COMPLETE CONVERGENCE OF WEIGHTED SUMS IN BANACH SPACES AND THE BOOTSTRAP MEAN

Manuel Ordóñez Cabrera[1], **Soo Hak Sung**[2], **and Andrei I. Volodin**[3,4]

Abstract. Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent random elements taking values in a real separable Banach space, and $\{a_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ an array of constants. Under some conditions of Chung [7] and Hu and Taylor [10] types for the arrays, and using a theorem of Hu et al. [9], the equivalence amongst various kinds of convergence of $\sum_{i=1}^{k_n} a_{ni}X_{ni}$ to zero is obtained. It leads to an unified vision of recent results in the literature. The authors use the main result in the paper in order to obtain the strong consistency of the bootstrapped mean of random elements in a Banach space from its weak consistency.

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1. INTRODUCTION

Hsu and Robbins [8] introduced the concept of complete convergence of a sequence $\{X_n, n \geq 1\}$ of random variables. $\{X_n, n \geq 1\}$ is said to converge completely to a constant c if for each $\epsilon > 0$

$$\sum_{n=1}^{\infty} P[|X_n - c| > \epsilon] < \infty.$$

This concept is extended to a sequence $\{X_n, n \geq 1\}$ of random elements taking values in a real separable Banach space $(\mathcal{B}, ||.||)$. It is said that $\{X_n\}$ converges completely to zero if for each $\epsilon > 0$

$$\sum_{n=1}^{\infty} P[||X_n|| > \epsilon] < \infty.$$

Classical Chung's [7] strong law of large numbers (SLLN) states that if $\{X_n, n \geq 1\}$ is a sequence of independent random variables with $EX_n = 0$ for all $n \geq 1$ and $\sum_{n=1}^{\infty} \frac{E\psi(|X_n|)}{\psi(n)} < \infty$ when ψ is a positive, even and continuous function such that

$$\frac{\psi(|t|)}{|t|} \uparrow \quad \text{and} \quad \frac{\psi(|t|)}{|t|^2} \downarrow \quad \text{as} \quad |t| \uparrow \tag{1.1}$$

then $\frac{1}{n}\sum_{i=1}^{n} X_i \longrightarrow 0$ almost surely.

Hu and Taylor [10] proved a Chung type SLLN for an array of rowwise independent random variables $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ weighted by a sequence $\{a_n, n \geq 1\}$ of real numbers with $0 < a_n \uparrow \infty$. One of the required conditions by the authors in order to prove that $\frac{1}{a_n}\sum_{i=1}^{n} X_{ni} \longrightarrow 0$ almost surely, is that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{E\psi(|X_{ni}|)}{\psi(a_n)} < \infty,$$

when ψ is a positive, even and continuous function such that

$$\frac{\psi(|t|)}{|t|^p}\uparrow \quad \text{and} \quad \frac{\psi(|t|)}{|t|^{p+1}}\downarrow \quad \text{as} \quad |t|\uparrow, \tag{1.2}$$

for some integer $p \geq 2$.

Many classical theorems hold for $\mathcal{B}$-valued random elements under the assumption that the weak law of large numbers (WLLN) holds (see Kuelbs and Zinn [12], de Acosta [1], Choi and Sung [5, 6], Wang et al. [16], Kuczmaszeva and Szynal [11] and Sung [14]).

Sung [15] obtained a Hu and Taylor's [10] result in a general Banach space under the assumption that WLLN holds, using a positive and even function ψ verifying the following condition, which is weaker than (1.2):

$$\frac{\psi(|t|)}{|t|}\uparrow \quad \text{and} \quad \frac{\psi(|t|)}{|t|^{p}}\downarrow \quad \text{as} \quad |t|\uparrow, \tag{1.3}$$

for some $p \geq 1$.

Ordóñez Cabrera and Sung [13] extend Sung's [15] result for a weighted sum $S_n = \sum_{i=1}^{k_n} a_{ni}X_{ni}$ of rowwise independent $\mathcal{B}$-valued random elements, where $\{k_n, n \geq 1\}$ is a sequence of positive integers, and $\{a_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ is an array of real constants. They use a function $\psi : (0, +\infty) \longrightarrow (0, +\infty)$ verifying that, for some $p \geq 1$, there exist constants $C, D > 0$ such that

$$u \geq v \implies C\frac{u}{v} \leq \frac{\psi(u)}{\psi(v)} \leq D\frac{u^p}{v^p} \tag{1.4}$$

This condition is weaker than (1.3).

