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A NOTE ON THE STRONG LAWS OF LARGE NUMBERS FOR RANDOM VARIABLES

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Abstract. Let $\{X_n, n \geq 1\}$ be a sequence of random variables and $\{b_n, n \geq 1\}$ a nondecreasing sequence of positive constants. No assumptions are imposed on the joint distributions of the random variables. Some sufficient conditions are given under which $\lim_{n\to\infty} \sum_{i=1}^n X_i/b_n = 0$ almost surely. Necessary conditions for the strong law of large numbers are also given.

1. Introduction

Throughout this paper, let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and $\{b_n, n \geq 1\}$ a nondecreasing sequence of positive constants. Set $S_n = \sum_{i=1}^n X_i, n \geq 1$ and $S_0 = 0$.

The purpose of this paper is to prove the strong law of large numbers

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \quad \text{almost surely (a.s.).}$$

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There are several approaches to prove the strong law of large numbers. The first approach is to use the method of subsequences. It is to prove the strong law of large numbers for a subsequence and then extend it to the whole sequence. This method is classical and has been used by numerous authors (see, for example, Stout [8, p. 17]. If there exists a strictly increasing sequence $\{m(k), k \geq 1\}$ of positive integers such that

$$\frac{S_{m(k)}}{b_{m(k)}} \to 0 \quad \text{a.s.}, \qquad \max_{m(k) \leq n < m(k+1)} \left| \frac{S_n}{b_n} - \frac{S_{m(k)}}{b_{m(k)}} \right| \to 0 \quad \text{a.s.},$$

then $S_n/b_n \to 0$ a.s.

The second approach is to use directly a Hájek–Rényi type maximal inequality. It is well known in probability theory that

$$S_n/b_n \to 0 \text{ a.s.} \ \Leftrightarrow \ \lim_{m\to\infty} \lim_{N\to\infty} P\Big(\max_{m \leq n \leq N} |S_n|/b_n > \varepsilon \Big) = 0$$

for all $\varepsilon > 0$. The Hájek–Rényi type inequality is to estimate the probability

$$P\Big(\max_{m \leq n \leq N} |S_n|/b_n > \varepsilon \Big).$$

Fazekas and Klesov [3] obtained a Hájek–Rényi type inequality for the moments and proved the strong law of large numbers. Their method has been improved and generalized by many authors. Hu and Hu [5], Sung et al. [9], and Tómács [11] have considered the rate of convergence in the strong law of large numbers. Tómács and Líbor [12] have used a Hájek–Rényi type inequality for the probabilities instead of the moments.

The third approach is to use a complete convergence result. The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [4]. A sequence $\{X_n, n \geq 1\}$ of random variables converges completely to the constant θ if

$$\sum_{n=1}^{\infty} P(|X_n - \theta| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0.$$

In view of the Borel–Cantelli lemma, this implies that $X_n \to \theta$ a.s. Klesov et al. [6] and Yang et al. [13] gave sufficient conditions on the complete convergence to prove the strong law of large numbers.

In this paper, we give equivalent conditions for the strong law of large numbers by using the method of subsequences. We also give sufficient conditions on the complete convergence to prove the strong law of large numbers. As corollaries, known results can be obtained.

2. Main results

Throughout this section, let $\{X_n, n \geq 1\}$ be a sequence of random variables, $\{b_n, n \geq 1\}$ a nondecreasing sequence of positive constants, and $\{m(k), k \geq 1\}$ a nondecreasing unbounded sequence of positive integers. No assumptions are imposed on the joint distributions of the random variables.

The following theorem gives equivalent conditions for the strong law of large numbers.

THEOREM 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables and $\{b_n, n \geq 1\}$ a nondecreasing unbounded sequence of positive constants. Suppose that there exists a nondecreasing unbounded sequence $\{m(k), k \geq 1\}$ of positive integers such that*

$$0 < \liminf_{k\to\infty} \frac{b_{m(k)}}{b_{m(k+1)}} \leq \limsup_{k\to\infty} \frac{b_{m(k)}}{b_{m(k+1)}} < 1.$$

Then the following three statements are equivalent:

(i) $S_n/b_n \to 0$ *a.s.*,

(ii) $\max_{m(k)\leq n<m(k+1)} |S_n - S_{m(k)-1}|/b_{m(k)} \to 0$ *a.s.*,

(iii) $\max_{1\leq i<m(k+1)} |S_i|/b_{m(k)} \to 0$ *a.s.*

PROOF. We first note that $\liminf_{k\to\infty} b_{m(k)}/b_{m(k+1)} > 0$ is equivalent to $b_{m(k+1)}/b_{m(k)} = O(1)$.

