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MARCINKIEWICZ-TYPE LAW OF LARGE NUMBERS FOR DOUBLE ARRAYS

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Abstract. Chatterji strengthened version of a theorem for martingales which is a generalization of a theorem of Marcinkiewicz proving that if X_n is a sequence of independent, identically distributed random variables with $E|X_n|^p < \infty, 0 < p < 2$ and $EX_1 = 0$ if $1 \leq p < 2$ then $n^{-1/p} \sum_{i=1}^n X_i \to 0$ a.s. and in L^p. In this paper, we prove a version of law of large numbers for double arrays. If $\{X_{ij}\}$ is a double sequence of random variables with $E|X_{11}|^p \log^+ |X_{11}|^p < \infty, 0 < p < 2$, then $\lim_{m \vee n \to \infty} \frac{\sum_{i=1}^m \sum_{j=1}^n (X_{ij} - a_{ij})}{(mn)^{\frac{1}{p}}} = 0$ a.s. and in L^p, where $a_{ij} = 0$ if $0 < p < 1$, and $a_{ij} = E[X_{ij}|\mathcal{F}_{ij}]$ if $1 \leq p \leq 2$, which is a generalization of Etemadi's Marcinkiewicz-type SLLN for double arrays. This also generalize earlier results of Smythe, and Gut for double arrays of i.i.d. r.v's.

Let N denote the set of positive integer. And we note $\prec$ the lexicographic order on $N \times N$, i.e., $(i,j) \prec (k,l)$ if and only if either $i < k$ or $i = k$ and $j < l$. And let $\mathcal{F}_{ij}$ be the σ-field generated by the family of random variables $\{X_{kl} | (k,l) \prec (i,j)\}$.

To prove the main theorem, we need the following lemma.

Lemma. *Let $\{X_{ij}\}$ be a double sequence of random variables with $E[X_{ij}|\mathcal{F}_{ij}] = 0$. Given any $\epsilon > 0$, we have*

$$P\left\{ \max_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \left| \sum_{k=1}^{i} \sum_{l=1}^{j} X_{kl} \right| > \epsilon \right\} \leq \frac{4}{\epsilon^2} \sum_{i=1}^{m} \sum_{j=1}^{n} E X_{ij}^2. \tag{1}$$

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Proof. Let $S_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}$. And let $Y_j = \max\{|S_{ij}| : i = 1, \cdots, m\}$, for each $j = 1, \cdots, n$. If σ_l is a σ-field generated by $\{X_{ij}|1 \le i \le m, 1 \le j \le l\}$,

$$\begin{aligned} E[S_{kl}|\sigma_{l-1}] &= E[S_{k,l-1} + X_{1l} + \cdots + X_{kl}|\sigma_{l-1}] \\ &= E[S_{k,l-1}|\sigma_{l-1}] + E[X_{1l}|\sigma_{l-1}] + \cdots + E[X_{kl}|\sigma_{l-1}] \\ &= S_{k(l-1)}. \end{aligned}$$

Since $E[X_{il}|\sigma_{l-1}] = E[E[X_{il}|\mathcal{F}_{ml}]|\sigma_{l-1}] = 0$ by hypothesis, which implies that $\{S_{kl}, \sigma_l\}_{l=1}^{n}$ is a martingale. Hence $\{|S_{kl}|, \sigma_l\}_{l=1}^{n}$ is a submartingale for each $k = 1, \cdots, m$, which follows that $\{Y_l^2, \sigma_l\}$ is a nonnegative submartingale. Applying Theorem 3.3.4 and Corollary 3.3.2 [8], we obtain the inequalities

$$\begin{aligned} P\left\{\max_{\substack{1\le i\le m\\ 1\le j\le n}} |S_{ij}| > \epsilon\right\} &= P\left\{\max_{1\le j\le n} Y_j^2 > \epsilon^2\right\} \\ &\le \frac{1}{\epsilon} EY_n^2 \\ &\le \frac{4}{\epsilon}\sum_{i=1}^{m}\sum_{j=1}^{n} EX_{ij}^2 \end{aligned}$$

