrability for any $\alpha > 0$ is weaker than Cesàro uniform integrability. Under the Cesàro α-integrability condition for some $\alpha > 1/2$, they obtain the weak law of large numbers for a sequence of pairwise independent random variables. They also prove that Cesàro α-integrability for appropriate α is also sufficient for the weak law of large numbers to hold for certain special dependent sequences of random variables.

In this paper, we introduce the notion of h-integrability for an array of random variables concerning an array of constant weights, and we prove that this concept is weaker than Cesàro uniform integrability, $\{a_{nk}\}$-uniform integrability and Cesàro α-integrability.

Under appropriate conditions on the weights, we prove that h-integrability concerning the weights is sufficient for a mean convergence theorem and a weak law of large numbers to hold for weighted sums of an array of random variables, when these random variables are subject to some special kind of rowwise dependence, and, of course, when the array of random variables is pairwise independent.

2. Definitions and relations between the new concept and some previous ones

The classical notion of uniform integrability of a sequence $\{X_n,\ n \geqslant 1\}$ of integrable random variables is defined through the condition

$$\lim_{a\to\infty} \sup_{n\geqslant 1} E|X_n| I\big[|X_n| > a\big] = 0.$$

Let $\Phi = \{\phi : (0,\infty) \to (0,\infty),\ \phi(t)/t \uparrow \infty \text{ as } t \uparrow \infty\}$. For $\phi \in \Phi$ we put $\phi(0) = 0$.

It is well known the classical result of La Vallée-Poussin: $\{X_n\}$ is uniformly integrable if, and only if, there exists $\phi \in \Phi$ such that $\sup_{n\geqslant 1} E\phi(|X_n|) < \infty$.

Chandra [3] introduces the notion of Cesàro uniform integrability, which is weaker than the notion of uniform integrability, through the condition

$$\lim_{a\to\infty} \sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} E|X_k| I\big[|X_k| > a\big] = 0.$$

Chandra and Goswami [4] obtained a characterization of this concept in the same way as the one of La Vallée-Poussin for uniform integrability.

Ordóñez Cabrera [10] introduces the notion of $\{a_{nk}\}$-uniform integrability or uniform integrability concerning an array $\{a_{nk}\}$ of weights, which is weaker than Cesàro uniform integrability.

In the following let $\{u_n,\ n \geqslant 1\}$ and $\{v_n,\ n \geqslant 1\}$ be two sequences of integers (not necessary positive or finite) such that $v_n > u_n$ for all $n \geqslant 1$ and $v_n - u_n \to \infty$ as $n \to \infty$.

Definition. Let $\{X_{nk},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ be an array of random variables and $\{a_{nk},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ an array of constants with $\sum_{k=u_n}^{v_n} |a_{nk}| \leqslant C$ for all $n \in \mathbf{N}$ and some constant $C > 0$. The array $\{X_{nk}\}$ is $\{a_{nk}\}$-uniformly integrable if

$$\lim_{a\to\infty} \sup_{n\geqslant 1} \sum_{k=u_n}^{v_n} |a_{nk}| E|X_{nk}| I\big[|X_{nk}| > a\big] = 0.$$

In a similar way to the classical extension of La Vallée-Poussin, Ordóñez Cabrera [10] proved that $\{X_{nk}\}$ is $\{a_{nk}\}$-uniformly integrable if and only if there exists $\phi \in \Phi$ such that

$$\sup_{n\geqslant 1} \sum_{k=u_n}^{v_n} |a_{nk}| E\phi\big(|X_{nk}|\big) < \infty.$$

Chandra and Goswami [5] introduce the following concept, weaker than Cesàro uniform integrability, which is again related with the tail probability of the random variables:

Definition. Let $\alpha > 0$. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be Cesàro α-integrable if the following two conditions hold:

$$\sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} E|X_k| < \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} E|X_k| I\big[|X_k| > k^{\alpha}\big] = 0.$$

We now introduce a new concept of integrability.

Definition. Let $\{X_{nk},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ be an array of random variables and $\{a_{nk},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ an array of constants with $\sum_{k=u_n}^{v_n} |a_{nk}| \leqslant C$ for all $n \in \mathbf{N}$ and some constant $C > 0$. Let moreover $\{h(n),\ n \geqslant 1\}$ be an increasing sequence of positive constants with $h(n) \uparrow \infty$ as $n \uparrow \infty$. The array $\{X_{nk}\}$ is said to be h-integrable with respect to the array of constants $\{a_{nk}\}$ if the following conditions hold:

$$\sup_{n\geqslant 1} \sum_{k=u_n}^{v_n} |a_{nk}| E|X_{nk}| < \infty \quad \text{and} \quad \lim_{n\to\infty} \sum_{k=u_n}^{v_n} |a_{nk}| E|X_{nk}| I\big[|X_{nk}| > h(n)\big] = 0.$$

