COMPLETE CONVERGENCE CRITERION FOR ARRAYS OF BANACH SPACE VALUED RANDOM ELEMENTS

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AMS 1991 Subject Classifications: 60B12 *Abstract:* We obtain an criterion

for complete convergence for arrays of rowwise independent Banach space valued random elements. In the main result no assumptions are made concerning the existence of expected values nor absolute moments of the random elements. Also no assumptions are made concerning the geometry of underlying Banach space. The corresponding convergence rates are also established.
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1 Introduction

A sequence $\{U_n,\ n \geq 1\}$ of random variables is said to *converge completely* to the constant C if $\sum\limits_{n=1}^{\infty} P\{|U_n - C| > \varepsilon\} < \infty$ for all $\varepsilon > 0$. In Hsu and Robbins (1947) it was proved there that the sequence of arithmetic means of i.i.d. random variables converges completely to the expected value if the variance of the summads is finite. That result has been generalized and extended in several directions we refer the reader to the Gut (1994) for a reference.

Etemadi (1987) obtained an elementary proof of the weak law of large numbers for separable Banach space valued random elements. The present note is devoted to obtaing the analogous extension of Hsu–Robbins theorem to general triangular arrays of rowwise independent, but not necessarily identically distributed Banach space valued random elements. Rowwise independence means that random elements within each row are independent, but that no indenpendence is assumed between the rows.

In Section 2 we give some inequalities and lemmas which will be used in the proofs of our main results. In Section 3 we obtain the complete convergence of row sums with the corresponding rates of convergence. However, in the main result no assumptions are made concerning the existence of expected values or absolute moments of the random elements. Also no assumptions are made concerning the geometry of underlying Banach space.

2 Definitions and preliminaries

Let (Ω, A, P) be a probability space and $(B, \|\cdot\|)$ be a real separable Banach space with the norm $\|\cdot\|$. A random element is defined as a measurable mapping from Ω with σ-algebra A into Banach space B with Borel σ-algebra. The concept of independent distributions has direct extension to B. For a detailed account of basic properties of random elements in separable Banach spaces can be founded in Taylor (1978).

Throughout this paper, we shall denote $\{(X_{nk},\ 1 \leq k \leq k_n),\ n \geq 1\}$ as an array of rowwise independent, but not necessarily identically distributed, random elements taking values in B. Here and in the sequel we denote $S_n = \sum\limits_{k=1}^{k_n} X_{nk}$. If $k_n = \infty$ we will assume that the series converges a.s. For $\delta > 0$ define $Y_{nk} = X_{nk} I\{\|X_{nk}\| \leq \delta\}$ and denote $S_n^\delta = \sum_{k=1}^{k_n} Y_{nk}$.

Before going futher, we first of all introduce some definitions on an array of random elements.

Definition 1. An array $\{(X_{nk},\ 1 \leq k \leq k_n),\ n \geq 1\}$ is said to be *symmetric* if each X_{nk} is symmetrically distributed for all $1 \leq k \leq k_n$ and all $n \geq 1$.

Definition 2. An array $\{(X_{nk},\ 1 \leq k \leq k_n),\ n \geq 1\}$ is said to be *infinitesimal* if for all $\varepsilon > 0$: $\lim\limits_{n\to\infty} \sup\limits_{1\leq k\leq k_n} P\{\|X_{nk}\| > \varepsilon\} = 0$.

Now we are able to formulate Etemadi (1987) result.

Degenerate convergence criteria. *Let $\{(X_{nk},\ 1 \leq k \leq k_n),\ n \geq 1\}$ be an array of rowwise independent random elements and $p \geq 1$. The array is infinitesimal and there exists a (nonrandom) sequence $\{A_n\}$ of vectors in B such that $S_n - A_n \to 0$ in probability as $n \to \infty$ if and only if there exists $\delta > 0$ such that:*

(i) $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{k_n} P\left\{\|X_{nk}\| > \delta\right\} = 0,$

(ii) $\lim\limits_{n\to\infty} E\left\|S_n^\delta - A_n\right\|^p = 0.$

A_n may be taken as ES_n^δ. Futhermore, if (i) and (ii) is true for some $\delta > 0$, they are true for all $\delta > 0$.

Now we shall present some well–known inequalities and lemmas which will be useful in the proof of the main result.

Hoffmann–Jorgensen (1974) inequality*If an array $\{(X_{nk},\ 1 \leq k \leq k_n),\ n \geq 1\}$ of rowwise independent random elements is symmetric, then for all $t > 0$:*

$$P\left\{\|S_n\| > 3t\right\} \leq P\left\{\sup_{1\leq k\leq k_n} \|X_{nk}\| > t\right\} + 4P\left\{\|S_n\| > t\right\},$$

Etemadi (1983) inequality*If an array $\{(X_{nk},\ 1 \leq k \leq k_n),\ n \geq 1\}$ of rowwise independent random elements is symmetric, then for any $\varepsilon > 0$:*

$$\sum_{k=1}^{k_n} P\{\|X_{nk}\| > \varepsilon\} \leq P\{\|S_n\| > \varepsilon/8\}/\left(1 - 8P\{\|S_n\| > \varepsilon/8\}\right)$$

The following lemma is only a slight modification of well–known result (cf. Kuelbs and Zinn (1979)).