In the main result of this paper, we obtain the equivalence between the convergence to zero in L^1, completely, almost surely and in probabillity, for a weighted sum $S_n = \sum_{i=1}^{k_n} a_{ni}X_{ni}$ of rowwise independent $\mathcal{B}$-valued random elements, using a function ψ

verifying a condition weaker than (1.4), and this allows us to give an unified overall view of recent results in the literature that we mentioned before.

The proof of our main result is based on the following theorem, proved in Hu et al. [9], Theorem 3.2 (we took $c_n = 1$ for all $n \geq 1$ and weights are built into random elements).

Theorem 1.1. Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent random elements in a real separable Banach space such that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} P\{||X_{ni}|| > \epsilon\} < \infty \text{ for all } \epsilon > 0,$$

there exist $s > 0$ such that for some $0 < r \leq 2$

$$\sum_{n=1}^{\infty}\left(\sum_{i=1}^{k_n} E||X_{ni}||^r\right)^s < \infty,$$

and

$$\sum_{i=1}^{k_n} X_{ni} \xrightarrow{P} 0.$$

Then

$$\sum_{n=1}^{\infty} P\{||\sum_{i=1}^{k_n} X_{ni}|| > \epsilon\} < \infty \text{ for all } \epsilon > 0.$$

To prove the main result, we also will need the following lemma in de Acosta [1] (cf. also Berger [3]):

Lemma 1.1. Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise independent random elements. Then for every $p \geq 1/2$, there is a positive constant A_p depending only on p such that for all $n \geq 1$

$$E\left| ||\sum_{i=1}^{k_n} X_{ni}|| - E||\sum_{i=1}^{k_n} X_{ni}||\right|^{2p} \leq A_p E\left(\sum_{k=1}^{k_n} ||X_{nk}||^2\right)^p.$$

In the last section, we apply the results of previous section to the field of bootstrapped means for random elements, after making a brief summary of the bootstrap procedure. Our result is in the scope of the previous ones of Bozorgnia et al. [4] and Hu and Taylor [10]. More specifically, we obtain strong consistency (concerning to an array of real constants $\{a_{ni}, 1 \leq i \leq k_n, n \geq 1\}$) for the bootstrapped mean, assuming the corresponding weak consistency (concerning the array). We impose neither conditions about the marginal or joint distributions of the random elements forming the sample nor geometric conditions on the Banach space where the random elements take values.

2. PRELIMINARIES

Definition 2.1. Let $\psi : (0, +\infty) \longrightarrow (0, +\infty)$ be a function such that there exists a constant $C > 0$ such that

$$u \geq v \quad \Longrightarrow \quad \psi(u) \geq C\psi(v) \tag{2.1}$$

and for any $\epsilon > 0$

$$\sup_{u>0} \frac{\psi(u)}{\psi(\epsilon u)} < \infty \tag{2.2}$$

By putting $u = v$ (or by using a continuity argument, if ψ is continuous), it is clear that $0 < C \leq 1$.

Conditions (2.1) and (2.2) are weaker than conditions (1.3) and (1.4). Consequently, the family of functions verifying (2.1) and (2.2) is wider than the family of functions verifying $\dfrac{\psi(x)}{x} \uparrow$ and $\dfrac{\psi(x)}{x^p} \downarrow$. The following Lemma shows a sufficient condition so that ψ should verify (2.1) and (2.2).

Lemma 2.1. Let $\psi : (0, +\infty) \longrightarrow (0, +\infty)$ be a function such that $Ax \leq \psi(x) \leq Bx$ for all $x \in (0, +\infty)$, for some constants $A, B > 0$. Then:

$$\frac{A}{B}\frac{u}{v} \leq \frac{\psi(u)}{\psi(v)} \leq \frac{B}{A}\frac{u}{v}$$

for all $u, v > 0$.

Throughout this paper, unless otherwise specified, ψ will be a function satisfying (2.1) and (2.2).