Remark 1. For the *classical triangular arrays*, as in [5]:

$$a_{nk} = \begin{cases} \frac{1}{n} & \text{if } 1 \leqslant k \leqslant n, \\ 0 & \text{if } k > n, \end{cases} \qquad X_{nk} = \begin{cases} X_k & \text{if } 1 \leqslant k \leqslant n, \\ 0 & \text{if } k > n, \end{cases}$$

and

$$h(n) = n^{\alpha}, \quad \alpha > 0,$$

we can write:

$$\sum_{k=1}^{n} |a_{nk}| E|X_{nk}| I\big[|X_{nk}| > h(n)\big] = \frac{1}{n} \sum_{k=1}^{n} E|X_k| I\big[|X_k| > n^{\alpha}\big] \leqslant \frac{1}{n} \sum_{k=1}^{n} E|X_k| I\big[|X_k| > k^{\alpha}\big]$$

for each $n \geqslant 1$. Moreover:

$$\sum_{k=1}^{n} |a_{nk}| E|X_{nk}| = \frac{1}{n} \sum_{k=1}^{n} E|X_k|$$

for each $n \geqslant 1$. Therefore if $\{X_n\}$ is Cesàro α-integrable, then $\{X_{nk}\}$ is h-integrable concerning the array of constants $\{a_{nk}\}$ for these arrays $\{X_{nk}\}$ and $\{a_{nk}\}$ and sequence h.

Hence, our condition of being h-integrable concerning the array of constants $\{a_{nk}\}$ is not only more general, it is also weaker than Cesàro α-integrability of Chandra and Goswami [5].

The following lemma relates the new concept with the concept of $\{a_{nk}\}$-uniform integrability.

Lemma 1. *If $\{X_{nk},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ is $\{a_{nk}\}$-uniformly integrable, then $\{X_{nk},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ is h-integrable concerning the array of constants $\{a_{nk}\}$ for all monotone increasing to infinity sequences h.*

Proof. (1) If $\{X_{nk}\}$ is $\{a_{nk}\}$-uniformly integrable, then there exists $a > 0$ such that

$$\sup_{n\geqslant 1}\sum_{k=u_n}^{v_n}|a_{nk}|E|X_{nk}|I\big[|X_{nk}|>a\big]<1$$
$$\Rightarrow\quad \sup_{n\geqslant 1}\sum_{k=u_n}^{v_n}|a_{nk}|E|X_{nk}|=\sup_{n\geqslant 1}\left(\sum_{k=u_n}^{v_n}|a_{nk}|E|X_{nk}|I\big[|X_{nk}|\leqslant a\big]\right.$$
$$\left.+\sup_{n\geqslant 1}\sum_{k=u_n}^{v_n}|a_{nk}|E|X_{nk}|I\big[|X_{nk}|>a\big]\right)$$
$$\leqslant aC+1<\infty.$$

The first condition in the definition is verified.

(2) An array $\{X_{nk}\}$ is $\{a_{nk}\}$-uniformly integrable if, and only if, there exists $\phi\in\Phi$ such that

$$\sup_{n\geqslant 1}\sum_{k=u_n}^{v_n}|a_{nk}|E\phi\big(|X_{nk}|\big)<\infty$$

(see [10]). Then, for all increasing to infinity sequence $\{h(n),\ n\geqslant 1\}$, note that

$$|X_{nk}|>h(n)\quad\Rightarrow\quad\frac{\phi(|X_{nk}|)}{|X_{nk}|}\geqslant\frac{\phi(h(n))}{h(n)},$$

and so

$$E|X_{nk}|I\big[|X_{nk}|>h(n)\big]\leqslant\frac{h(n)}{\phi(h(n))}E\phi\big(|X_{nk}|\big).$$

It follows that, for every $n\geqslant 1$:

$$\sum_{k=u_n}^{v_n}|a_{nk}|E|X_{nk}|I\big[|X_{nk}|>h(n)\big]\leqslant\frac{h(n)}{\phi(h(n))}\sum_{k=u_n}^{v_n}|a_{nk}|E\phi\big(|X_{nk}|\big)$$

and, since

$$\sup_{n\geqslant 1}\sum_{k=u_n}^{v_n}|a_{nk}|E\phi\big(|X_{nk}|\big)<\infty\quad\text{and}\quad\lim_{n\to\infty}\frac{h(n)}{\phi(h(n))}=0,$$

we have

$$\lim_{n\to\infty}\sum_{k=u_n}^{v_n}|a_{nk}|E|X_{nk}|I\big[|X_{nk}|>h(n)\big]=0,$$

which is the second condition in the definition. □

Remark 2. The concept of h-integrability concerning the array $\{a_{nk}\}$ is strictly weaker than the concept of $\{a_{nk}\}$-uniform integrability, i.e., there exist arrays $\{X_{nk}\}$ and $\{a_{nk}\}$ such that $\{X_{nk}\}$ is h-integrable concerning the array of constants $\{a_{nk}\}$, but not $\{a_{nk}\}$-uniformly integrable.

We refer to [5, Example 2.2] for such an example for the case of the classical triangular arrays from Remark 1.

Remark 3. Let $\{h_1(n), n \geqslant 1\}$ and $\{h_2(n), n \geqslant 1\}$ be two monotonically increasing to infinity positive sequences such that $h_2(n) \geqslant h_1(n)$ for all sufficiently large n. Then h_1-integrability concerning the array of constants $\{a_{nk}\}$ implies h_2-integrability concerning the same array of constants.

3. Mean convergence theorems and weak laws of large numbers for weighted sums of random variables with some conditions of dependence

In this section we obtain some weak laws of large numbers for weighted sums of arrays of h-integrable random variables under some conditions of dependence. Namely, we consider the following rowwise dependence structures for an array: low case negative dependence, non-positive correlation, and φ-mixing.