Lemma 1 *Let an array* $\{(X_{nk},\ 1 \le k \le k_n),\ n \ge 1\}$ *of rowwise independent random elements be symmetric and suppose there exists* $\delta > 0$ *such that* $\|X_{nk}\| \le \delta$ *for all* $1 \le k \le k_n, n \ge 1$*. If* $S_n \to 0$ *in probability then for any* $\varepsilon > 0$*, any* $p \ge 1$ *any all sufficiently large* n *:*

$$E\|S_n\|^p I\{\|S_n\| > \varepsilon\} \le 2(3\delta)^p \sum_{k=1}^{k_n} P\left\{\|X_{nk}\| > \varepsilon/3\right\} + 2\varepsilon^p P\left\{\|S_n\| > \varepsilon/3\right\}.$$

Proof. Fix any $\varepsilon > 0$ and $A > 0$. Since $S_n \to 0$ in probability, we can choose N large enought such that $\sup\limits_{n \ge N} P\{\|S_n\| \ge \varepsilon\} \le 1/3^p 8$. Using the integral by part we have:

$$E\|S_n\|^p I\{\|S_n\| > \varepsilon\} = \int_{\varepsilon}^{A} P\{\|S_n\| \ge t\}\ dt^p + \varepsilon^p P\left\{\|S_n\| > \varepsilon\right\}.$$

Now by the change of variable, letting $t = 3u$ and by Hoffmann–Jorgensen inequality with $n \ge N$, we have:

$$\int_{\varepsilon}^{A} P\{\|S_n\| \ge t\}\ dt^p = 3^p \int_{\varepsilon/3}^{A/3} P\{\|S_n\| \ge 3u\}\ du^p$$

$$\le 3^p \left(4 \int_{\varepsilon/3}^{A/3} P^2\{\|S_n\| \ge u\}\ du^p + \int_{\varepsilon/3}^{A/3} P\{\sup_{1 \le k \le k_n} \|X_{nk}\| \ge u\}\ du^p \right)$$

$$\le 3^p 4 \int_{\varepsilon/3}^{A/3} \frac{1}{3^p 8} P\{\|S_n\| \ge u\}\ du^p + 3^p \int_{\varepsilon/3}^{\delta} P\{\sup_{1 \le k \le k_n} \|X_{nk}\| \ge u\}\ du^p$$

$$\le \frac{1}{2} \int_{\varepsilon/3}^{A} P\{\|S_n\| \ge u\}\ du^p + 3^p \int_{\varepsilon/3}^{\delta} P\{\sup_{1 \le k \le k_n} \|X_{nk}\| \ge u\}\ du^p$$

$$\leq \frac{1}{2}\int_{\varepsilon}^{A} P\{\|S_n\| \geq t\}\, dt^p + \frac{1}{2}\int_{\varepsilon/3}^{\varepsilon} P\{\|S_n\| \geq u\}\, du^p$$

$$+3^p\delta^p P\{\sup_{1\leq k\leq k_n} \|X_{nk}\| \geq \varepsilon/3\}.$$

Hence, we obtain

$$\int_{\varepsilon}^{A} P\{\|S_n\| \geq t\}\, dt^p \leq 3^p 2\delta^p \sum_{k=1}^{k_n} P\{\|X_{nk}\| \geq \varepsilon/3\} + \varepsilon^p P\{\|S_n\| > \varepsilon/3\}.$$

Finally, letting $A \to \infty$ the proof can be easily established.

We also need the following simple symmetrization inequalities in the proof of Section 3. For a random element Y, denote its symmetrization $Y^s = Y - \widetilde{Y}$, where $\widetilde{Y}$ is an independent copy of Y.

