Stochastic Analysis and Applications, 28: 928–936, 2010

ISSN 0736-2994 print/1532-9356 online
DOI: 10.1080/07362994.2010.513671

L_1-Convergence for Weighted Sums of Some Dependent Random Variables

PINGYAN CHEN,[1] MANUEL ORDÓÑEZ CABRERA,[2] AND ANDREI VOLODIN[3]

[1]Department of Mathematics, Jinan University, Guangzhou, P. R. China
[2]Department of Mathematical Analysis, University of Seville, Seville, Spain
[3]School of Mathematics and Statistics (M019), University of Western Australia, Crawley, Western Australia, Australia

In this article, the authors discuss the L_1-convergence for weighted sums of some dependent random variables under the condition of h-integrability with respect to an array of weights. The dependence structure of the random variables includes pairwise lower case negative dependence and conditions on the mixing coefficient, the maximal correlation coefficient, or the ρ^*-mixing coefficient. They prove that all the weighted sums have similar limiting behaviour.

Keywords Cesàro uniform integrability; *h*-integrability; L_1-convergence; maximal correlation coefficient; Mixing coefficient; Pairwise lower case negatively dependent random variables; ρ^*-mixing coefficient.

Mathematics Subject Classification 60F05; 60F15.

Received October 19, 2009; Accepted July 27, 2010

The authors are grateful to the referee for carefully reading the manuscript and for providing some comments and suggestions, which led to improvements in this article.

The research of P.C. has been supported by the National Natural Science Foundation of China.

The research of M.O.C. has been partially supported by DGICYT grant BFM2003-03893-C02-01 and Junta de Andalucia FQM 127.

The research of A.V. has been partially supported by the National Sciences and Engineering Research Council of Canada.

Address correspondence to Manuel Ordóñez Cabrera, Department of Mathematical Analysis, University of Seville, Seville 41080, Spain; E-mail: cabrera@us.es

1. Introduction

Pyke and Root [5] proved the following result: Let $\{X, X_n, n \geq 1\}$ be a sequence of independent identically distributed random variables with $E|X| < \infty$, then

$$\lim_{n\to\infty} E\left|n^{-1}\sum_{i=1}^{n}(X_i - EX_i)\right| = 0.$$

Note that Pyke and Root [5] proved a more general result, that is, L_r-convergence, but this is irrelevant to the present article since we are mostly concerned with L_1-convergence. The result above has been generalized in different directions: for example, for martingale differences and nonidentically distributed random variables. Of course, in the case of nonidentically distributed random variables, the moment condition $E|X| < \infty$ is replaced by Cesàro uniform integrability (see [1, 3]). In the case of independent or martingale difference sequences of random variables, the proofs are based on the following famous Marcinkiewicz-Zygmund's inequality:

$$E\left|\sum_{i=1}^{n} Y_i\right|^p \leq C_p \sum_{i=1}^{n} E|Y_i|^p, \quad \text{for all } n \geq 1,$$

where $\{Y_n, n \geq 1\}$ is a sequence of independent random variables or martingale difference with zero mean, $1 < p \leq 2$, and $C_p > 0$ depends only on p.

But for some dependent random variables, such as pairwise independent random variables, pairwise negatively quadrant dependent random variables, φ-mixing sequence, ρ-mixing sequence, etc., we only know that this inequality holds for exponent $p = 2$. According to our knowledge, no one discusses the Marcinkiewicz–Zygmund inequality for these classes of dependent random variables for exponents $1 < p < 2$. In this article, we find that only exponent 2 for the Marcinkiewicz–Zygmund inequality is important for the proofs of these results and provide some laws of large numbers in the L_1 sense for the case of dependent random variables under the assumption of h-integrability.