which gives us the desired results. □

Theorem 1. *Let $\{X_{ij}\}$ be a double sequence of random variables such that either $E|X|^p \log^+ |X|^p < \infty,\ 0 < p < 2,\ p \ne 1$ and $P\{|X_{ij}| \ge x\} \le P\{|X| \ge x\},\ 0 \le x < \infty$ or $E|X|\log^+|X| < \infty$ and $P(|X_{ij}| \ge x|\mathcal{F}_{ij}) \le P(|X| \ge x|\mathcal{F}_{ij})$ a.s. Then*

$$\lim_{m\vee n\to\infty} \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij} - a_{ij})}{mn^{\frac{1}{p}}} = 0 \quad a.s. \tag{2}$$

where $a_{ij} = 0$ if $0 < p < 1$, and $a_{ij} = E[X_{ij}|\mathcal{F}_{ij}]$ if $1 \le p \le 2$.

Proof. Let $\mathcal{F}$ be the distribution function of X and let $X'_{ij} = X_{ij} I\{|X_{ij}| \le (ij)^{\frac{1}{p}}\}$ with I the indicator function, $X''_{ij} = X_{ij} - X'_{ij}$, and let d_k be the number of divisors of k. Denote by $S_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}$, $S'_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n} X'_{ij}$. By using the fact that $\sum_{k=1}^{n} d_k = O(n\log n)$, we obtain the inequalities

$$
\begin{aligned}
\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} P\{X_{ij} \neq X'_{ij}\} &\leq \sum_{k=1}^{\infty} d_k P\{|X| > k^{\frac{1}{p}}\} \\
&= \sum_{i=1}^{\infty}\left(\sum_{k=1}^{i} d_k\right)\int_{i^{\frac{1}{p}}}^{(i+1)^{\frac{1}{p}}} dF(x) \qquad (3) \\
&\leq c\sum_{i=1}^{\infty} i\log i \int_{i^{\frac{1}{p}}}^{(i+1)^{\frac{1}{p}}} dF(x) \\
&\leq cE|X|^p \log^+ |X|^p < \infty,
\end{aligned}
$$

where c is an unimportant positive constant which is allowed to change. Hence by the Borel-Cantelli lemma

$$
\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij} - X'_{ij})}{(mn)^{\frac{1}{p}}} \Rightarrow 0 \quad \text{a.s.} \tag{4}
$$

Since $I_d(\cdot) - E[\cdot|\mathcal{F}_{ij}]$ is a contraction on L^2, we obtain

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} P\left\{\left|\frac{\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}(X'_{ij} - E[X'_{ij}|\mathcal{F}_{ij}])}{(2^k 2^l)^{\frac{1}{p}}}\right| > \epsilon\right\} \\
&\leq c\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{E(\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}(X'_{ij} - E[X'_{ij}|\mathcal{F}_{ij}]))^2}{(2^k 2^l)^{\frac{2}{p}}} \\
&= c\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} E(X'_{ij} - E[X'_{ij}|\mathcal{F}_{ij}])^2}{(2^k 2^l)^{\frac{2}{p}}} \\
&\leq c\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} E(X'_{ij})^2}{(2^k 2^l)^{\frac{2}{p}}} \\
&\leq c\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E(X'_{ij})^2}{(ij)^{\frac{2}{p}}}.
\end{aligned}
$$

If we use the fact that $\sum_{k=i+1}^{\infty} \frac{d_k}{k^{\frac{2}{p}}} = O(\frac{\log i}{(i+1)^{\frac{2}{p}-1}})$,

$$\begin{aligned} \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E(X'_{ij})^2}{(ij)^{\frac{2}{p}}} &\leq c\sum_{k=1}^{\infty} \frac{d_k}{k^{\frac{2}{p}}} \int_0^{k^{\frac{1}{p}}} x^2 dF(x) \\ &\leq c\sum_{i=0}^{\infty} \left(\sum_{k=i+1}^{\infty} \frac{d_k}{k^{\frac{2}{p}}} \right) \int_{i^{\frac{1}{p}}}^{(i+1)^{\frac{1}{p}}} x^2 df(x) \\ &\leq c\sum_{i=0}^{\infty} \frac{\log i}{(i+1)^{\frac{2}{p}-1}} \int_{i^{\frac{1}{p}}}^{(i+1)^{\frac{1}{p}}} x^2 dF(x) \\ &\leq cE|X|^p \log^+ |X|^p < \infty, \end{aligned} \tag{5}$$

which follows easily by summation by part. It follows that

$$\frac{\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}(X'_{ij} - E[X'_{ij}|\mathcal{F}_{ij}])}{(2^k 2^l)^{\frac{1}{p}}} \to 0 \quad \text{a.s.} \tag{6}$$