3. MAIN RESULT

Theorem 3.1. Let $k_n \to \infty$ be a sequence of positive integers. Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent $\mathcal{B}$-valued random elements. Assume that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} P\{||X_{ni}|| > \epsilon\} < \infty \text{ for all } \epsilon > 0, \tag{3.1}$$

and

$$\sum_{n=1}^{\infty}\left(\sum_{i=1}^{k_n} E||X_{ni}||^r\right)^s < \infty \tag{3.2}$$

for some $s > 0$ and some $1 < r \leq 2$. Then the following statements are equivalent:

(i) $\displaystyle\sum_{i=1}^{k_n} X_{ni} \longrightarrow 0$ in L^1.

(ii) $\sum_{i=1}^{k_n} X_{ni} \longrightarrow 0$ completely.

(iii) $\sum_{i=1}^{k_n} X_{ni} \longrightarrow 0$ almost surely.

(iv) $\sum_{i=1}^{k_n} X_{ni} \longrightarrow 0$ in probability.

Proof.$(ii) \Longrightarrow (iii)$, $(iii) \Longrightarrow (iv)$ and $(i) \Longrightarrow (iv)$ are immediate. $(iv) \Longrightarrow (ii)$ is proved in Theorem 1.1.

$(iv) \Longrightarrow (i)$. Assume that (iv) holds. From (3.2) and Lemma 1.1 with $p = r/2$:

$$E\left| ||\sum_{i=1}^{k_n} X_{ni}|| - E||\sum_{i=1}^{k_n} X_{ni}|| \right|^r \leq A_{r/2} E(\sum_{i=1}^{k_n} ||X_{ni}||^2)^{r/2}$$

$$\leq A_{r/2} \sum_{i=1}^{k_n} E||X_{ni}||^r \longrightarrow 0$$

and so

$$||\sum_{i=1}^{k_n} X_{ni}|| - E||\sum_{i=1}^{k_n} X_{ni}|| \longrightarrow 0 \text{ in probability.}$$

Consequently, $E||\sum_{i=1}^{k_n} X_{ni}|| \longrightarrow 0$, and so (i) holds. □

Theorem 3.2. Let $k_n \to \infty$ be a sequence of positive integers. Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent $\mathcal{B}$-valued random elements, and $\{a_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ an array of constants. Assume that

$$\sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \frac{E\psi(||X_{ni}||)}{\psi(|a_{ni}|^{-1})} < \infty \tag{3.3}$$

and

$$\sum_{n=1}^{\infty} \left(\sum_{i=1}^{k_n} |a_{ni}|^r E||X_{ni}||^r \right)^s < \infty \tag{3.4}$$

for some $1 \leq r \leq 2$ and some $s > 0$. Then the following statements are equivalent:

(i) $\sum_{i=1}^{k_n} a_{ni} X_{ni} \longrightarrow 0$ in L^1.

(ii) $\sum_{i=1}^{k_n} a_{ni} X_{ni} \longrightarrow 0$ completely.

(iii) $\sum_{i=1}^{k_n} a_{ni} X_{ni} \longrightarrow 0$ almost surely.

(iv) $\sum_{i=1}^{k_n} a_{ni}X_{ni} \longrightarrow 0$ in probability.

Proof. Consider $a_{ni}X_{ni}$ insterd of X_{ni} in Theorem 3.1. The only thing we need to prove is that (3.3) $\Longrightarrow$ (3.1).

For each $\epsilon > 0$ by (2.1), Markov inequality and (2.2):

$$\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} P[||a_{ni}X_{ni}|| > \epsilon]$$

$$\leq \sum_{n=1}^{\infty}\sum_{i=1}^{k_n} P[\psi(||X_{ni}||) > C\psi(\epsilon|a_{ni}|^{-1})] \leq \frac{1}{C}\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} \frac{E[\psi(||X_{ni}||)]}{\psi(\epsilon|a_{ni}|^{-1})}$$

$$\leq C'\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} \frac{E[\psi(||X_{ni}||)]}{\psi(|a_{ni}|^{-1})}.\square$$

Remark 3.1. An analysis of the proof of the theorem 3.2 shows that the condition (2.2) can be simplified as

$$\sup_{n,i} \frac{\psi(|a_{ni}|^{-1})}{\psi(\epsilon|a_{ni}|^{-1})} < \infty$$

for any $\epsilon > 0$

We obtain the complete convergence of weighted sums taking values in a Banach space of type r $(1 \leq r \leq 2)$ as a corollary of this theorem.