Now we specify some notation that will be used in this article. In the following, let $\{u_n, n \geq 1\}$ and $\{v_n, n \geq 1\}$ be two sequences of integers (not necessary positive or finite) such that $v_n > u_n$ for all $n \geq 1$, $v_n - u_n \to \infty$ as $n \to \infty$. We note that in Theorem 2, Corollary 5, and Corollary 6 we require that the sequences $\{u_n, n \geq 1\}$ and $\{v_n, n \geq 1\}$ are finite, that is, $+\infty > v_n > u_n > -\infty$. Let $\{h(n), n \geq 1\}$ be a sequence of positive constants increasing to infinity and let $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of constants such that

$$h(n) \max_{v_n \leq k \leq u_n} |a_{nk}| \to 0 \quad \text{as } n \to \infty. \tag{1}$$

For an array of random variables $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$, denote $S_n = \sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - EX_{nk})$, $n \geq 1$.

Next, for $u_n \leq k \leq v_n, n \geq 1$ we denote

$$Y_{nk} = X_{nk} I[|X_{nk}| \leq h(n)] - h(n) I[X_{nk} < -h(n)] + h(n) I[X_{nk} > h(n)],$$

the monotone truncation of the array $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$.

Note that

$$Y_{nk}^2 = X_{nk}^2 I[|X_{nk}| \leq h(n)] + h^2(n) I[|X_{nk}| > h(n)] \leq h(n)|X_{nk}|. \tag{2}$$

The following notion is introduced in [4].

Definition. Let $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of random variables and let $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of constants with $\sum_{k=u_n}^{v_n} |a_{nk}| \leq C$ for all $n \geq 1$ and some constant $C > 0$. Moreover, let $\{h(n),\ n \geq 1\}$ be an increasing sequence of positive constants with $h(n) \uparrow \infty$ as $n \uparrow \infty$. The array $\{X_{nk}\}$ is said to be *h-integrable with respect to the array of constants* $\{a_{nk}\}$ if the following conditions hold:

$$\text{a)} \quad \sup_{n \geq 1} \sum_{k=u_n}^{v_n} |a_{nk}| E|X_{nk}| < \infty$$

$$\text{b)} \quad \lim_{n \to \infty} \sum_{k=u_n}^{v_n} |a_{nk}| E|X_{nk}| I[|X_{nk}| > h(n)] = 0. \tag{3}$$

We refer the reader to [4] and [8] for properties of the notion of h-integrability, as well as for the relationship between this concept and some previous ones.

Remark. The condition $\sum_{k=u_n}^{v_n} |a_{nk}| \leq C$ was imposed in [4] in order to prove that the property of h-integrability with respect to $\{a_{nk}\}$ is weaker than the property of $\{a_{nk}\}$-uniform integrability, and this one, for its part, is weaker than the property of uniform integrability.

When h-integrability of $\{X_{nk}\}$ with respect to $\{a_{nk}\}$ is required in the hypothesis of theorems and corollaries below, the condition $\sum_{k=u_n}^{v_n} |a_{nk}| \leq C$ on the array of weights can be suppressed.

Let $\{X_i, -\infty < i < \infty\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$ and denote σ-algebras

$$\mathscr{F}_n^m = \sigma(X_i, n \leq i \leq m), \quad n \leq m \leq +\infty.$$

As usual, for a σ-algebra $\mathscr{F}$ we denote by $\mathscr{L}^2(\mathscr{F})$ the class of all $\mathscr{F}$ -measurable random variables with the finite second moment.

Now we will provide a series of definitions of dependence structures that we explore in the article.

Definition. Random variables X and Y are said to be *lower case negatively dependent*, if

$$P[X \leq x, Y \leq y] \leq P[X \leq x] P[Y \leq y]$$

for all real x and y.

A sequence of random variables $\{X_n, n \geq 1\}$ is said to be *pairwise lower case negatively dependent* if every pair of random variables in the sequence are lower case negatively dependent.

Definition. A sequence of random variables $\{X_n, n \geq 1\}$ is called *φ-mixing* if the mixing coefficient

$$\varphi(m) = \sup_{k\geq 1} \sup\{|P\{B|A\} - P\{B\}|, A \in \mathscr{F}_1^k, P\{A\} \neq 0, B \in \mathscr{F}_{k+m}^{\infty}\} \to 0$$

as $m \to \infty$.