And

$$\begin{aligned} T_{kl} &= \max_{\substack{2^k \leq m < 2^{k+1} \\ 2^l \leq n < 2^{l+1}}} \left\| \frac{S^*_{2^k 2^l}}{(2^k 2^l)^{\frac{1}{p}}} - \frac{S^*_{mn}}{(mn)^{\frac{1}{p}}} \right\| \\ &\leq \frac{|S^*_{2^k 2^l}|}{(2^k 2^l)^{\frac{1}{p}}} + \max_{\substack{2^k \leq m < 2^{k+1} \\ 2^l \leq n < 2^{l+1}}} \frac{|S^*_{mn}|}{(mn)^{\frac{1}{p}}}, \end{aligned}$$

where $S^*_{mn} = \sum_{i=1}^{m}\sum_{j=1}^{n}(X'_{ij} - E[X'_{ij}|\mathcal{F}_{ij}])$. From Lemma we obtain, for any $\epsilon > 0$,

$$\begin{aligned} P\{|T_{kl}| > \epsilon\} &\leq P\left\{ \frac{|S^*_{2^k 2^l}|}{(2^k 2^l)^{\frac{1}{p}}} > \frac{\epsilon}{2} \right\} + P\left\{ \max_{\substack{2^k \leq m < 2^{k+1} \\ 2^l \leq n < 2^{l+1}}} \frac{|S^*_{mn}}{(mn)^{\frac{1}{p}}} > \frac{\epsilon}{2} \right\} \\ &\leq P\left\{ \frac{|S^*_{2^k 2^l}|}{(2^k 2^l)^{\frac{1}{p}}} > \frac{\epsilon}{2} \right\} + \frac{16}{\epsilon^2} \sum_{i=1}^{2^{k+1}}\sum_{j=1}^{2^{l+1}} \frac{E|X'_{ij}|^2}{(2^k 2^l)^{\frac{2}{p}}}. \end{aligned} \tag{7}$$

Since $\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} P\{|T_{kl}| > \epsilon\} < \infty$ from the inequalities in (5), by Borel-Cantelli lemma, we obtain

$$T_{kl} \to 0 \quad \text{a.s.} \tag{8}$$

Hence (6) and (8) give us

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X'_{ij}-E[X'_{ij}|\mathcal{F}_{ij}])}{(mn)^{\frac{1}{p}}}\to 0 \quad \text{a.s.} \tag{9}$$

Combining (4) and (9), we get

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-E[X'_{ij}|\mathcal{F}_{ij}])}{(mn)^{\frac{1}{p}}}\to 0 \quad \text{a.s.} \tag{10}$$

Suppose that $1<p<2$. Since

$$\begin{aligned}\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-E[X_{ij}|\mathcal{F}_{ij}])}{(mn)^{\frac{1}{p}}} &= \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-E[X'_{ij}|\mathcal{F}_{ij}])}{(mn)^{\frac{1}{p}}}\\ &- \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}E[X''_{ij}|\mathcal{F}_{ij}]}{(mn)^{\frac{1}{p}}},\end{aligned}$$

it remains to prove that the second term of right-hand side converges to 0. By Markov's inequality

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}P\left\{\frac{\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}E[|X''_{ij}|\mid\mathcal{F}_{ij}]}{(2^k2^l)^{\frac{1}{p}}}>\epsilon\right\}\le\frac{1}{\epsilon}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}E|X''_{ij}|}{(2^k2^l)^{\frac{1}{p}}}.$$