(i) $\Rightarrow$ (ii). Since $0 < b_n \uparrow$ and $b_{m(k+1)}/b_{m(k)} = O(1)$, we have that

$$\begin{aligned}
(2.1)\qquad & \frac{1}{b_{m(k)}} \max_{m(k)\leq n<m(k+1)} |S_n - S_{m(k)-1}| \\
&\leq \frac{|S_{m(k)-1}|}{b_{m(k)}} + \frac{1}{b_{m(k)}} \max_{m(k)\leq n<m(k+1)} |S_n| \\
&\leq \frac{|S_{m(k)-1}|}{b_{m(k)-1}} + \frac{b_{m(k+1)}}{b_{m(k)}} \max_{m(k)\leq n<m(k+1)} \frac{|S_n|}{b_n} \\
&\leq \frac{|S_{m(k)-1}|}{b_{m(k)-1}} + O(1) \max_{m(k)\leq n<m(k+1)} \frac{|S_n|}{b_n}.
\end{aligned}$$

Hence (i) implies that the right-hand side of (2.1) converges to zero a.s. and so (ii) holds.

(ii) $\Rightarrow$ (iii). We can assume, without loss of generality, that

$$\sup_{k\geq 1} \frac{b_{m(k)}}{b_{m(k+1)}} = \alpha < 1.$$

Noting that

$$\max_{1\le i<m(k+1)}|S_i| \le \max_{1\le i<m(1)}|S_i|$$
$$+\max_{m(1)\le i<m(2)}|S_i - S_{m(1)-1}| + \dots + \max_{m(k)\le i<m(k+1)}|S_i - S_{m(k)-1}|,$$

we get that

$$\frac{1}{b_{m(k)}}\max_{1\le i<m(k+1)}|S_i| \le \frac{1}{b_{m(k)}}\sum_{i=0}^{k} b_{m(i)}T_i, \tag{2.2}$$

where

$$T_i = \max_{m(i)\le j<m(i+1)}|S_j - S_{m(i)-1}|/b_{m(i)} \quad \text{and} \quad m(0)=1.$$

For $0\le i\le k$,

$$\frac{b_{m(i)}}{b_{m(k)}} = \frac{b_{m(i)}}{b_{m(i+1)}}\frac{b_{m(i+1)}}{b_{m(i+2)}}\cdots\frac{b_{m(k-1)}}{b_{m(k)}} \le \alpha^{k-i}.$$

It follows that

$$\frac{1}{b_{m(k)}}\sum_{i=0}^{k} b_{m(i)} \le \sum_{i=0}^{k}\alpha^{k-i} \le \frac{1}{1-\alpha}\,.$$

Clearly we have that, for each i, $b_{m(i)}/b_{m(k)} \to 0$ as $k\to\infty$, since $0<b_n\uparrow\infty$. Hence, by the Toeplitz lemma, (ii) implies that the right-hand side of (2.2) converges to zero a.s. and so (iii) holds.

(iii) $\Rightarrow$ (i). For $m(k)\le n<m(k+1)$,

$$\frac{|S_n|}{b_n} \le \max_{m(k)\le i<m(k+1)}\frac{|S_i|}{b_i} \le \frac{1}{b_{m(k)}}\max_{1\le i<m(k+1)}|S_i|.$$

Hence (iii) implies (i). $\square$

Remark 2.1. Under the additional assumption of independency of the symmetric random variables, the following equivalence is well known (see, for example, Loève [7, p. 264]).

$$\frac{S_n}{b_n}\to 0 \ \text{ a.s.} \quad \Leftrightarrow \quad \frac{S_{m(k)} - S_{m(k-1)}}{b_{m(k)}} \to 0 \ \text{ a.s.}$$

Chobanyan et al. [2] proved the equivalence of (i) and (ii) when $b_n = n$. Thanh [10] extended the result of Chobanyan et al. [2] to d-dimensional array of random variables.

As special cases of Theorem 2.1, the following corollaries can be obtained.