Definition. A sequence of random variables $\{X_n, n \geq 1\}$ is called *ρ-mixing* if the maximal correlation coefficient

$$\rho(m) = \sup_{k\geq 1} \sup\left\{ \frac{\mathrm{cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}, X \in \mathscr{L}^2(\mathscr{F}_1^k), Y \in \mathscr{L}^2(\mathscr{F}_{m+k}^{\infty}) \right\} \to 0$$

as $m \to \infty$.

Definition. A sequence of random variables $\{X_n, n \geq 1\}$ is called ρ^*-mixing if for some integer $s \geq 1$ the mixing coefficient

$$\rho(s)^* = \sup \sup\{\mathrm{Corr}(X, Y) : X \in \mathscr{L}^2(\mathscr{F}_S), Y \in \mathscr{L}^2(\mathscr{F}_T)\} < 1,$$

where the outside sup is taken over all pairs of nonempty finite sets S, T of integers, such that $\min\{|s - t|, s \in S, t \in T\} \geq s$ and $\mathscr{F}_S = \sigma\{X_i, i \in S\}$.

Definition. We say that a sequence $\{Y_n, n \geq 1\}$ of random variables satisfies the *Marcinkiewicz–Zygmund inequality with exponent* 2, if for all $n \geq 1$

$$E\left|\sum_{i=1}^{n} Y_i\right|^2 \leq C \sum_{i=1}^{n} E|Y_i|^2.$$

We say that an array $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ of random variables satisfies the *Marcinkiewicz–Zygmund inequality with exponent* 2, if for all $n \geq 1$

$$E\left|\sum_{k=u_n}^{v_n} X_{nk}\right|^2 \leq C \sum_{k=u_n}^{v_n} E|X_{nk}|^2,$$

where C is a positive constant independent of n.

Note that the following sequences of mean zero random variables satisfy the Marcinkiewicz–Zygmund inequality with exponent 2, and with the indicated value of C:

1. Independent identically distributed ($C = 1$): Pyke and Root [5].
2. Martingale difference ($C = 1$): Adler, Rosalsky, and Volodin [1] and Chen, Liu, and Gan [3].
3. Pairwise lower case negatively dependent ($C = 1$): Ordóñez Cabrera and Volodin [4].
4. φ-mixing random variables with $\sum_{n=1}^{\infty} \varphi^{1/2}(n) < \infty$ $(C = 1 + 4\sum_{n=1}^{\infty} \varphi^{1/2}(n))$: Yang [9].
5. ρ-mixing random variables with $\sum_{n=1}^{\infty} \rho(n) < \infty$ $(C = 1 + 4\sum_{n=1}^{\infty} \rho(n))$: Yang [9].
6. ρ^*-mixing random variables with $\rho^*(s) < 1$ for some integer $s \geq 1$ $(C = 64s(1 - \rho^*(s))^{-2})$: Bryc and Smoleński [2] (Lemma 1 and Lemma 2).

Definition. We say that a sequence $\{Y_n, n \geq 1\}$ of random variables satisfies the *maximal inequality with exponent* 2, if for all $n \geq 1$

$$E\left|\sum_{k=1}^{n} Y_k\right|^2 \leq Cn \max_{1\leq k\leq n} E|Y_k|^2.$$

Let the sequences $\{u_n, n \geq 1\}$ and $\{v_n, n \geq 1\}$ be finite, that is, $+\infty > v_n > u_n > -\infty$. We say that an array $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ of random variables satisfies the *maximal inequality with exponent* 2, if for all $n \geq 1$

$$E\left|\sum_{k=u_n}^{v_n} X_{nk}\right|^2 \leq C(v_n - u_n) \max_{u_n\leq k\leq v_n} E|X_{nk}|^2,$$

where C is a positive constant independent of n.