Recall that a separable Banach space $\mathcal{B}$ is of type r, $1 \leq r \leq 2$, if, and only if, there exists a constant C_r such that

$$E||\sum_{i=1}^{n} X_i||^r \leq C_r \sum_{i=1}^{n} E||X_i||^r$$

for all independent $\mathcal{B}$-valued random elements $X_1, ..., X_n$ with mean zero and finite r-th moments.

Corollary 3.1. Let $k_n \to \infty$ be a sequence of positive integers. Let $\{X_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ be an array of rowwise independent $\mathcal{B}$-valued random elements, and $\{a_{ni}, 1 \leq i \leq k_n, n \geq 1\}$ an array of constants. Assume that $EX_{ni} = 0$ for all $1 \leq i \leq k_n$ and $n \geq 1$, and that $\mathcal{B}$ is of type r $(1 \leq r \leq 2)$. Assume (3.1) and (3.4) hold for this r and some $s > 0$. Then,

$$\sum_{i=1}^{k_n} a_{ni}X_{ni} \longrightarrow 0 \text{ completely.}$$

Proof. By Theorem 3.2, it is enough to prove that $\sum_{i=1}^{k_n} a_{ni}X_{ni} \longrightarrow 0$ in L^1.

Since $\mathcal{B}$ is of type r and $E(a_{ni}X_{ni}) = 0$, we have

$$E||\sum_{i=1}^{k_n} a_{ni}X_{ni}||^r \le C_r \sum_{i=1}^{k_n} |a_{ni}|^r E||X_{ni}||^r \longrightarrow 0$$

as a consequence of (3.4).

Convergence in L^r implies convergence in L^1, so Theorem 3.2 implies that

$$\sum_{i=1}^{k_n} a_{ni}X_{ni} \longrightarrow 0 \text{ completely.}\square$$

Remark 3.2. If $C = 1, k_n = n$ and $a_{ni} = \frac{1}{a_n}, 1 \le i \le k_n, n \ge 1$, then Theorem 3.2 and Corollary 3.1 become Theorem 2.2, Theorem 2.3 and Corollary 2.4 in Sung [15].

By using functions ψ satisfyimg the hypothesis of Lemma 2.1, we can weaken slightly the conditions (3.2) and (3.4) in Theorem 3.2.

Theorem 3.3. Let $k_n \to \infty$ be a sequence of positive integers. Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise independent $\mathcal{B}$-valued random elements, and $\{a_{ni}, 1 \le i \le k_n, n \ge 1\}$ an array of constants. Let $\psi : (0, +\infty) \longrightarrow (0, +\infty)$ be a function such that $Ax \le \psi(x) \le Bx$ for all $x \in (0, +\infty)$, for some constants $A, B > 0$. Assume that

$$\sum_{n=1}^{\infty}\sum_{i=1}^{k_n} \frac{E\psi(||X_{ni}||)}{\psi(|a_{ni}|^{-1})} < \infty.$$

Then the following statements are equivalent:

(i) $\sum_{i=1}^{k_n} a_{ni}X_{ni} \longrightarrow 0$ in L^1.

(ii) $\sum_{i=1}^{k_n} a_{ni}X_{ni} \longrightarrow 0$ completely.

(iii) $\sum_{i=1}^{k_n} a_{ni}X_{ni} \longrightarrow 0$ almost surely.

(iv) $\sum_{i=1}^{k_n} a_{ni}X_{ni} \longrightarrow 0$ in probability.

Proof. Mention that in this case

$$\frac{A}{B}||a_{ni}X_{ni}|| \le \frac{\psi(||X_{ni}||)}{\psi(|a_{ni}|^{-1})}.$$

For $(i) \Longrightarrow (ii)$ we refer to Corollary 4.7 in [9]. $(iv) \Longrightarrow (i)$. $E||\sum_{i=1}^{k_n} a_{ni}X_{ni}|| \le \sum_{i=1}^{k_n} |a_{ni}|E||X_{ni}|| \longrightarrow 0$, and (i) holds. $\square$

Similarly, it is easy to check the following result of complete convergence in a Banach space of type $1 \le r \le 2$.