If we use the fact that $\sum_{k=1}^{n} \frac{d_k}{k^{\frac{1}{p}}} = O(n^{1-\frac{1}{p}} \log n)$, we obtain the following inequalities

$$\begin{aligned}
\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} E|X''_{ij}|}{(2^k 2^l)^{\frac{1}{p}}} &\leq c\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{E|X'_{ij}|}{(ij)^{\frac{1}{p}}} \\
&\leq c\sum_{k=1}^{\infty} \frac{d_k}{k^{\frac{1}{p}}} \int_{k^{\frac{1}{p}}}^{\infty} |x| dF(x) \\
&= c\sum_{k=1}^{\infty} \left(\sum_{k=1}^{i} \frac{d_k}{k^{\frac{1}{p}}} \right) \int_{i^{\frac{1}{p}}}^{(i+1)^{\frac{1}{p}}} |x| dF(x) \\
&\leq c\sum_{i=1}^{\infty} i^{1-\frac{1}{p}} \log i \int_{i^{\frac{1}{p}}}^{(i+1)^{\frac{1}{p}}} |x| dF(x) \\
&\leq c\int_{1}^{\infty} |x|^p \log^+ |x|^p dF(x) \\
&\leq cE|X_{11}|^p \log^+ |X_{11}|^p < \infty.
\end{aligned}$$

If follows that

$$\frac{\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} E[|X''_{ij}| | \mathcal{F}_{ij}]}{(2^k 2^l)^{\frac{1}{p}}} \to 0 \quad \text{a.s.} \tag{11}$$

But

$$\begin{aligned}
T'_{kl} &= \max_{\substack{2^k \leq m \leq 2^{k+1} \\ 2^l \leq n \leq 2^{l+1}}} \left| \frac{\sum_{i=1}^{m}\sum_{j=1}^{n} E[|X''_{ij}| | \mathcal{F}_{ij}]}{(mn)^{\frac{1}{p}}} - \frac{\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} E[|X''_{ij}| | \mathcal{F}_{ij}]}{(2^k 2^l)^{\frac{1}{p}}} \right| \\
&\leq \frac{2^{\frac{2}{p}}}{(2^{k+1} 2^{l+1})^{\frac{1}{p}}} \sum_{i=1}^{2^{k+1}}\sum_{j=1}^{2^{l+1}} E[|X''_{ij}| \mid \mathcal{F}_{ij}]. \qquad (12)
\end{aligned}$$

Since $\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} E|T'_{kl}| < \infty, T'_{kl}$ converges to 0 which implies by (11)

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n} E[|X''_{ij}| \mid \mathcal{F}_{ij}]}{(mn)^{\frac{1}{p}}} \to 0 \quad \text{a.s.}$$

Therefore the equation (2) holds for $1 < p < 2$. For $0 < p < 1$, by using the fact that $\sum_{k=i+1}^{\infty} \frac{d_k}{k^{\frac{1}{p}}} = O\left(\frac{\log i}{(i+1)^{\frac{1}{p}-1}}\right)$, we can show that

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n} E[|X'_{ij}|\|\mathcal{F}_{ij}]}{(mn)^{\frac{1}{p}}} \to 0 \quad \text{a.s.}$$

by the argument similar to the case $1 < p < 2$. Hence by (10)

$$\frac{\sum_{i=1}^{m}\sum_{j=1}^{n} X_{ij}}{(mn)^{\frac{1}{p}}} \to 0 \quad \text{a.s.}$$

Now for $p = 1$, we note that

$$E[X''_{ij}|\mathcal{F}_{ij}] \leq E[XI\{|X| > ij\}|\mathcal{F}_{ij}].$$

In the identity

$$\begin{aligned}
&(mn)^{-1}\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij} - E[X_{ij}|\mathcal{F}_{ij}]) \\
&= (mn)^{-1}\sum_{i=1}^{m}\sum_{j=1}^{n}(X'_{ij} - E[X'_{ij}|\mathcal{F}_{ij}]) \\
&\quad +(mn)^{-1}\sum_{i=1}^{m}\sum_{j=1}^{n}[X''_{ij} + (E[X'_{ij}|\mathcal{F}_{ij}] - E[X_{ij}|\mathcal{F}^{ij})],
\end{aligned}$$