COROLLARY 2.1. *The following three statements are equivalent*:

(i) $S_n/n \to 0$ *a.s.*,

(ii) $\max_{2^k \le n < 2^{k+1}} |S_n - S_{2^k - 1}|/2^k \to 0$ *a.s.*,

(iii) $\max_{1 \le i < 2^{k+1}} |S_i|/2^k \to 0$ *a.s.*

PROOF. Let $b_n = n$, $n \ge 1$ and $m(k) = 2^k$, $k \ge 1$. Then the result follows directly from Theorem 2.1. □

Recall that a positive function $l(x)$, defined on $[0, \infty)$, is said to be regularly varying with exponent δ if for all $\lambda > 0$,

$$\lim_{x \to \infty} \frac{l(\lambda x)}{l(x)} = \lambda^{\delta}.$$

COROLLARY 2.2. *Let $l(x)$ be a positive nondecreasing regularly varying function with exponent $\delta > 0$. Then the following three statements are equivalent*:

(i) $S_n/l(n) \to 0$ *a.s.*,

(ii) $\max_{2^k \le n < 2^{k+1}} |S_n - S_{2^k - 1}|/l(2^k) \to 0$ *a.s.*,

(iii) $\max_{1 \le i < 2^{k+1}} |S_i|/l(2^k) \to 0$ *a.s.*

PROOF. Let $b_n = l(n)$, $n \ge 1$ and $m(k) = 2^k$, $k \ge 1$. Then

$$\frac{b_{m(k)}}{b_{m(k+1)}} = \frac{l(2^k)}{l(2^{k+1})} \to 2^{-\delta}.$$

Hence the result follows directly from Theorem 2.1 □

The following lemma will be used in the proofs of Theorems 2.2 and 2.3.

LEMMA 2.1. *Let $\{m(k), k \ge 1\}$ be a nondecreasing unbounded sequence of positive integers and let $S = \{k : m(k) < m(k+1)\}$. Let k_1, k_2, ... denote the elements of S in increasing order. Then the followings hold*:

(i) *$\{m(k_j), j \ge 1\}$ is an increasing sequence of positive integers*;

(ii) *if $k_j < i \le k_{j+1}$, then $m(i) = m(k_{j+1})$, in particular, $m(k_j + 1) = m(k_{j+1})$.*

The proof of Lemma 2.1 is obvious. Instead of giving the proof, we give an example illustrating 2.1.

EXAMPLE 2.1. If $m(1) < m(2) = m(3) < m(4) = m(5) = m(6) < m(7) = m(8) = m(9) < m(10) < \cdots$, then $k_1 = 1$, $k_2 = 3$, $k_3 = 6$, $k_4 = 9$, ..., and $m(k_1) < m(k_2) < m(k_3) < m(k_4) < \cdots$.

THEOREM 2.2. *Let $\{X_n, n \ge 1\}$ be a sequence of random variables, $\{b_n, n \ge 1\}$ a nondecreasing sequence of positive constants, and $\{a_n, n \ge 1\}$ a sequence of nonnegative constants. Suppose that there exists a nondecreasing unbounded sequence $\{m(k), k \ge 1\}$ of positive integers such that*

$$\frac{b_{m(k+1)}}{b_{m(k)}} = O(1) \quad \text{and} \quad \liminf_{\substack{k\to\infty \\ m(k)<m(k+1)}} \sum_{i=m(k)}^{m(k+1)-1} a_i > 0.$$

If

$$\sum_{n=1}^{\infty} a_n P\Big(\max_{1\le i\le n} |S_i| > b_n\varepsilon\Big) < \infty \qquad \text{for all } \varepsilon > 0,$$

then $S_n/b_n \to 0$ a.s.