3.1. Low case negatively dependent random variables

In the first theorem of this section, we are going to show that, for an array of rowwise pairwise low case negatively dependent random variables, a certain technique of truncation which preserves the negative dependence can be used to obtain a weak law of large numbers for the weighted sums under the condition of h-integrability concerning the array of constants $\{a_{nk}\}$.

Definition. Random variables X and Y are *lower case negatively dependent* (LCND, in short) if

$$P[X \leqslant x,\ Y \leqslant y] \leqslant P[X \leqslant x]P[Y \leqslant y] \quad \text{for all } x, y \in \mathbf{R}.$$

A sequence of random variables $\{X_n,\ n \geqslant 1\}$ is said to be *pairwise lower case negatively dependent* if every pair of random variables in the sequence are LCND.

The following lemmas are well known (cf., for example, [6]). Lemma 2 states that pairwise LCND random variables are non-positive correlated, while Lemma 3 gives a certain technique of truncation that preserves the negative dependence property.

Lemma 2. *If $\{X_n,\ n \geqslant 1\}$ is a sequence of pairwise LCND random variables, then*

$$E(X_i X_j) \leqslant EX_i EX_j, \quad i \neq j.$$

Lemma 3. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of pairwise LCND random variables. Then, for any sequences $\{a_n,\ n \geqslant 1\}$ and $\{b_n,\ n \geqslant 1\}$ of constants such that $a_n < b_n$ for all $n \in \mathbf{N}$, the sequence $\{Y_n,\ n \geqslant 1\}$ is a sequence of pairwise LCND random variables, where*

$$Y_n = X_n I[a_n \leqslant X_n \leqslant b_n] + a_n I[X_n < a_n] + b_n I[X_n > b_n].$$

Theorem 1. *Let $\{X_{nk},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ be an array of rowwise pairwise LCND random variables and $\{a_{nk},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ be an array of non-negative constants such that $\sum_{k=u_n}^{v_n} a_{nk} \leqslant C$ for all $n \geqslant 1$ and some constant $C > 0$. Let moreover $\{h(n),\ n \geqslant 1\}$ be a sequence of increasing to infinity positive constants. Suppose that*

(a) $\{X_{nk}\}$ *is* h*-integrable concerning the array of constants* $\{a_{nk}\}$,
(b) $h^2(n)\sum_{k=u_n}^{v_n} a_{nk}^2 \to 0$ *as* $n \to \infty$.

Let $S_n = \sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - EX_{nk})$, *for all* $n \geqslant 1$. *Then* $S_n \to 0$ *as* $n \to \infty$ *in* L_1 *and hence in probability.*

Proof. For each $n \in \mathbf{N}$, $u_n \leqslant k \leqslant v_n$, let

$$Y_{nk} = X_{nk} I\big[|X_{nk}| \leqslant h(n)\big] - h(n) I\big[X_{nk} < -h(n)\big] + h(n) I\big[X_{nk} > h(n)\big],$$
$$A_{1n} = \sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - Y_{nk}), \qquad A_{2n} = \sum_{k=u_n}^{v_n} a_{nk}(Y_{nk} - EY_{nk}), \quad \text{and}$$
$$A_{3n} = \sum_{k=u_n}^{v_n} a_{nk} E(Y_{nk} - X_{nk}).$$

It follows from the following that in the case of infinite u_n and/or v_n, the series A_{1n}, A_{2n}, and A_{3n} converge absolutely in L_1. Hence, we can write that

$$S_n = A_{1n} + A_{2n} + A_{3n}$$

and we will estimate each of these terms separately.

The series A_{1n} and A_{3n} can be estimated:

$$|EA_{1n}| = |A_{3n}| \leqslant \sum_{k=u_n}^{v_n} a_{nk} E|X_{nk} - Y_{nk}| \leqslant \sum_{k=u_n}^{v_n} a_{nk} E|X_{nk}| I\big[|X_{nk}| > h(n)\big] \to 0$$

as $n \to \infty$, because

$$|X_{nk} - Y_{nk}| = \begin{cases} 0 & \text{if } |X_{nk}| \leqslant h(n), \\ |X_{nk} + h(n)| & \text{if } X_{nk} < -h(n), \\ |X_{nk} - h(n)| & \text{if } X_{nk} > h(n). \end{cases}$$

Hence $A_{1n} \to 0$ in L_1, and, in the same way, $A_{3n} \to 0$.