Corollary 3.2. Let $k_n \to \infty$ be a sequence of positive integers. Let $\{X_{ni}, 1 \le i \le k_n, n \ge 1\}$ be an array of rowwise independent $\mathcal{B}$-valued random elements, and $\{a_{ni}, 1 \le i \le k_n, n \ge 1\}$ an array of constants. Let $\psi : (0, +\infty) \longrightarrow (0, +\infty)$ be a function such that $Ax \le \psi(x) \le Bx$ for all $x \in (0, +\infty)$, for some constants $A, B > 0$. Assume that $EX_{ni} = 0$ for all $1 \le i \le k_n$ and $n \ge 1$, and $\mathcal{B}$ is of type r $(1 \le r \le 2)$. Assume (3.2) holds.Then

$$\sum_{i=1}^{k_n} a_{ni} X_{ni} \longrightarrow 0 \text{ completely.}$$

4. APPLICATION TO THE STUDY OF THE CONSISTENCY OF THE BOOTSTRAPPED MEAN

We now outline the bootstrap procedure. Let $\{X_n;\ n \ge 1\}$ be a sequence of (not necessarily independent or identically distributed) random elements defined on some complete probability space $(\Omega, \mathcal{F}, P)$ which take values in a real separable Banach space. For $\omega \in \Omega$ and $n \ge 1$, let $P_n(\omega) = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i(\omega)}$ denote the empirical measure. For $n \ge 1$, let $\{\hat{X}^{\omega}_{n,j};\ 1 \le j \le k_n\}$ be i.i.d. random elements with law $P_n(\omega)$ where k_n is a positive integer. Let $\bar{X}_n(\omega)$ denote the sample mean of $\{X_i(\omega);\ 1 \le i \le n\}$, $n \ge 1$, that is, $\bar{X}_n(\omega) = \frac{1}{n}\sum_{i=1}^{n} X_i(\omega)$.

In order to prove Theorem 4.1, we need the following lemma:
Lemma 4.1. If $s > 0$ then for almost every $\omega \in \Omega$

$$E||\hat{X}^{\omega}_{n,1} - \bar{X}_n(\omega)||^s \le A_s \left[\frac{1}{n}\sum_{i=1}^{n} ||X_i(\omega)||^s + ||\bar{X}_n(\omega)||^s\right],$$

where $A_s = 2^{s-1}$ for $s \ge 1$ and $A_s = 1$ for $0 < s < 1$.
Proof. For almost every $\omega \in \Omega$,

$$E||\hat{X}^{\omega}_{n,j} - \bar{X}_n(\omega)||^s = \frac{1}{n}\sum_{i=1}^{n} ||X_i(\omega) - \bar{X}_n(\omega)||^s$$

$$\le A_s \left[\frac{1}{n}\sum_{i=1}^{n} \Big(||X_i(\omega)||^s + ||\bar{X}_n(\omega)||^s\Big)\right],$$

by the c_r-inequalities. □

Theorem 4.1. Let $\{X_n, n \ge 1\}$ be a sequence of random elements taking values in a real separable Banach space. Let $k_n \to \infty$ be a sequence of positive integers, and let $\{a_{nk}, 1 \le k \le k_n, n \ge 1\}$ be an array of real constants. Let $1 \le r \le 2$. Denote $h_n = \sum_{k=1}^{k_n} |a_{nk}|^r$, for every $n \ge 1$. Let $\{b_n, n \ge 1\}$ and $\{d_n, n \ge 1\}$ be sequences of positive constants. Suppose that:

(1) $\sup_{n\ge1} \frac{1}{d_n} ||\bar{X}_n|| < \infty$ a.e. and $\sup_{n\ge1} \frac{1}{b_n} \sum_{i=1}^{n} ||X_i||^r < \infty$ a.e.,

(2) $\sum_{n=1}^{\infty} \frac{h_n b_n}{n} < \infty$ and $\sum_{n=1}^{\infty} h_n d_n^r < \infty$,

