Again on the weak law in martingale type p Banach spaces

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AMS 1991 Subject Classifications: 60B11, 60B12, 60F05, 60F25, 60G42.

Introduction

Consider an array of constants $\{a_{nj}, j \geq 1, n \geq 1\}$ and an array of random elements $\{V_{nj}, j \geq 1, n \geq 1\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a real separable Banach space $\mathcal{X}$ with norm $||\cdot||$. Let $\{c_{nj}, j \geq 1, n \geq 1\}$ be a "centering" array consisting of (suitably selected) conditional expectations and $\{N_n, n \geq 1\}$ be a sequence of positive integer-valued random variables. In this paper, a general weak law of large numbers (WLLN) will be established. This convergence result is of the form

$$\sum_{j=1}^{N_n} a_{nj}(V_{nj} - c_{nj}) \xrightarrow{P} 0$$

as $n \to \infty$. This expression is referred to as *weighted sums* with *weights* $\{a_{nj}, j \geq 1, n \geq 1\}$. The hypotheses to the main result impose conditions on the growth behavior of the weights $\{a_{nj}, j \geq 1, n \geq 1\}$ and on the marginal distributions of the random variables $\{||V_{nj}||, j \geq 1, n \geq 1\}$. The random elements in the array under consideration are not assumed to be rowwise independent. Indeed, no conditions are imposed on the joint distributions of the random elements comprising the array. Also, no conditions are imposed on the joint distributions of $\{N_n, n \geq 1\}$. Moreover, no conditions are imposed on the joint distribution of the sequence $\{V_{nj}, j \geq 1, n \geq 1\}$ and the sequence $\{N_n, n \geq 1\}$. However, the Banach space $\mathcal{X}$ is assumed to be of martingale type p, i.e., there exists a finite constant C such that for all martingales $\{S_n, n \geq 1\}$ with values in $\mathcal{X}$, $\sup_{n\geq 1} E||S_n||^p \leq C \sum_{n=1}^{\infty} E||S_n - S_{n-1}||^p where \mathrm{S}_0 \equiv 0$. Clearly every real separable Banach space is of martingale type 1, while the L_p-spaces and l_p-spaces $(1 \leq p < \infty)$ are of martingale type $p \wedge 2$. It is well known that if a Banach space is of martingale type p, then it is of Rademacher type p. The notion of martingale type p is only superficially similar to that of Rademacher type p, but it has a geometric characterization in terms of smoothness: a Banach space is of martingale type p $(1 \leq p \leq 2)$ if, and only if, it is p-smooth. For more details, the reader may refer to [9]. The main result is an extension, generalization and improvement to a martingale type p Banach space setting and weighted sums of results in [3], [4], [5], [6] and [10] which were proved for arrays of (real-valued) random variables. Moreover, it is an extension on randomly indexed sums of results in [1] and [7]. The WLLN is proved assuming a Cesàro type condition of Hong and Oh ([5]) which is weaker than Cesàro uniform integrability.

Mainstream

In [6], the following Theorem was proved.

THEOREM 1. *Let $\{V_{nj}, j \geq 1, n \geq 1\}$ be an array of random elements in a real separable, martingale type p $(1 \leq p \leq 2)$ Banach space, and $\{N_n, n \geq 1\}$ be a sequence of positive integer-valued random variables such that for some nonrandom sequence of positive integers $k_n \to \infty$ we have*

$$P\{N_n > k_n\} = o(1) \text{ as } n \to \infty. \tag{1}$$

Let $\{a_{nj}, j \geq 1, n \geq 1\}$ be an array of constants and let the sequence $\{f(n), n \geq 1\}$, where $f(n) = 1/\max\limits_{1 \leq j \leq k_n} |a_{nj}|$, satisfying

$$k_n f^{-p}(n) = o(1) \text{as } n \to \infty. \tag{2}$$

Suppose that there exists a positive nondecreasing sequence $\{g(m), m \geq 0\}$ such that

$$\sum_{m=1}^{k_n - 1} \frac{g^p(m+1) - g^p(m)}{m} = O(f^p(n)/k_n) \text{ as } n \to \infty. \tag{3}$$

Suppose the uniform Cesàro type condition

$$\lim_{m \to \infty} \sup_{n \geq 1} \frac{1}{k_n} \sum_{j=1}^{k_n} m P\{||V_{nj}|| > g(m)\} = 0 \tag{4}$$

holds. Then the WLLN

$$\sum_{j=1}^{N_n} a_{nj}(V_{nj} - E(V'_{nj}|\mathcal{F}_{n,j-1})) \xrightarrow{P} 0 \text{ as } n \to \infty \tag{5}$$

follows where $V'_{nj} = V_{nj} I(||V_{nj}|| \leq g(k_n)), \mathcal{F}_{nj} = \sigma(V_{ni}, 1 \leq i \leq j), j \geq 1, n \geq 1,$ and $\mathcal{F}_{n0} = \{\emptyset, \Omega\}, n \geq 1$.

Now we formulate the main result of the paper.

THEOREM 2. *Let all the hypotheses of Theorem 1 hold, except that (3) is changed on:*

$$\sum_{m=1}^{g(k_n)-1} \frac{m^{p-1}}{h(m)} = O(f^p(n)/k_n) \text{ as } n \to \infty \tag{6}$$

where $h(m) = \min\{n \geq 0 : g(n) \geq m\}$ is the inverse sequence to $g(m)$. Then (5) holds.

We need the following lemma in order to prove the Theorem 2.