For A_{2n} we actually prove that $A_{2n} \to 0$ in L_2 and hence in L_1.

$$\begin{aligned} 0 &\leqslant E\left[\sum_{k=u_n}^{v_n} a_{nk}(Y_{nk} - EY_{nk})\right]^2 \\ &= \sum_k a_{nk}^2 E(Y_{nk} - EY_{nk})^2 + \sum_{j \neq k} a_{nj} a_{nk}\big[E(Y_{nj} Y_{nk}) - EY_{nj} EY_{nk}\big] \\ &\leqslant \sum_k a_{nk}^2 EY_{nk}^2 + \sum_{j \neq k} a_{nj} a_{nk}\big[E(Y_{nj} Y_{nk}) - EY_{nj} EY_{nk}\big] = B_{1n} + B_{2n}, \quad \text{say.} \end{aligned}$$

But

$$B_{1n} = \sum_{k=u_n}^{v_n} a_{nk}^2 E\big[X_{nk}^2 I\big[|X_{nk}| \leqslant h(n)\big] + h^2(n) I\big[|X_{nk}| > h(n)\big]\big]$$
$$\leqslant 2h^2(n) \sum_{k=u_n}^{v_n} a_{nk}^2 \to 0 \quad \text{as } n \to \infty.$$

With regard to B_{2n}, by applying Lemmas 2 and 3, we have:

$$E(Y_{nj}Y_{nk}) - EY_{nj}EY_{nk} \leqslant 0, \quad j \neq k, \text{ for each } n \geqslant 1,$$

and hence

$$0 \leqslant EA_{2n}^2 \leqslant \sum_k a_{nk}^2 EY_{nk}^2 \to 0 \quad \text{as } n \to \infty,$$

and $A_{2n} \to 0$ in L_2, and hence in L_1. □

A particular case of pairwise LCND random variables is the case of pairwise independent random variables. Therefore, we get the following corollary of Theorem 1, which is, in various senses, an extension of [5, Theorem 2.2(a)].

Corollary 1. *Let $\{X_{nk}, u_n \leqslant k \leqslant v_n, n \geqslant 1\}$ be an array of rowwise pairwise independent random variables. Let $\{a_{nk}, u_n \leqslant k \leqslant v_n, n \geqslant 1\}$ be an array of non-negative constants such that $\sum_{k=u_n}^{v_n} a_{nk} \leqslant C$ for all $n \geqslant 1$ and some constant $C > 0$ and $\{h(n), n \geqslant 1\}$ be a sequence of increasing to infinity positive constants. Suppose that*

(a) *$\{X_{nk}\}$ is h-integrable concerning the array of constants $\{a_{nk}\}$,*
(b) *$h^2(n) \sum_{k=u_n}^{v_n} a_{nk}^2 \to 0$ as $n \to \infty$.*

Let $S_n = \sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - EX_{nk})$, for all $n \geqslant 1$. Then $S_n \to 0$ as $n \to \infty$, in L_1 and hence in probability.

3.2. Non-positively correlated random variables

The careful analysis of the proof of Theorem 1 shows that the most important property of LCND random variables we use in the proof of Theorem 1 is that the specially truncated random variables are non-positive correlated (cf. estimation of the term B_2 applying Lemmas 2 and 3). Hence it is interesting to establish the weak law of large numbers for non-positively correlated random variables. In the following theorem we show that the condition of negative dependence can be replaced by the conditions of non-positive correlation and non-negativity of random variables.

Theorem 2. *Let $\{X_{nk}, u_n \leqslant k \leqslant v_n, n \geqslant 1\}$ be an array of non-negative random variables satisfying $E(X_{nj}X_{nk}) - EX_{nj}EX_{nk} \leqslant 0$, $j \neq k$, for each $n \geqslant 1$.*

Let $\{a_{nk}, u_n \leqslant k \leqslant v_n, n \geqslant 1\}$ be an array of non-negative constants such that $\sum_{k=u_n}^{v_n} a_{nk} \leqslant C$ for all $n \geqslant 1$ and some constant $C > 0$ and $\{h(n), n \geqslant 1\}$ be a sequence of increasing to infinity positive constants. Suppose that

(a) $\{X_{nk}\}$ *is* h*-integrable concerning the array of constants* $\{a_{nk}\}$,
(b) $h^2(n)\sum_{k=u_n}^{v_n} a_{nk}^2 \to 0$ *as* $n \to \infty$.

Let $S_n = \sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - EX_{nk})$, *for all* $n \geqslant 1$. *Then* $S_n \to 0$ *as* $n \to \infty$, *in* L_1 *and hence in probability.*

Proof. The proof is similar to the proof of Theorem 1, only now we can use usual truncation technique. Hence, for each $n \geqslant 1$, $u_n \leqslant k \leqslant v_n$, let

$$Y_{nk} = X_{nk} I\big[X_{nk} \leqslant h(n)\big],$$
$$A_{1n} = \sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - Y_{nk}), \qquad A_{2n} = \sum_{k=u_n}^{v_n} a_{nk}(Y_{nk} - EY_{nk}), \quad \text{and}$$
$$A_{3n} = \sum_{k=u_n}^{v_n} a_{nk} E(Y_{nk} - X_{nk}).$$

It follows from the following that in the case of infinite u_n and/or v_n, the series A_{1n}, A_{2n}, and A_{3n} converge absolutely in L_1. Hence, we can write that

$$S_n = A_{1n} + A_{2n} + A_{3n}$$

and we will estimate each of these terms separately.

The series A_{1n} and A_{3n} can be estimated:

$$E|A_{1n}| = EA_{1n} = -A_{3n} = |A_{3n}| = \sum_{k=u_n}^{v_n} a_{nk} EX_{nk} I\big[X_{nk} > h(n)\big] \to 0$$

as $n \to \infty$. So, $A_{1n} \to 0$ in L_1 and, in the same way $A_{3n} \to 0$.