the first term on the right converges a.s. to 0 by (9). Since $(mn)^{-1}$ $\sum_{i=1}^{m}\sum_{j=1}^{n} X''_{ij} \to 0$ a.s by (4), it will be enough to show that $(mn)^{-1}$ $\sum_{i=1}^{m}\sum_{j=1}^{n}(E[X'_{ij}|\mathcal{F}_{ij}] - E[X_{ij}|\mathcal{F}_{ij}]) \to 0$ a.s. We note that by the stronger hypothesis for $p = 1$

$$\begin{aligned}
|E[X'_{ij}|\mathcal{F}_{ij}] - E[X_{ij}|\mathcal{F}_{ij}]| &\leq E[|X''_{ij}|\mathcal{F}_{ij}] \\
&\leq 2E[|X|I\{|X| > ij\}|\mathcal{F}_{ij}].
\end{aligned}$$

And using the fact that $|X|I\{|X| > n+1\} < |X|I\{|X| > n\}$, we see that both $E[|X|I\{|X| > n\}|\mathcal{F}_{1n}], n = 1, 2\cdots$ and $E[|X|I\{|X| > n\}|\mathcal{F}_{in}], n = 1, 2, \cdots$ are positive super-martingale. Since every positive super-martingale converges a.s., $\lim_{k\to\infty} E[|X|I\{|X| > k\}|\mathcal{F}_{1k}] = Y$ exist a.s. But $\int Y \leq \lim_{k\to\infty} E\{|X|I\{|X| > k\} = 0$ so that Y being non-negative must be zero a.s. and similarly $\lim_{k\to\infty} E[|X|I\{|X| > k\}|\mathcal{F}_{k1}] = 0$ a.s.

Now, noting that

$$\begin{aligned} E[|X|I\{|X|>ij\}|\mathcal{F}_{ik}] &\geq E[|X|I\{|X|>i(j+k)\}|\mathcal{F}_{i(j+k)}\}], \\ &k=0,1,2,\cdots, \\ E[|X|I\{|X|>ij\}|\mathcal{F}_{ik}] &\geq E[|X|I\{|X|>(i+k)j\}|\mathcal{F}_{(i+k)j}\}], \\ &k=0,1,2,\cdots, \end{aligned}$$

we have

$$\begin{aligned} &\sum_{i=1}^{m}\sum_{j=1}^{n} E[|X''_{ij}|\mathcal{F}_{ij}] \\ &\leq \min\left\{ m\sum_{j=1}^{n} E[|X|I\{|X|>j\}|\mathcal{F}_{1j}], n\sum_{i=1}^{m} E[|X|I\{|X|>i\}|\mathcal{F}_{i1}] \right\}. \end{aligned}$$

If $m \vee n \to \infty$, then either $m \to \infty$ or $n \to \infty$. Without loss of generality we assume $n \to \infty$. Then

$$(mn)^{-1}\sum_{i=1}^{m}\sum_{j=1}^{n} E[|X''_{ij}|\|\mathcal{F}_{ij}] \leq \frac{1}{n}\sum_{j=1}^{n} E[|X|I\{|X|>i\}|\mathcal{F}_{1j}] \to 0 \quad \text{a.s.}$$

The theorem is thus completely proved. □

THEOREM 2. *Under the same conditions of* Theorem 1, *we also have L^p-convergence.*

Proof. We first consider the case $1 < p < 2$. In the identity

$$\begin{aligned} &\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X_{ij}-E[X_{ij}|\mathcal{F}_{ij}])}{(mn)^{\frac{1}{p}}} \\ &= \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X'_{ij}-E[X'_{ij}|\mathcal{F}_{ij}])}{(mn)^{\frac{1}{p}}} + \frac{\sum_{i=1}^{m}\sum_{j=1}^{n}(X''_{ij}-E[X''_{ij}|\mathcal{F}_{ij}])}{(mn)^{\frac{1}{p}}}, \end{aligned}$$

the first term on the right hand side converges in L^p. For

$$\begin{aligned} &\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{E\big(\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}(X'_{ij}-E[X'_{ij}|\mathcal{F}_{ij}])\big)^2}{(2^k2^l)^{\frac{2}{p}}} \\ &\leq \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \frac{\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} E(X'_{ij})^2}{(2^k2^l)^{\frac{2}{p}}} < \infty \quad \text{by (5)}, \end{aligned}$$