Note that the following sequences of mean zero random variables satisfy the maximal inequality with exponent 2, and with the indicated value of C:

1. φ-mixing random variables with $\sum_{n=1}^{\infty} \varphi^{1/2}(2^n) < \infty$ $(C = 8000 \exp(6 \sum_{n=0}^{\infty} \varphi^{1/2}(2^n)))$: Shao [6] (Lemma 3.1).
2. ρ-mixing random variables with $\sum_{n=1}^{\infty} \rho(2^n) < \infty$ $(C = K \exp(2 \sum_{n=0}^{\infty} \rho(2^n))$ where $K > 0$ is a absolute constant): Shao [7] (Lemma 2).

2. Main Results, Corollaries, and Their Proofs

Theorem 1. *Let $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of random variables that is h-integrable with respect to the array of constants $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ satisfying assumption* (1). *Suppose that the array $\{a_{nk}(Y_{nk} - EY_{nk}), u_n \leq k \leq v_n, n \geq 1\}$ satisfies the Marcinkiewicz-Zygmund inequality with exponent* 2. *Then $S_n \to 0$ as $n \to \infty$, in L_1 and hence in probability.*

Proof. For each $n \geq 1$, $u_n \leq k \leq v_n$, let

$$A_{1n} = \sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - Y_{nk}),$$

$$A_{2n} = \sum_{k=u_n}^{v_n} a_{nk}(Y_{nk} - EY_{nk}), \quad \text{and}$$

$$A_{3n} = \sum_{k=u_n}^{v_n} a_{nk}E(Y_{nk} - X_{nk}).$$

It follows from the following that in the case of infinite u_n and/or v_n, the series A_{1n}, and A_{2n} converge absolutely almost surely and A_{3n} converges absolutely. Hence, we can write that

$$S_n = A_{1n} + A_{2n} + A_{3n}$$

and we will estimate each of these terms separately.

The series A_{1n} and A_{3n} can be estimated as follows:

$$|EA_{1n}| = |A_{3n}| \leq \sum_{k=u_n}^{v_n} |a_{nk}|E|X_{nk} - Y_{nk}| \leq 2 \sum_{k=u_n}^{v_n} |a_{nk}|E|X_{nk}|I[|X_{nk}| > h(n)] \to 0$$

as $n \to \infty$ by (3) and

$$|X_{nk} - Y_{nk}| = \begin{cases} 0 & \text{if } |X_{nk}| \leq h(n) \\ |X_{nk} + h(n)| & \text{if } X_{nk} < -h(n) \\ |X_{nk} - h(n)| & \text{if } X_{nk} > h(n). \end{cases} \leq 2|X_{nk}|I[|X_{nk}| > h(n)]$$

Hence, $A_{1n} \to 0$ in L_1, and $A_{3n} \to 0$.

For A_{2n}, we actually prove that $A_{2n} \to 0$ in L_2 and hence in L_1. Since the array $\{a_{nk}(Y_{nk} - EY_{nk}), u_n \leq k \leq v_n, n \geq 1\}$ satisfies the Marcinkiewicz-Zygmund inequality with exponent 2,

$$\begin{aligned}
0 &\leq E[\sum_{k=u_n}^{v_n} a_{nk}(Y_{nk} - EY_{nk})]^2 \\
&\leq C \sum_{k=u_n}^{v_n} a_{nk}^2 E(Y_{nk} - EY_{nk})^2 \\
&\leq C \sum_{k=u_n}^{v_n} a_{nk}^2 EY_{nk}^2 \\
&\leq C \sum_{k=u_n}^{v_n} a_{nk}^2 h(n)E|X_{nk}| \quad \text{(by (2))} \\
&\leq Ch(n) \max_{u_n \leq k \leq v_n} |a_{nk}| \sum_{k=u_n}^{v_n} |a_{nk}|E|X_{nk}| \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

by the assumptions of the theorem. □

Corollary 1 ([4]). *Let $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of rowwise lower case negatively dependent random variables that is h-integrable with respect to the array of nonnegative constants $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ satisfying assumption* (1). *Then $S_n \to 0$ as $n \to \infty$, in L_1 and hence in probability.*

Proof. Note that the only statement we need to prove is that the array $\{a_{nk}(Y_{nk} - EY_{nk}), u_n \leq k \leq v_n, n \geq 1\}$ satisfies the Marcinkiewicz-Zygmund inequality with exponent 2 and with $C = 1$.