For A_{2n} we actually prove that $A_{2n} \to 0$ in L_2 and hence in L_1.

$$0 \leqslant E\left[\sum_{k=u_n}^{v_n} a_{nk}(Y_{nk} - EY_{nk})\right]^2$$
$$\leqslant \sum_k a_{nk}^2 EY_{nk}^2 + \sum_{j \neq k} a_{nj} a_{nk}\big[E(Y_{nj} Y_{nk}) - EY_{nj} EY_{nk}\big] = B_{1n} + B_{2n}, \quad \text{say.}$$

But

$$B_{1n} = \sum_{k=u_n}^{v_n} a_{nk}^2 EY_{nk}^2 \leqslant h^2(n) \sum_{k=u_n}^{v_n} a_{nk}^2 \to 0, \quad \text{when } n \to \infty.$$

Next, it suffices to show that $\limsup_{n\to\infty} B_{2n} \leqslant 0$. Since every random variable X_{nk} and every constant a_{nk} are non-negative:

$$
\begin{aligned}
&\sum_{j\neq k} a_{nj}a_{nk}E(Y_{nj}Y_{nk} - EY_{nj}EY_{nk}) \\
&\quad\leqslant \sum_{j\neq k} a_{nj}a_{nk}E(X_{nj}X_{nk} - EY_{nj}EY_{nk}) \\
&\quad\quad \text{(by the hypothesis of non-positive correlation)} \\
&\quad\leqslant \sum_{j\neq k} a_{nj}a_{nk}(EX_{nj}EX_{nk} - EY_{nj}EY_{nk}) \\
&\quad\leqslant \sum_{j,k=u_n}^{v_n} a_{nj}a_{nk}(EX_{nj}EX_{nk} - EY_{nj}EY_{nk}) \\
&\quad= \sum_{j,k=u_n}^{v_n} a_{nj}a_{nk}\big[(EX_{nj} - EY_{nj})EX_{nk} + (EX_{nk} - EY_{nk})EY_{nj}\big] \\
&\quad= \sum_{j,k=u_n}^{v_n} a_{nj}a_{nk}\big[EX_{nk}EX_{nj}I\big[X_{nj} > h(n)\big] + EY_{nj}EX_{nk}I\big[X_{nk} > h(n)\big]\big] \\
&\quad= \left(\sum_{j=u_n}^{v_n} a_{nj}EX_{nj}I\big[X_{nj} > h(n)\big]\right)\left(\sum_{k=u_n}^{v_n} a_{nk}EX_{nk}\right) \\
&\quad\quad + \left(\sum_{j=u_n}^{v_n} a_{nj}EY_{nj}\right)\left(\sum_{k=u_n}^{v_n} a_{nk}EX_{nk}I\big[X_{nk} > h(n)\big]\right) \\
&\quad\leqslant 2\left(\sum_{j=u_n}^{v_n} a_{nj}EX_{nj}\right)\left(\sum_{k=u_n}^{v_n} a_{nk}EX_{nk}I\big[X_{nk} > h(n)\big]\right) \to 0
\end{aligned}
$$

as $n \to \infty$, because $\{X_{nk}\}$ is h-integrable concerning the array of constants $\{a_{nk}\}$. □

Remark 4. The condition h-integrability concerning the array of constants $\{a_{nk}\}$ is weaker than the condition Cesàro α-integrability in the case of the *classical weights* in Remark 2. Moreover, if $\alpha \in (0, 1/2)$ and since for the classical weights $h(n) = n^\alpha$ we have:

$$
h^2(n)\sum_{k=1}^{n} a_{nk}^2 = h^2(n)\sum_{k=1}^{n}\frac{1}{n^2} = \frac{h^2(n)}{n} \to 0 \quad \text{as } n \to \infty.
$$

Hence Theorem 2 contains as a particular case [5, Theorem 2.1(a)].

3.3. φ-mixing random variables

Definition. Let $\{X_n,\ -\infty < n < \infty\}$ be a sequence of random variables. Let $\mathcal{B}^k$ be the σ-algebra generated by $\{X_n,\ n \leqslant k\}$, and $\mathcal{B}_k$ the σ-algebra generated by $\{X_n,\ n \geqslant k\}$. We say that $\{X_n,\ -\infty < n < \infty\}$ is *φ-mixing* if there exists a non-negative sequence $\{\varphi(i),\ i \geqslant 1\}$, with $\lim_{i\to\infty}\varphi(i) = 0$, such that, for each $-\infty < k < \infty$ and for each $i \geqslant 1$,

$$
\big|P(E_2|E_1) - P(E_2)\big| \leqslant \varphi(i) \quad \text{for } E_1 \in \mathcal{B}^k,\ E_2 \in \mathcal{B}_{k+i}.
$$

The following lemma is in Billingsley [2]:

Lemma 4. *Let ζ be a $\mathcal{B}^k$-measurable random variable, and η be a $\mathcal{B}_{i+k}$-measurable random variable, with $|\zeta| \leqslant C_1$ and $|\eta| \leqslant C_2$. Then*:

$$\bigl|\mathrm{Cov}(\zeta, \eta)\bigr| \leqslant 2C_1C_2\varphi(i).$$

Definition. Let m be a positive integer. The sequence of random variables $\{X_n,\ -\infty < n < \infty\}$ is said to be *m-dependent* if the random vectors $(X_l, \dots, X_k)$ and $(X_{k+i}, \dots, X_j)$ are independent for all integers i, j, k, l satisfying $i > m$, $l < k$ and $k + i < j$. It is immediate that such a sequence is φ-mixing with $\varphi(i) = 0$ for $i > m$.