which implies $\frac{\sum_{i=1}^{2^k}\sum_{j=1}^{2^l}(X'_{ij}-E[X'_{ij}|\mathcal{F}_{ij}])}{(2^k2^l)} \to 0$ in L^2 and

$$\begin{aligned} T'_{kl} &= \max_{\substack{2^k\le m\le 2^{k+1}\\ 2^l\le n\le 2^{l+1}}} \left| \frac{\sum_{i=1}^m\sum_{j=1}^n E(X'_{ij})^2}{(mn)^{\frac{2}{p}}} - \frac{\sum_{i=1}^{2^k}\sum_{j=1}^{2^l} E(X'_{ij})^2}{(2^k2^l)^{\frac{2}{p}}} \right| \\ &\le 4^{\frac{2}{p}} \frac{\sum_{i=1}^{2^{k+1}}\sum_{j=1}^{2^{l+1}} E(X'_{ij})^2}{(2^{k+1}2^{l+1})^{\frac{2}{p}}} \to 0 \end{aligned}$$

by (5) again, which implies the first term converges to 0 in L^2 and hence in L^p. For the second term,

$$\begin{aligned} \frac{\sum_{i=1}^m\sum_{j=1}^n E|X''_{ij}-E[X''_{ij}|\mathcal{F}_{ij}]|^p}{mn} &\le \frac{\alpha}{mn}\sum_{i=1}^m\sum_{j=1}^n E|X''_{ij}|^p \\ &\le \frac{\alpha}{mn} m \sum_{j=1}^n E|X''_{1j}|^p \\ &= \frac{\alpha}{n}\sum_{j=1}^n E|X''_{1j}|^p \to 0 \quad \text{as } n\to\infty, \end{aligned}$$

where the first inequality comes from Essen-Von Bahr inequality.
Consider now the case $0 < p < 1$ and note that

$$\frac{\sum_{i=1}^m\sum_{j=1}^n E|X_{ij}|}{(mn)^{\frac{1}{p}}} \le \frac{\sum_{i=1}^m\sum_{j=1}^n E|X'_{ij}|}{(mn)^{\frac{1}{p}}} + \frac{\sum_{i=1}^m\sum_{j=1}^n E|X''_{ij}|}{(mn)^{\frac{2}{p}}}.$$

We already show that in the processes of the proof of Theorem 1 that the two terms on the right hand side converges 0.

The proof for $p = 1$ is as before except for one small detail. In the identity

$$\begin{aligned} &\frac{\sum_{i=1}^m\sum_{j=1}^n (X_{ij}-E[X_{ij}|\mathcal{F}_{ij}])}{(mn)} \\ &= \frac{\sum_{i=1}^m\sum_{j=1}^n (X'_{ij}-E[X'_{ij}|\mathcal{F}_{ij}])}{mn} + \frac{\sum_{i=1}^m\sum_{j=1}^n (X''_{ij}-E[X''_{ij}|\mathcal{F}_{ij}])}{(mn)} \end{aligned}$$

the first term on the right converges in L^2 to 0 as before and the second converges to 0 in L^1. Since

$$
\begin{aligned}
E\left|\frac{\sum_{i=1}^m \sum_{j=1}^n (X''_{ij} - E[X''_{ij}|\mathcal{F}_{ij}])}{mn}\right| &\leq \frac{\sum_{i=1}^m \sum_{j=1}^n E|X''_{ij} - E[X''_{ij}|\mathcal{F}_{ij}]|}{(mn)} \\
&\leq \frac{2\sum_{i=1}^m \sum_{j=1}^n E|X''_{ij}|}{(mn)} \\
&\leq \frac{2m\sum_{j=1}^n E|X''_{1j}|}{(mn)} \\
&= \frac{2}{n}\sum_{j=1}^n E|X''_{1j}|,
\end{aligned}
$$

and the last term goes to 0 as $n \to \infty$, by the fact that $E|X_{1j}| \to 0$ as $j \to \infty$. The theorem is thus completed. □

REMARK 1. The generalization to r dimensional array of random variables can be obtained under the condition

$$E|X|^p(\log^+|X|^p)^{(r-1)} < \infty, \quad \text{for} \quad 0 < p < 2.$$

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