Lemma 2. *(a) If an array $\{(X_{nk},\ 1 \leq k \leq k_n),\ n \geq 1\}$ of rowwise independent random elements is infinitesimal then for all $t > 0$ sufficiently large n and all $1 \leq k \leq k_n$ we have:*

$$P\{\|X_{nk}\| > t\} \leq 2P\{\|X^s_{nk}\| > t/2\}.$$

(b) If a sequence $\{Y_n,\ n \geq 1\}$ of random elements is such that $Y_n \to 0$ in probability, as $n \to \infty$, then for all $t > 0$ and sufficiently large n:

$$P\left\{\|Y_n\| > t\right\} \leq 2P\left\{\|Y^s_n\| > t/2\right\}.$$

(c) If $A \in B$ is nonrandom then for any random element Y and $t > 0$ we have:

$$P\left\{\|Y^s\| > t\right\} \leq P\left\{\|Y - A\| > t/2\right\}.$$

Proof Denote m_{nk} the median of a random variable $\|X_{nk}\|$. Since the array is infinitesimal we have $\sup_{1\leq k\leq k_n} m_{nk} \to 0$ as $n \to \infty$. Let N be so large that $\sup_{n\geq N} \sup_{1\leq k\leq k_n} m_{nk} \leq t/2$. Then for $n \geq N$:

$$P\left\{\|X_{nk}\| > t\right\} \leq P\left\{\|X_{nk}\| - m_{nk}) > t/2\right\}$$

$$\leq P\{|\|X_{nk}\| - m_{nk})| > t/2\}$$
$$\leq 2P\left\{\left|\|X_{nk}\| - \|\widetilde{X_{nk}}\|\right| > t/2\right\}$$
$$\leq 2P\{\|X^s_{nk}\| > t/2\}$$

(cf. Loève (1977), p.257). We mention that the proof can be also found in Etemadi (1987), p.249.
Note that part (b) can be proved by the same arguments as the proof of part (a) and (c) can be found in Loève (1977), p.257.

The following lemma will be also usefull in the proofs of our main result. Before the proof of Lemma 3, we recall that $S_n^\delta = \sum_{k=1}^{k_n} X_{nk} I\{\|X_{nk}\| \leq \delta\}$
Lemma 3. *Let* $\{(X_{nk},\ 1 \leq k \leq k_n),\ n \geq 1\}$ *be an array of rowwise independent random elements and* $\{A_n\}$ *be a (nonrandom) sequence of vectors in* B *Then for all* $\varepsilon > 0$ *and* $\delta > 0$ *we have that*

$$P\{\|S_n - A_n\| \geq \varepsilon\} \leq P\left\{\|S_n^\delta - A_n\| \geq \varepsilon/2\right\} + \sum_{k=1}^{k_n} P\{\|X_{nk}\| > \delta\}$$

and

$$P\left\{\|S_n^\delta - A_n\| \geq \varepsilon\right\} \leq P\{\|S_n - A_n\| \geq \varepsilon\} + \sum_{k=1}^{k_n} P\{\|X_{nk}\| > \delta\}$$

Proof. For any $\delta > 0$ we denote

$$S_n^{''} = \sum_{k=1}^{k_n} X_{nk} I\{\|X_{nk}\| > \delta\}.$$

We mention that first part of Lemma can be obtained by

$$P\{\|S_n - A_n\| \geq \varepsilon\} \leq P\left\{\|S_n^\delta - A_n\| \geq \varepsilon/2\right\} + P\left\{\|S_n^{''}\| \geq \varepsilon/2\right\}$$
$$\leq P\left\{\|S_n^\delta - A_n\| \geq \varepsilon/2\right\} + \textstyle\sum_{k=1}^{k_n} P\{\|X_{nk}\| > \delta\}.$$

For the second part, we shall estimate $P\left\{\|S_n^\delta - A_n\| \geq \varepsilon\right\}$. (the proof can be also found in Etemadi (1987), p.249)

$$
\begin{aligned}
P\{\|S_n - A_n\| \geq \varepsilon\} &\geq P\left\{\|S_n - A_n\| \geq \varepsilon,\ \sup\nolimits_{1\leq k\leq k_n}\|X_{nk}\| \leq \delta\right\} \\
&\geq P\left\{\|S_n^\delta - A_n\| \geq \varepsilon,\ \sup\nolimits_{1\leq k\leq k_n}\|X_{nk}\| \leq \delta\right\} \\
&\geq P\left\{\|S_n^\delta - A_n\| \geq \varepsilon\right\} - P\left\{\sup\nolimits_{1\leq k\leq k_n}\|X_{nk}\| > \delta\right\}.
\end{aligned}
$$

Hence, we have

$$
P\left\{\|S_n^\delta - A_n\| \geq \varepsilon\right\} \leq P\{\|S_n - A_n\| \geq \varepsilon\} + \sum_{k=1}^{k_n} P\{\|X_{nk}\| > \delta\}
$$

3 The main result

In this section, we shall extend Etemadi's criterion on the convergence in probability (see Degenerate convergence criterion in Section 1) to the complete convergence with the rate of convergence. With the preliminaries accounted for, the main theorem can be presented.