Theorem 3. *Let $\{X_{nk},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ be an array of random variables such that for each $n \geqslant 1$ the row $\{X_{nk},\ u_n \leqslant k \leqslant v_n\}$ is a φ_n-mixing sequence of random variables with*

$$\limsup_{n\to\infty} \sum_{i=1}^{v_n-u_n} \varphi_n(i) < \infty.$$

Let $\{a_{nk},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ be an array of non-negative constants such that $\sum_{k=u_n}^{v_n} a_{nk} \leqslant C$ for all $n \in \mathbf{N}$ and some constant $C > 0$, and $a_{nj} \leqslant a_{ni}$ if $i < j$ for all $n \geqslant 1$. Let moreover $\{h(n),\ n \geqslant 1\}$ be a sequence of increasing to infinity positive constants. Suppose that

(a) *$\{X_{nk}\}$ is h-integrable concerning the array of constants $\{a_{nk}\}$,*
(b) *$h^2(n) \sum_{k=u_n}^{v_n} a_{nk}^2 \to 0$ as $n \to \infty$.*

Let $S_n = \sum_{k=u_n}^{v_n} a_{nk} (X_{nk} - EX_{nk})$, for all $n \geqslant 1$. Then $S_n \to 0$ as $n \to \infty$, in L_1 and hence in probability.

Proof. Let Y_{nk}, A_{1n}, A_{2n}, A_{3n}, B_{1n}, and B_{2n} be the same as in the proof of Theorem 2. We proceed as in the proof of Theorem 2 in order to prove that $A_{1n} \to 0$ in L_1, $A_{3n} \to 0$, and $B_{1n} \to 0$ when $n \to \infty$.

We only need to prove that

$$\limsup_{n\to\infty} \sum_{\substack{k,j=u_n\\k<j}}^{v_n} a_{nk}a_{nj}\mathrm{Cov}(Y_{nk}Y_{nj}) \leqslant 0.$$

By applying Lemma 4:

$$\sum_{\substack{k,j=u_n\\k<j}}^{v_n} a_{nk}a_{nj}\mathrm{Cov}(Y_{nk}Y_{nj})$$
$$= \sum_{i=1}^{v_n-u_n} \sum_{k=u_n}^{v_n-i} a_{nk}a_{n(k+i)}\mathrm{Cov}(Y_{nk}Y_{nj}) \leqslant 2h^2(n) \sum_{i=1}^{v_n-u_n} \sum_{k=u_n}^{v_n-i} a_{nk}^2\varphi_n(i)$$

$$\leqslant 2h^2(n)\sum_{k=u_n}^{v_n} a_{nk}^2 \sum_{i=1}^{v_n-u_n} \varphi_n(i) \to 0. \qquad \square$$

Corollary 2. *Let $\{X_{nk},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ be an array of random variables such that for each $n \geqslant 1$, $\{X_{nk},\ u_n \leqslant k \leqslant v_n\}$ is a $m(n)$-dependent sequence of random variables with $\limsup_{n\to\infty} m(n) < \infty$. Let the other conditions of Theorem 3 be satisfied.*

Then $S_n \to 0$ as $n \to \infty$, in L_1 and hence in probability.

Proof. We only have to note that we can consider $\varphi_n(i) = 0$ for $i > m(n)$ and $\varphi_n(i) = 1$ for $i \leqslant m(n)$, and so $\sum_{i=1}^{v_n-u_n} \varphi_n(i) \leqslant m(n)$ for all $n \geqslant 1$. $\square$

4. Gut's general mean convergence theorem and weak law of large numbers in a Banach space setting

Consider an array of random elements $\{V_{nk},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a real separable Banach space $\mathcal{X}$ with norm $\|\cdot\|$. Let $\{c_{nk},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ be a "centering" array consisting of (suitably selected) conditional expectations. In this section, a general mean convergence theorem will be established. This convergence result is of the form

$$\left\| \sum_{k=u_n}^{v_n} a_{nk}(V_{nk} - c_{nk}) \right\| \xrightarrow{L_r} 0.$$

The hypotheses to Theorem 4 impose conditions on the growth behavior of the weights $\{a_{nk},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ and on the marginal distributions of the random variables $\{\|V_{nk}\|,\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$. The random elements in the array under our consideration are not assumed to be rowwise independent. Indeed, no conditions are imposed on the joint distributions of the random elements comprising the array. However, the Banach space $\mathcal{X}$ is assumed to be of martingale type p. (Technical definitions such as this will be discussed below.)

Theorem 4 is an extension to a martingale type-p Banach space setting of results of Gut [7] which were proved for arrays of (real-valued) random variables and the generalization to h-integrable arrays of results of Adler, Rosalsky, and Volodin [1] which were proved under the uniform integrability concerning an array of weights condition. Theorem 4, which establishes the L_r convergence result and hence the weak law of large numbers concerns arrays of random elements satisfying the h-integrability concerning the array of constants condition which is more general than the Cesàro uniform integrability condition employed by Gut [7] and the condition of uniform integrability concerning an array of weights employed by Adler, Rosalsky, and Volodin [1]. As will be apparent, the proof of Theorem 4 owes much to this earlier article.