LEMMA. *If (4) holds then*

$$\lim_{m\to\infty} \sup_{n\geq 1} \frac{1}{k_n} \sum_{j=1}^{k_n} h(m) P\{||V_{nj}|| > m\} = 0 \tag{4'}$$

Proof. First, we consider that $\{g(n), n \geq 1\}$ is bounded. Say, $g(n) \leq \alpha$. By (4),

$$\frac{1}{k_n} \sum_{j=1}^{k_n} mP\{\|V_{nj}\| > \alpha\} \leq \sup_{n\geq 1} \frac{1}{k_n} \sum_{j=1}^{k_n} mP\{\|V_{nj}\| > g(m)\} \to 0$$

as $m \to \infty$. Hence, we have $P\{\|V_{nj}\| > \alpha\} = 0$ for all j and n. So, (4′) follows easily. Now, we consider $g(m) \to \infty$. By the definition of $h(m)$,

$$g(h(m) - 1) < m \leq g(h(m)) \text{ for } m \geq 1.$$

Since $g(m) \to \infty$, $h(m) \to \infty$. Thus, there exists an integer M such that $h(m) \geq 2$ if $m \geq M$. For $m \geq M$, it follows that

$$\begin{aligned} &\frac{1}{k_n} \sum_{j=1}^{k_n} h(m) P\{\|V_{nj}\| > m\} \\ &\leq \frac{1}{k_n} \sum_{j=1}^{k_n} 2(h(m) - 1) P\{\|V_{nj}\| > g(h(m) - 1)\}. \end{aligned}$$

Thus (4′) follows by (4). □

Remark. The implication (4′) ⇒ (4) does not necessarily hold (see the following counterexample).

Counterexample. Let $\{V_{nj}, j \geq 1, n \geq 1\}$ be an array of i.i.d. with $P\{\|V_{11}\| = n\} = c/2^{2^n}$ for $n \geq 1$, where $c = 1/\sum_{n=1}^{\infty}(\frac{1}{2^{2^n}})$. Define $g(n) = [\log\log n]$ for $n \geq 1$, where $\log x$ has base 2. Then, it follows that $h(n) = 2^{2^n}$. Thus, we have

$$\sup_{n\geq 1} \frac{1}{k_n} \sum_{j=1}^{k_n} h(m) P\{\|V_{nj}\| > m\} = 2^{2^m} (\sum_{i=m+1}^{\infty} \frac{c}{2^{2^i}}) \leq 2^{2^m} \frac{2c}{2^{2^{m+1}}} \to 0$$

as $m \to \infty$. Hence (4′) holds. Now, we show that (4) does not hold for $m = 2^{2^l} - 1$. Noting that $g(m) = l - 1$, we have

$$\sup_{n\geq 1} \frac{1}{k_n} \sum_{j=1}^{k_n} mP\{\|V_{nj}\| > g(m)\} = (2^{2^l} - 1)(\sum_{i=l}^{\infty} \frac{c}{2^{2^i}}) \geq (2^{2^l} - 1) \frac{c}{2^{2^l}} \to c$$

as $l \to \infty$. Thus (4) does not hold for $m = 2^{2^l} - 1$.

Sketch of proof of THEOREM 2. By (1) and (4), it suffices to show that

$$\sum_{j=1}^{N_n} a_{nj} V'_{nj} - \sum_{j=1}^{N_n} a_{nj} c_{nj} \xrightarrow{P} 0,$$

where $c_{nj} = E(V'_{nj} | \mathcal{F}_{n,j-1})$. For arbitrary $\epsilon > 0$ and $n \geq 1$, denote

$$D_n = \bigcup_{m=1}^{k_n} B_m^n = \bigcup_{m=1}^{k_n} \left\{ || \sum_{j=1}^{m} a_{nj} V'_{nj} - \sum_{j=1}^{m} a_{nj} c_{nj} || > \epsilon \right\}$$

Since for every $n \geq 1$ the sequence $\{V'_{nj} - c_{nj}, 1 \leq j \leq k_n\}$ is a martingale difference sequence and since underlying Banach space is of martingale type p,

$$P\{D_n\} \leq C \sum_{j=1}^{k_n} E(|a_{nj}|\ ||V'_{nj} - c_{nj}||)^p \leq C 2^p f^{-p}(n) \sum_{j=1}^{k_n} E||V'_{nj}||^p.$$

Moreover

$$\sum_{j=1}^{k_n} E||V'_{nj}||^p = \sum_{j=1}^{k_n} \sum_{m=1}^{g(k_n)} E||V_{nj}||^p I(m-1 < ||V_{nj}|| \leq m)$$

$$\leq \sum_{j=1}^{k_n} \sum_{m=1}^{g(k_n)} m^p \left(P\{||V_{nj}|| > m-1\} - P\{||V_{nj}|| > m\} \right)$$

$$= \sum_{j=1}^{k_n} [P\{||V_{nj}|| > 0\} + g^p(k_n) P\{||V_{nj}|| > k_n\}$$

$$+ \sum_{m=1}^{g(k_n)-1} \left((m+1)^p - m^p \right) P\{||V_{nj}|| > m\}]$$

$$\leq k_n + p 2^{p-1} \sum_{j=1}^{k_n} \sum_{m=1}^{g(k_n)-1} m^{p-1} P\{||V_{nj}|| > m\}]$$

$$= k_n + C k_n \sum_{m=1}^{g(k_n)-1} \frac{m^{p-1}}{h(m)} b_{nm},$$

where $b_{nm} = \frac{1}{k_n} \sum_{j=1}^{k_n} h(m) P\{||V_{nj}|| > m\}$. By (2), $k_n f^{-p}(n) = o(1)$ as $n \to \infty$. By uniform Cesàro type condition (4'), $\sup_{n \geq 1} b_{nm} = o(1)$ as $m \to \infty$. Then, by (6) and the Toeplitz lemma, the expression $P\{D_n\}$ is $o(1)$ thereby completing the proof of Theorem 2.

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