Theorem *Let $\{(X_{nk},\ 1\leq k\leq k_n),\ n\geq 1\}$ be an array of rowwise independent random elements and $\{c_n, n\geq 1\}$ be a sequence of positive constants such that $\sum_{n=1}^{\infty} c_n = \infty$. Let $p\geq 1$. The array is infinitesimal and there exists a (nonrandom) sequence $\{A_n\}$ of vectors in B such that $\sum_{n=1}^{\infty} c_n P\{\|S_n - A_n\| > \varepsilon\} < \infty$ for all $\varepsilon > 0$ if and only if there exists $\delta > 0$ such that for all $\varepsilon > 0$:*
(i) $\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P\{\|X_{nk}\| > \varepsilon\} < \infty$,
(ii) $\sum_{n=1}^{\infty} c_n E\left\|S_n^\delta - A_n\right\|^p I\{\left\|S_n^\delta - A_n\right\| > \varepsilon\} < \infty$,
where $S_n^\delta = \sum_{k=1}^{k_n} X_{nk} I\{\|X_{nk}\| \leq \delta\}$.
A_n may be taken as ES_n^δ. Futhermore, if (ii) is true for some $\delta > 0$, it is true for all $\delta > 0$.

Proof. First of all, in the case $k_n = \infty$, we assume that series $\sum_{k=1}^{k_n} X_{nk}$ converges a.s.

For $\delta > 0$, from assumption and by Chebyshev's inequality, we have

$$P\{\|S_n - A_n\| > \varepsilon\} \le P\left\{\|S_n^\delta - A_n\| > \varepsilon\right\} + P\left\{\sup_{1\le k\le k_n} \|X_{nk}\| > \delta\right\}$$

$$\le P\left\{\|S_n^\delta - A_n\| I\{\|S_n - A_n\| > \varepsilon\} > \varepsilon\right\} + \sum_{k=1}^{k_n} P\{\|X_{nk}\| > \delta\}$$

$$\le E\|S_n^\delta - A_n\|^p I\{\|S_n^\delta - A_n\| > \varepsilon\}/\varepsilon^p + \sum_{k=1}^{k_n} P\{\|X_{nk}\| > \delta\}.$$

Thus the sufficient part can be easily established.

For the necessary part first of all we shall estimate assumption (i). By Lemma 2(a), Etemadi's inquality and Lemma 2(c) we have:

$$\sum_{k=1}^{k_n} P\{\|X_{nk}\| > \varepsilon\} \le 2\sum_{k=1}^{k_n} P\{\|X_{nk}^s\| > \varepsilon\}$$

$$\le \frac{2P\{\|S_n - A_n\| > \varepsilon/8\}}{1 - 8P\{\|S_n - A_n\| > \varepsilon/8\}}.$$

Since $\sum_{n=1}^{\infty} c_n P\{\|S_n - A_n\| > \varepsilon\} < \infty$ and $\sum_{n=1}^{\infty} c_n = \infty$, it follows that $P\{\|S_n - A_n\| > \varepsilon\} \to 0$. Let N be so large that for all $n > N$:

$$P\{\|S_n - A_n\| > \varepsilon\} < 1/16.$$

Therefore, $\sum_{k=1}^{k_n} P\{\|X_{nk}\| > \varepsilon\} \le 4P\{\|S_n - A_n\| > \varepsilon/8\}$.

In order to estimate the assumption (ii) we mention that by Lemma 3 and assumption (i) we have that $\sum_{n=1}^{\infty} c_n P\left\{\|S_n^\delta - A_n\| > \varepsilon\right\} < \infty$ and $S_n^\delta - A_n \to 0$ in probability. By Lemma 2(b), Lemma 1 and Lemma 2 (c):

$$E\left\|S_n^\delta - A_n\right\|^p I\{\left\|S_n^\delta - A_n\right\| > \varepsilon\}$$

$$= \int_\varepsilon^\infty P\{\left\|S_n^\delta - A_n\right\| \ge t\}\ dt^p + \varepsilon^p P\left\{\|S_n^\delta - A_n\| > \varepsilon\right\}$$

So, it is sufficient to estimate the integral $\int_\varepsilon^\infty P\{\left\|S_n^\delta - A_n\right\| \geq t\}\ dt^p$

$$\leq 2^{p+1}\int_{\varepsilon/2}^\infty P\{\left\|S_n^{\delta s}\right\| \geq t\}\ dt^p \leq 2^{p+1}E\left\|S_n^{\delta s}\right\|^p I\{\left\|S_n^{\delta s}\right\| \geq \varepsilon/2\}$$

$$\leq 2^{p+1}\left(6\delta^p P\left\{\|Y_{nk}^s\| > \varepsilon/6\right\} + \varepsilon^p P\left\{\|S_n^{\delta s}\| > \varepsilon/6\right\}\right).$$

$$\leq 2^{p+2}\left(6\delta^p P\left\{\|X_{nk}\| > \varepsilon/6\right\} + \varepsilon^p P\left\{\|S_n^{\delta} - A_n\| > \varepsilon/6\right\}\right).$$

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