PROOF. Let

$$\{k : m(k) < m(k+1)\} = \{k_1, k_2, \dots\} \quad \text{and} \quad u_j = \sum_{i=m(k_j)}^{m(k_j+1)-1} a_i, \quad j \ge 1.$$

Then $\liminf_{j\to\infty} u_j > 0$ and so there exists a positive real number $\alpha > 0$ and a positive integer J such that $u_j \ge \alpha > 0$ if $j \ge J$. It follows that

$$\begin{aligned}
\infty > \sum_{k=1, m(k)<m(k+1)}^{\infty} \sum_{n=m(k)}^{m(k+1)-1} a_n P\big(\max_{1\le i\le n} |S_i| > b_n\varepsilon\big) \\
= \sum_{j=1}^{\infty} \sum_{n=m(k_j)}^{m(k_j+1)-1} a_n P\big(\max_{1\le i\le n} |S_i| > b_n\varepsilon\big) \\
\ge \sum_{j=1}^{\infty} P\big(\max_{1\le i\le m(k_j)} |S_i| > b_{m(k_j+1)-1}\varepsilon\big) u_j \\
\ge \alpha \sum_{j=J}^{\infty} P\big(\max_{1\le i\le m(k_j)} |S_i| > b_{m(k_j+1)-1}\varepsilon\big).
\end{aligned}$$

Noting that $m(k_j + 1) = m(k_{j+1})$ (see Lemma 2.1), we have by the Borel–Cantelli lemma that

$$\frac{1}{b_{m(k_{j+1})-1}} \max_{1\le i\le m(k_j)} |S_i| \to 0 \ \text{ a.s.} \tag{2.3}$$

For $m(k_j) \le n < m(k_{j+1})$,

$$\frac{|S_n|}{b_n} \le \frac{1}{b_{m(k_j)}} \max_{m(k_j)\le i<m(k_{j+1})} |S_i| \le \frac{1}{b_{m(k_j)}} \max_{1\le i\le m(k_{j+1})} |S_i| \tag{2.4}$$

$$\le \frac{b_{m(k_{j+2})}}{b_{m(k_j)}} \frac{1}{b_{m(k_{j+2})-1}} \max_{1\le i\le m(k_{j+1})} |S_i| \le O(1) \frac{1}{b_{m(k_{j+2})-1}} \max_{1\le i\le m(k_{j+1})} |S_i|.$$

The last inequality follows by the fact that

$$\frac{b_{m(k_{j+2})}}{b_{m(k_j)}} = \frac{b_{m(k_{j+1})}}{b_{m(k_j)}} \frac{b_{m(k_{j+2})}}{b_{m(k_{j+1})}} = \frac{b_{m(k_j+1)}}{b_{m(k_j)}} \frac{b_{m(k_{j+1}+1)}}{b_{m(k_{j+1})}} = O(1).$$

The right-hand side of (2.4) converges to 0 a.s. by (2.3) and so the result is proved. □

Remark 2.2. As noted in the proof of Theorem 2.1, $b_{m(k+1)}/b_{m(k)} = O(1)$ is equivalent to $\liminf_{k\to\infty} b_{m(k)}/b_{m(k+1)} > 0$.

Corollary 2.3. *Let* $r > 0$. *If*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\Big(\max_{1\le i\le n} |S_i| > n^{1/r}\varepsilon\Big) < \infty \quad \textit{for all } \varepsilon > 0, \tag{2.5}$$

then $S_n/n^{1/r} \to 0$ *a.s.*

Proof. Let $b_n = n^{1/r}, a_n = 1/n$, and $m(k) = 2^k$. Then the conditions of Theorem 2.2 are satisfied. Hence the result follows from Theorem 2.2. □

Remark 2.3. When $1 \le r < 2$ and $\{X_n\}$ is a sequence of i.i.d. random variables with $EX_1 = 0$, Marcinkiewicz and Zygmund proved the strong law of large numbers

$$E|X_1|^r < \infty \quad \Leftrightarrow \quad S_n/n^{1/r} \to 0 \text{ a.s.} \tag{2.6}$$

and Baum and Katz [1] proved the rate of convergence in the strong law of large numbers

$$E|X_1|^r < \infty \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n} P\Big(|S_n| > n^{1/r}\varepsilon\Big) < \infty \text{ for all } \varepsilon > 0. \tag{2.7}$$

In fact, the following equivalence holds.

$$E|X_1|^r < \infty \quad \Leftrightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n} P\Big(\max_{1\le i\le n} |S_i| > n^{1/r}\varepsilon\Big) < \infty \text{ for all } \varepsilon > 0. \tag{2.8}$$

From (2.6) and (2.8), we see that the converse of Corollary 2.3 holds.

COROLLARY 2.4. *Let $r>0$. If*

$$\sum_{k=1}^{\infty} P\Big(\max_{1\le i\le 2^k} |S_i| > (2^k)^{1/r}\varepsilon\Big) < \infty \quad \text{for all } \varepsilon>0, \tag{2.9}$$

then $S_n/n^{1/r} \to 0$ a.s.