Technical definitions relevant to the current section will now be discussed.

Scalora [12] introduced the idea of the *conditional expectation* of a random element in a Banach space. For a random element V and sub σ-algebra $\mathcal{G}$ of $\mathcal{F}$, the conditional expectation $E(V \mid \mathcal{G})$ is defined analogously to that in the random variable case and enjoys similar properties. See [12] for a complete development, as well as for a development of Banach space valued martingales including martingale convergence theorems.

Definition. A real separable Banach space $\mathcal{X}$ is said to be of *martingale type p* $(1 \leqslant p \leqslant 2)$ if there exists a finite constant C such that for all martingales $\{S_n, n \geqslant 1\}$ with values in $\mathcal{X}$,

$$\sup_{n \geqslant 1} E\|S_n\|^p \leqslant C \sum_{n=1}^{\infty} E\|S_n - S_{n-1}\|^p \quad \text{where } S_0 \equiv 0.$$

It can be shown using classical methods from martingale theory that if $\mathcal{X}$ is of martingale type p, then for all $1 \leqslant r < \infty$ there exists a finite constant C' such that for all $\mathcal{X}$-valued martingales $\{S_n, n \geqslant 1\}$

$$E \sup_{n \geqslant 1} \|S_n\|^r \leqslant C' E\left(\sum_{n=1}^{\infty} \|S_n - S_{n-1}\|^p\right)^{r/p}.$$

Clearly every real separable Banach space is of martingale type 1. It follows from the Hoffmann–Jørgensen and Pisier [8] characterization of Rademacher type p Banach spaces that if a Banach space is of martingale type p, then it is of Rademacher type p. But the notion of martingale type p is only superficially similar to that of Rademacher type p and has a geometric characterization in terms of smoothness. For proofs and more details, the reader may refer to [11].

With these preliminaries accounted for, the theorem may now be stated and proved.

Theorem 4. *Let $1 \leqslant r \leqslant p \leqslant 2$ and $\{h(n), n \geqslant 1\}$ be a sequence of increasing to infinity positive constants. Let $\{V_{nk}, u_n \leqslant k \leqslant v_n, n \geqslant 1\}$ be an array of random elements in a real separable, martingale type p Banach space and let $\{a_{nk}, u_n \leqslant k \leqslant v_n, n \geqslant 1\}$ be an array of constants such that $\sum_{k=u_n}^{v_n} |a_{nk}| \leqslant C$ for all $n \geqslant 1$ and some constant $C > 0$. Suppose that*

(a) *$\{\|V_{nk}\|^r, u_n \leqslant k \leqslant v_n, n \geqslant 1\}$ is h^r-integrable concerning the array of constants $\{|a_{nk}|^r\}$,*
(b) *$h^p(n) \sum_{k=u_n}^{v_n} |a_{nk}|^p \to 0$ as $n \to \infty$.*

Then

$$\left\| \sum_{k=u_n}^{v_n} a_{nk}\big(V_{nk} - E(V_{nk} \mid \mathcal{F}_{n,k-1})\big) \right\| \to 0$$

in L_r and, a fortiori, in probability, where $\mathcal{F}_{nk} = \sigma(V_{ni}, u_n \leqslant i \leqslant k)$, $u_n \leqslant k \leqslant v_n$, $n \geqslant 1$, and $\mathcal{F}_{n,u_n-1} = \{\emptyset, \Omega\}$, $n \geqslant 1$.

Proof. For any $n \geqslant 1$, set

$$V'_{nk} = (V_{nk} - \mu_{nk}) I\big[\|V_{nk} - \mu_{nk}\| \leqslant h(n)\big], \quad u_n \leqslant k \leqslant v_n, \ n \geqslant 1,$$
$$V''_{nk} = (V_{nk} - \mu_{nk}) I\big[\|V_{nk} - \mu_{nk}\| > h(n)\big], \quad u_n \leqslant k \leqslant v_n, \ n \geqslant 1,$$

where $\mu_{nk} = E(V_{nk} \mid \mathcal{F}_{n,k-1})$, $u_n \leqslant k \leqslant v_n$, $n \geqslant 1$. Note that for each $n \geqslant 1$ the row $\{V_{nk} - \mu_{nk},\ u_n \leqslant k \leqslant v_n\}$ forms a martingale difference sequence. Then

$$
\begin{aligned}
&E\left\| \sum_{k=u_n}^{v_n} a_{nk}\bigl(V_{nk} - E(V_{nk} \mid \mathcal{F}_{n,k-1})\bigr) \right\|^r \\
&\leqslant C' E\left(\sum_{k=u_n}^{v_n} \bigl(|a_{nk}|\,\|V_{nk} - \mu_{nk}\|\bigr)^p \right)^{r/p} \quad \text{(by martingale type property)} \\
&\leqslant C' 2^{(p-1)r/p} E\left(\sum_{k=u_n}^{v_n} |a_{nk}|^p \|V'_{nk}\|^p + \sum_{k=u_n}^{v_n} |a_{nk}|^p \|V''_{nk}\|^p \right)^{r/p} \\
&\leqslant C'' E\left(\left(\sum_{k=u_n}^{v_n} |a_{nk}|^p \|V'_{nk}\|^p \right)^{r/p} + \sum_{k=u_n}^{v_n} |a_{nk}|^r \|V''_{nk}\|^r \right) \quad \left(\text{since } \frac{r}{p} \leqslant 1\right) \\
&\leqslant C'' \left(h^p(n) \sum_{k=u_n}^{v_n} |a_{nk}|^p \right)^{r/p} + C'' \sum_{k=u_n}^{v_n} |a_{nk}|^r E\|V''_{nk}\|^r.
\end{aligned}
$$