The following two facts are well known (cf., e.g., Lemmas 2 and 3 [4]). Fact 1 states that pairwise lower case negatively dependent random variables are non-positively correlated, while Fact 2 can be applied to the technique of truncation used in this article (so-called monotone truncation) to preserve the negative dependence property.

Fact 1. If $\{X_n, n \geq 1\}$ is a sequence of mean zero pairwise lower case negatively dependent random variables, then

$$E(X_i X_j) \leq 0, \quad i \neq j.$$

Fact 2. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise lower case negatively dependent random variables and $\{f_n, n \geq 1\}$ be a sequence of nondecreasing functions. Then the sequence $\{f_n(X_n), n \geq 1\}$ is a sequence of pairwise lower case negatively dependent random variables.

Since $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ is an array of rowwise lower case negatively dependent random variables, by Fact 2, $\{a_{nk}(Y_{nk} - EY_{nk}), u_n \leq k \leq v_n, n \geq 1\}$ is also an array of rowwise lower case negatively dependent random variables, by the assumption $a_{nk} \geq 0$ for all $n \geq 1$, $u_n \leq k \leq v_n$.

Hence, it suffices to show that if $\{Z_n, n \geq 1\}$ is a sequence of mean zero pairwise lower case negatively dependent random variables, then for all $n \geq 1$

$$E\left|\sum_{i=1}^{n} Z_i\right|^2 \leq \sum_{i=1}^{n} E|Z_i|^2.$$

This follows from

$$\begin{aligned} E\left[\sum_{k=1}^{n} Z_k\right]^2 &= \sum_k E(Z_k)^2 + \sum_{j \neq k} [E(Z_j Z_k)] \\ &\leq \sum_k EZ_k^2 \quad \text{by Fact 1.} \end{aligned}$$

□

Corollary 2. *Let $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of random variables that is h-integrable with respect to the array of constants $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ satisfying assumption* (1). *For each fixed $n \geq 1$ denote by φ_n the mixing coefficient for the row $\{X_{nk}, u_n \leq k \leq v_n\}$. If $\sup_{n \geq 1} \sum_{k=1}^{v_n - u_n} \varphi_n^{1/2}(k) < \infty$, then $S_n \to 0$ as $n \to \infty$, in L_1 and hence in probability.*

Proof. Denote by $\tilde{\varphi}_n$ the φ-mixing coefficient for the row $\{a_{nk}(Y_{nk} - EY_{nk}), u_n \leq k \leq v_n\}$. Then $\tilde{\varphi}_n \leq \varphi_n$, and, hence, $\sup_{n \geq 1} \sum_{k=1}^{v_n - u_n} \tilde{\varphi}_n^{1/2}(k) < \infty$. This follows from the definition of the mixing coefficient which is only based on the generated σ-algebras. The fact that the array of random variables with $\sup_{n \geq 1} \sum_{k=1}^{v_n - u_n} \varphi_n^{1/2}(k) < \infty$ satisfies the Marcinkiewicz-Zygmund inequality with exponent 2 and with $C = 1 + 4 \sup_{n \geq 1} \sum_{k=1}^{v_n - u_n} \varphi_n^{1/2}(k)$ is proved in [9]. □