PROOF. Let $b_n = n^{1/r}$, $n\ge 1$ and $m(k)=2^k$, $k\ge 1$. Define $a_n=1$ if $n=m(k)$ and $a_n=0$ otherwise. Then the conditions of Theorem 2.2 are satisfied. Hence the result follows from Theorem 2.2. □

REMARK 2.4. Condition (2.5) is equivalent to condition (2.9).

COROLLARY 2.5 (Yang et al. [13]). *Let $\{b_n, n\ge 1\}$ be a nondecreasing sequence of positive constants satisfying $b_{2n}/b_n = O(1)$. If*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\Big(\max_{1\le i\le n} |S_i| > b_n\varepsilon\Big) < \infty \quad \text{for all } \varepsilon>0,$$

then $S_n/b_n \to 0$ a.s.

PROOF. Let $a_n=1/n$, $n\ge 1$ and $m(k)=2^k$, $k\ge 1$. Then the conditions of Theorem 2.2 are satisfied. Hence the result follows from Theorem 2.2. □

THEOREM 2.3. *Let $\{X_n, n\ge 1\}$ be a sequence of random variables and $\{b_n, n\ge 1\}$ a nondecreasing sequence of positive constants. Suppose that there exists a nondecreasing unbounded sequence $\{m(k), k\ge 1\}$ of positive integers satisfying*

$$\frac{b_{m(k+1)-1}}{b_{m(k)}} = O(1).$$

If

$$\sum_{\substack{k=1\\ m(k)<m(k+1)}}^{\infty} P\Big(\max_{1\le i<m(k)} |S_i| > b_{m(k)-1}\varepsilon\Big) < \infty \quad \text{for all } \varepsilon>0, \tag{2.10}$$

then $S_n/b_n \to 0$ a.s.

PROOF. Let k_1, k_2, … denote the elements of the set

$$\{k : m(k) < m(k+1)\}$$

in increasing order. Observing that

$$\sum_{\substack{k=1\\ m(k)<m(k+1)}}^{\infty} P\Big(\max_{1\le i<m(k)} |S_i| > b_{m(k)-1}\varepsilon\Big) = \sum_{j=1}^{\infty} P\Big(\max_{1\le i<m(k_j)} |S_i| > b_{m(k_j)-1}\varepsilon\Big),$$

(2.10) implies by the Borel–Cantelli lemma that

$$\frac{1}{b_{m(k_j)-1}} \max_{1\le i<m(k_j)} |S_i| \to 0 \ \text{ a.s.} \tag{2.11}$$

For $m(k_j) \le n < m(k_{j+1})$,

$$\frac{|S_n|}{b_n} \le \frac{1}{b_{m(k_j)}} \max_{m(k_j)\le i<m(k_{j+1})} |S_i| \le \frac{1}{b_{m(k_j)}} \max_{1\le i<m(k_{j+1})} |S_i| \tag{2.12}$$

$$= \frac{b_{m(k_j+1)-1}}{b_{m(k_j)}} \frac{1}{b_{m(k_{j+1})-1}} \max_{1\le i<m(k_{j+1})} |S_i| \quad (\text{since } m(k_j+1) = m(k_{j+1}))$$

$$\le O(1) \frac{1}{b_{m(k_{j+1})-1}} \max_{1\le i<m(k_{j+1})} |S_i|.$$

The right-hand side of (2.12) converges to 0 a.s. by (2.11) and so the result is proved. □

REMARK 2.5. The interested reader may notice that the conditions on the sequence $\{b_n, n \ge 1\}$ in Theorems 2.2 and 2.3 look very similar. In fact, the condition in Theorem 2.2 is stronger than that in Theorem 2.3. In many cases, the two conditions are equivalent. Generally, it is not easy to check the condition in Theorem 2.2, while the condition in Theorem 2.3 is easier to check.

Using Theorem 2.3, we can obtain the following corollary.