(3) The bootstrapped mean is weakly consistent concerning to $\{a_{nk}, 1 \le k \le k_n, n \ge 1\}$, that is, for almost every $\omega \in \Omega$

$$|| \sum_{k=1}^{k_n} a_{nk} \left(\hat{X}^{\omega}_{n,k} - \bar{X}_n(\omega) \right) || \xrightarrow{P} 0.$$

Then the bootstrapped mean is strongly consistent concerning to $\{a_{nk}, 1 \le k \le k_n, n \ge 1\}$, that is, for almost every $\omega \in \Omega$ and all $\epsilon > 0$,

$$\sum_{n=1}^{\infty} P \left\{ || \sum_{k=1}^{k_n} a_{nk} \left(\hat{X}^{\omega}_{n,k} - \bar{X}_n(\omega) \right) || > \epsilon \right\} < \infty.$$

Proof. If we consider the function $\psi(t) = t^r$, the condition (3.3) in Theorem 3.2 becomes $\sum_{n=1}^{\infty} \sum_{k=1}^{k_n} |a_{nk}|^r E||X_{nk}||^r < \infty$, which implies condition (3.4) with $s = 1$. Therefore, we need only check condition (3.3) for the array $\{Z_{nk} = \hat{X}^{\omega}_{n,k} - \bar{X}_n(\omega), 1 \le k \le k_n, n \ge 1\}$ with $\psi(t) = t^r$.

An application of the above lemma yields for almost every $\omega \in \Omega$:

$$\sum_{n=1}^{\infty} \sum_{k=1}^{k_n} |a_{nk}|^r E||Z_{nk}||^r = \sum_{n=1}^{\infty} \sum_{k=1}^{k_n} |a_{nk}|^r E||\hat{X}^{\omega}_{n,k} - \bar{X}_n(\omega)||^r$$

$$\le 2^{r-1} \sum_{n=1}^{\infty} \sum_{k=1}^{k_n} |a_{nk}|^r \left[\frac{1}{n} \sum_{i=1}^{n} ||X_i(\omega)||^r + ||\bar{X}_n(\omega)||^r \right]$$

$$= 2^{r-1} \left(\sum_{n=1}^{\infty} (\frac{1}{n} \sum_{i=1}^{n} ||X_i(\omega)||^r) \sum_{k=1}^{k_n} |a_{nk}|^r + \sum_{n=1}^{\infty} ||\bar{X}_n(\omega)||^r \sum_{k=1}^{k_n} |a_{nk}|^r \right)$$

$$= 2^{r-1} \left(\sum_{n=1}^{\infty} \frac{h_n}{n} \sum_{i=1}^{n} ||X_i(\omega)||^r + \sum_{n=1}^{\infty} h_n ||\bar{X}_n(\omega)||^r \right)$$

$$= 2^{r-1} \left(\sum_{n=1}^{\infty} \frac{h_n b_n}{n} \left[\frac{1}{b_n} \sum_{i=1}^{n} ||X_i(\omega)||^r \right] + \sum_{n=1}^{\infty} h_n d_n^r \left[\frac{1}{d_n} ||\bar{X}_n(\omega)|| \right]^r \right) < \infty$$

in view of (1) and (2).

By applying Theorem 3.2, we have that $\sum_{k=1}^{k_n} a_{nk} Z_{nk} \longrightarrow 0$ completely, that is, the bootstrapped mean is strongly consistent concerning to $\{a_{nk}, 1 \leq k \leq k_n, n \geq 1\}$. $\square$

Remark 4.1. In the particular case $a_{nk} = \dfrac{1}{a_n}$ for all $1 \leq k \leq k_n$ and every $n \geq 1$, where $\{a_n, n \geq 1\}$ is a sequence of positive constants, we have Theorem 3 of Ahmed, Hu and Volodin [2], with $q = r \in [1, 2]$ and $c_n = 1$ for every $n \geq 1$.

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[1]*Department of Mathematical Analysis, University of Sevilla, Sevila 41080, Spain*
[2]*Department of Applied Mathematics, Pai Chai University, Taejon 302-735, South Korea*
[3]*Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, S4S 0A2 Canada*
[4]*Research Institute of Mathematics and Mechanics, Kazan State University, Universitetskaya st. 17, Kazan, 420008, Russia*