Corollary 3. *Let $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of random variables that is h-integrable with respect to the array of nonnegative constants $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ satisfying assumption* (1). *For each fixed $n \geq 1$ denote by ρ_n the mixing coefficient for the row $\{X_{nk}, u_n \leq k \leq v_n\}$. If $\sup_{n \geq 1} \sum_{k=1}^{v_n - u_n} \rho_n(k) < \infty$, then $S_n \to 0$ as $n \to \infty$, in L_1 and, hence, in probability.*

Proof. The proof follows along the same lines as that of Corollary 2. □

Corollary 4. *Let $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ be an array of random variables that is h-integrable with respect to the array of nonnegative constants $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$*

satisfying assumption (1). *For each fixed* $n \geq 1$ *denote by* ρ_n^* *the mixing coefficient for the row* $\{X_{nk}, u_n \leq k \leq v_n\}$. *If* $\sup_{n\geq 1} \rho_n^*(s) < 1$ *for some integer* $s \geq 1$, *then* $S_n \to 0$ *as* $n \to \infty$, *in* L_1 *and hence in probability.*

Proof. Denote by $\tilde{\rho}_n^*$ the mixing coefficient for the row $\{a_{nk}(Y_{nk} - EY_{nk}), u_n \leq k \leq v_n\}$. Then $\tilde{\rho}_n^* \leq \rho_n^*$, and hence $\sup_{n\geq 1} \tilde{\rho}_n^*(s) < 1$. Hence, for all $n \geq 1$, $\{a_{nk}(Y_{nk} - EY_{nk}), 1 \leq k \leq n\}$ is the array of ρ^*-mixing random variables, which satisfies the Marcinkiewicz-Zygmund inequality with exponent 2 and with $C = 64s(1 - \sup_{n\geq 1} \rho_n^*(s))^{-2}$ ([2]). □

Theorem 2. *Let the sequences* $\{u_n, n \geq 1\}$ *and* $\{v_n, n \geq 1\}$ *be finite, that is,* $+\infty > v_n > u_n > -\infty$ *and let* $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ *be an array of random variables that is h-integrable with respect to the array of nonnegative constants* $\{a_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ *satisfying*

$$(v_n - u_n)h(n) \max_{v_n \leq k \leq u_n} a_{nk} \to 0 \quad \text{as } n \to \infty. \tag{4}$$

Suppose that the array $\{a_{nk}(Y_{nk} - EY_{nk}), u_n \leq k \leq v_n, n \geq 1\}$ *satisfies the maximal inequality with exponent* 2. *Then* $S_n \to 0$ *as* $n \to \infty$, *in* L_1 *and hence in probability.*

Proof. We proceed in the same way as in the proof of Theorem 1. For each $n \geq 1, u_n \leq k \leq v_n$, let

$$A_{1n} = \sum_{k=u_n}^{v_n} a_{nk}(X_{nk} - Y_{nk}),$$

$$A_{2n} = \sum_{k=u_n}^{v_n} a_{nk}(Y_{nk} - EY_{nk}), \quad \text{and}$$

$$A_{3n} = \sum_{k=u_n}^{v_n} a_{nk}E(Y_{nk} - X_{nk}).$$

Since we assume that u_n and v_n are finite, we can write that

$$S_n = A_{1n} + A_{2n} + A_{3n}$$

and estimate each of these terms separately.

Arguing as in the proof of Theorem 1, we obtain

$$A_{1n} \to 0 \quad \text{in } L_1, \quad \text{and } A_{3n} \to 0.$$

For A_{2n}, we actually prove that $A_{2n} \to 0$ in L_2 and hence in L_1. Since the array $\{a_{nk}(Y_{nk} - EY_{nk}), u_n \leq k \leq v_n, n \geq 1\}$ satisfies maximal inequality with exponent 2,

$$\begin{aligned} 0 &\leq E[\sum_{k=u_n}^{v_n} a_{nk}(Y_{nk} - EY_{nk})]^2 \\ &\leq C(v_n - u_n) \max_{u_n \leq k \leq v_n} a_{nk}^2 EY_{nk}^2 \end{aligned}$$