COROLLARY 2.6 (Fazekas and Klesov [3]). *Let $r > 0$. Let $\{b_n, n \ge 1\}$ be a nondecreasing unbounded sequence of positive constants and $\{\alpha_n, n \ge 1\}$ a sequence of nonnegative constants. Suppose that for all $n \ge 1$,*

$$E\Big(\max_{1\le i\le n} |S_i|^r\Big) \le \sum_{i=1}^{n} \alpha_i. \tag{2.13}$$

If $\sum_{n=1}^{\infty} \alpha_n / b_n^r < \infty$, then $S_n/b_n \to 0$ a.s.

PROOF. For $k \ge 1$, define $m(k) = \inf\{i : b_i \ge 2^k\}$. Then $\{m(k), k \ge 1\}$ is a nondecreasing unbounded sequence of positive integers. By the definition of $m(k)$, we get $b_{m(k+1)-1}/b_{m(k)} < 2$. Let k_1, k_2, … denote the elements of the set $\{k : m(k) < m(k+1)\}$ in increasing order. Then we have by Markov's inequality and (2.13) that

$$\sum_{\substack{k=1\\ m(k)<m(k+1)}}^{\infty} P\Big(\max_{1\le i<m(k)} |S_i| > b_{m(k)-1}\varepsilon\Big) \tag{2.14}$$

$$= \sum_{j=1}^{\infty} P\Big(\max_{1\le i<m(k_j)} |S_i| > b_{m(k_j)-1}\varepsilon\Big) \le \frac{1}{\varepsilon^r}\sum_{j=1}^{\infty} \frac{1}{b^r_{m(k_j)-1}} E\Big(\max_{1\le i<m(k_j)} |S_i|^r\Big)$$

$$\le \frac{1}{\varepsilon^r}\sum_{j=1}^{\infty}\frac{1}{b^r_{m(k_j)-1}}\sum_{i=1}^{m(k_j)-1}\alpha_i = \frac{1}{\varepsilon^r}\sum_{i=1}^{\infty}\alpha_i \sum_{\{j:\, m(k_j)-1\ge i\}} 1/b^r_{m(k_j)-1}.$$

Now we estimate $\sum_{\{j:\, m(k_j)-1\ge i\}} 1/b^r_{m(k_j)-1}$. Let $j1 = \min\{j : m(k_j) - 1 \ge i\}$. Then $i \le m(k_{j1}) - 1 < m(k_{j1})$ and so $b_i \le b_{m(k_{j1})-1}$ and $b_i < 2^{k_{j1}}$. It follows that

$$\sum_{\{j:\, m(k_j)-1\ge i\}} \frac{1}{b^r_{m(k_j)-1}} = \sum_{j=j1}^{\infty}\frac{1}{b^r_{m(k_j)-1}} = \frac{1}{b^r_{m(k_{j1})-1}} + \sum_{j=j1}^{\infty}\frac{1}{b^r_{m(k_{j+1})-1}} \tag{2.15}$$

$$\le \frac{1}{b^r_{m(k_{j1})-1}} + \sum_{j=j1}^{\infty}\frac{1}{b^r_{m(k_j)}} \le \frac{1}{b^r_{m(k_{j1})-1}} + \sum_{j=j1}^{\infty}\frac{1}{2^{rk_j}}$$

$$= \frac{1}{b^r_{m(k_{j1})-1}} + \frac{2^r}{(2^r-1)2^{rk_{j1}}} \le \Big(1 + \frac{2^r}{2^r-1}\Big)\frac{1}{b_i^r}.$$

Substituting (2.15) into (2.14), condition (2.10) of Theorem 2.3 is satisfied. Hence the result follows from Theorem 2.3. □

COROLLARY 2.7 (Tómács and Líbor, [12]). *Let $r > 0$. Let $\{b_n, n \ge 1\}$ be a nondecreasing unbounded sequence of positive constants and $\{\alpha_n, n \ge 1\}$ a sequence of nonnegative constants. Suppose that for any $n \ge 1$ and any $\varepsilon > 0$,*

$$P\Big(\max_{1\le i\le n} |S_i| > \varepsilon\Big) \le \varepsilon^{-r}\sum_{i=1}^{n}\alpha_i. \tag{2.16}$$

If $\sum_{n=1}^{\infty}\alpha_n/b_n^r < \infty$, then $S_n/b_n \to 0$ a.s.

PROOF. In this case, equation (2.14) in the proof of Corollary 2.6 holds without the first inequality concerning the moments. The rest of the proof is same as that of Corollary 2.6 except using (2.16) instead of (2.13) and is omitted. □

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