Next, the array $\{\|V_{nk} - \mu_{nk}\|^r,\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ is h^r-integrable concerning the array $\{|a_{nj}|^r\}$ by using the approach of Gut [7]. (The only modification needed in the argument is that the definition of h integrability concerning the array is used instead of [3, Theorem 3].) The conclusion follows by letting $n \to \infty$. □

Remark 5. Theorem 4 should be compared with [10, Theorem 6] which establishes an L_r convergence result $(0 < r < 1)$ for weighted sums of random elements where there is not any martingale type p condition (or any other geometric condition) given on the Banach space.

The real line, as is any Hilbert space, is of martingale type 2. Hence we can formulate the following corollary to Theorem 4.

Corollary 3. *Let $\{X_{nk},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ be an array of random variables and $\{a_{nk},\ u_n \leqslant k \leqslant v_n,\ n \geqslant 1\}$ be an array of constants such that $\sum_{k=u_n}^{v_n} |a_{nk}| \leqslant C$ for all $n \geqslant 1$ and some constant $C > 0$. Let $1 \leqslant r \leqslant 2$ and $\{h(n),\ n \geqslant 1\}$ be a sequence of increasing to infinity positive constants. Suppose that*

(a) *$\{|X_{nk}|^r\}$ is h^r-integrable concerning the array of constants $\{|a_{nk}|^r\}$,*
(b) *$h^2(n) \sum_{k=u_n}^{v_n} a_{nk}^2 \to 0$ as $n \to \infty$.*

Let $S_n = \sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - E(X_{nk} \mid \mathcal{F}_{n,k-1}))$, for all $n \geqslant 1$, where $\mathcal{F}_{nk} = \sigma(X_{ni},\ u_n \leqslant i \leqslant k)$, $u_n \leqslant k \leqslant v_n$, $n \geqslant 1$ and $\mathcal{F}_{n,u_n-1} = \{\emptyset, \Omega\}$, $n \geqslant 1$. Then $S_n \to 0$ as $n \to \infty$, in L_r and, a fortiori, in probability.

Finally, we would like to mention that there are two special cases ensuring for us that the conditions (b) of Theorems 1–3 and Corollaries 1–3 is satisfied. A proof of the following proposition is obvious and hence omitted.

Proposition. *If* (i) $\sup_{u_n \leqslant k \leqslant v_n} |a_{nk}| = o(h^{-2}(n))$ *or* (ii) u_n *and* v_n *are finite for all* $n \geqslant 1$ *and* $(v_n - u_n) \sup_{u_n \leqslant k \leqslant v_n} a_{nk}^2 = o(h^{-2}(n))$, *then condition* (b) *holds, that is,*

$$h^2(n) \sum_{k=u_n}^{v_n} a_{nk}^2 \to 0 \quad \text{as } n \to \infty.$$

Acknowledgment

The authors are grateful to the referee for his/her helpful and important remarks.

References

[1] A. Adler, A. Rosalsky, A. Volodin, A mean convergence theorem and weak law for arrays of random elements in martingale type p Banach spaces, Statist. Probab. Lett. 32 (1997) 167–174.
[2] P. Billingsley, Convergence of Probability Measures, Wiley, New York, 1968.
[3] T.K. Chandra, Uniform integrability in the Cesàro sense and the weak law of large numbers, Sankhyā Ser. A 51 (1989) 309–317.
[4] T.K. Chandra, A. Goswami, Cesàro uniform integrability and the strong law of large numbers, Sankhyā Ser. A 54 (1992) 215–231.
[5] T.K. Chandra, A. Goswami, Cesàro α-integrability and laws of large numbers I, J. Theoret. Probab. 16 (2003) 655–669.
[6] K. Joag-Dev, F. Proschan, Negative association of random variables with applications, Ann. Statist. 11 (1983) 286–295.
[7] A. Gut, The weak law of large numbers for arrays, Statist. Probab. Lett. 14 (1992) 49–52.
[8] J. Hoffmann-Jørgensen, G. Pisier, The law of large numbers and the central limit theorem in Banach spaces, Ann. Probab. 4 (1976) 587–599.
[9] D. Landers, L. Rogge, Laws of large numbers for pairwise independent uniformly integrable random variables, Math. Nachr. 130 (1987) 189–192.
[10] M. Ordóñez Cabrera, Convergence of weighted sums of random variables and uniform integrability concerning the weights, Collect. Math. 45 (1994) 121–132.
[11] G. Pisier, Martingales with values in uniformly convex spaces, Israel J. Math. 20 (1975) 326–350.
[12] F.S. Scalora, Abstract martingale convergence theorems, Pacific J. Math. 11 (1961) 347–374.