$$\leq C(v_n - u_n)h(n)\left(\max_{u_n \leq k \leq v_n} |a_{nk}|\right)\left(\max_{u_n \leq k \leq v_n} |a_{nk}|E|X_{nk}|\right)$$
$$\leq C(v_n - u_n)h(n)\left(\max_{u_n \leq k \leq v_n} |a_{nk}|\right)\sum_{k=u_n}^{v_n} |a_{nk}|E|X_{nk}|$$
$$\leq C(v_n - u_n)h(n)\max_{u_n \leq k \leq v_n} |a_{nk}| \to 0 \quad \text{as } n \to \infty$$

by the assumptions of the theorem. □

Corollary 5. *Let the sequences* $\{u_n, n \geq 1\}$ *and* $\{v_n, n \geq 1\}$ *be finite, that is,* $+\infty > v_n > u_n > -\infty$ *and* $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ *be an array of random variables that are h-integrable with respect to the array of constants* $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ *satisfying assumption* (4). *For each fixed* $n \geq 1$ *denote by* φ_n *the mixing coefficient for the row* $\{X_{nk}, u_n \leq k \leq v_n\}$. *If* $\sup_{n\geq 1} \sum_{k=1}^{v_n - u_n} \varphi_n^{1/2}(2^k) < \infty$, *then* $S_n \to 0$ *as* $n \to \infty$, *in* L_1 *and hence in probability.*

Proof. Similar to Corollary 2, except we use [6]. □

Corollary 6. *Let the sequences* $\{u_n, n \geq 1\}$ *and* $\{v_n, n \geq 1\}$ *be finite, that is,* $+\infty > v_n > u_n > -\infty$ *and* $\{X_{nk}, u_n \leq k \leq v_n, n \geq 1\}$ *be an array of random variables that are h-integrable with respect to the array of constants* $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ *satisfying assumption* (4). *For each fixed* $n \geq 1$ *denote by* ρ_n *the mixing coefficient for the row* $\{X_{nk}, u_n \leq k \leq v_n\}$. *If* $\sup_{n\geq 1} \sum_{k=1}^{v_n - u_n} \rho_n(2^k) < \infty$, *then* $S_n \to 0$ *as* $n \to \infty$, *in* L_1 *and hence in probability.*

Proof. Similar to Corollary 3, except we use [7]. □

References

1. Adler, A., Rosalsky, A., and Volodin, A. 1997. A mean convergence theorem and weak law for arrays of random elements in martingale type p Banach spaces. *Statist. Probab. Lett.* 32:167–174.
2. Bryc, W., and Smolénski, W. 1993. Moment conditions for almost sure convergence of weakly correlated random variables. *Proceedings of the American Mathematical Society* 119(2):629–635.
3. Chen P.Y., Liu J.J., and Gan S.X. 1998. Convergence for B-valued random fields and the type of Banach spaces. *J. Wuhan Univ.* (Natural Science Edition), 44(1):11–14. (in Chinese)
4. Ordóñez Cabrera, M., and Volodin, A. 2005. Mean convergence theorems and weak laws of large numbers for weighted sums of random variables under a condition of weighted integrability. *Journal of Mathematical Analysis and Applications* 305:644–658.
5. Pyke, R., and Root D. 1968. On convergence in r-mean of normalized partial sums. *Ann Math Statist.* 39:379–381.
6. Shao, Q.M. 1988. A moment inequality and its applications. *Acta Math. Sinica* 31:736–747. (in Chinese)
7. Shao, Q.M. 1995. Maximal inequalities for partial sums of ρ-mixing sequences. *Annals of Probability* 23:948–965.
8. Sung, S.H., Lisawadi, S., and Volodin, A. 2008. Weak laws of large numbers for arrays under a condition of uniform integrability. *Journal of the Korean Mathematical Society* 45:289–300.
9. Yang, S.-C. 1995. Almost sure convergence of weighted sums of mixing sequences. *Journal of Systems Science and Mathematical Sciences* 15:254–265. (in Chinese)