ON THE GOLDEN RATIO, STRONG LAW, AND FIRST PASSAGE PROBLEM

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ABSTRACT

For a sequence of correlated square integrable random variables $\{X_n, n \geq 1\}$, conditions are provided for the strong law of large numbers $\lim_{n\to\infty} \frac{S_n - ES_n}{n} = 0$ almost surely to hold where $S_n = \sum_{i=1}^{n} X_i, n \geq 1$. The hypotheses stipulate that two series converge, the terms of which involve, respectively, both the Golden Ratio $\varphi = \frac{1+\sqrt{5}}{2}$ and bounds on Var X_n (respectively, bounds on Cov (X_n, X_{n+m})). An application to first passage times is provided.

Key Words and Phrases: Strong law of large numbers, Golden Ratio, Sums of correlated square integrable random variables, First passage times.

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1. INTRODUCTION

The purpose of the current investigation is to establish a link between the Golden Ratio φ and a limit theorem in probability theory. More specifically, we establish a strong law of large numbers (SLLN) for a sequence of correlated random variables $\{X_n, n \geq 1\}$. The hypotheses stipulate that two series converge, the terms of which involve, respectively, both φ and bounds on Var X_n (respectively, bounds on Cov (X_n, X_{n+m})).

We begin with a brief discussion concerning the Golden Ratio. The Golden Ratio $\varphi = \frac{1+\sqrt{5}}{2}$ (=1.61803398875...) can be derived in a variety of ways but is perhaps most simply defined as the positive solution of the equation $\varphi = 1/\varphi + 1$. This equation arises from a simple problem in geometry, namely: If we split a line segment into two smaller line segments in such a way that the ratio of the length of the original line segment to that of the larger of the two resulting line segments is equal to the ratio of the length of the larger of the two resulting line segments to the length of the smaller of the two resulting line segments, then what is this common ratio? The answer is of course the positive solution of the above equation, namely the Golden Ratio.

The Golden Ratio enjoys many remarkable properties. For example, letting F_n denote the n^{th} Fibonacci number, $n \geq 0$, then it is well known that

$$\varphi^n = F_n\varphi + F_{n-1}, n \geq 1 \text{ and } \lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \varphi.$$

Consequently,

$$\lim_{n\to\infty} \frac{\varphi^n}{F_n} = 2\varphi - 1.$$

The Golden Ratio has found numerous applications in many diverse fields including biology, astronomy, physics, architecture, fractal geometry, psychology, and financial mathematics, among others. The reader may refer to the interesting books by Dunlap [4], Koshy [7], and Livio [8] for further information on the Golden Ratio and its relation to the Fibonacci numbers, for a discussion of the numerous and diverse applications of the Golden Ratio, and for an exposé of its rich and facinating history.

The setting of the current paper is as follows. Let $\{X_n, n \geq 1\}$ be a sequence of square integrable random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. As usual, the partial sums will be denoted by $S_n = \sum_{i=1}^{n} X_i, n \geq 1$. We assume throughout this paper the following:

(i) There exists a function $H(x)$ on $(0, \infty)$ such that

$$0 < H(x) \uparrow, \tag{1.1}$$

$$\text{Var}\, X_n \leq H(n), n \geq 1, \tag{1.2}$$

and

$$\sum_{n=1}^{\infty} \frac{H(n^{\varphi+1})}{n^2} < \infty. \tag{1.3}$$

(ii) There exists a sequence of constants $\{\rho_m, m \geq 1\}$ such that

$$\sup_{n\geq 1} |\operatorname{Cov}(X_n, X_{n+m})| \leq \rho_m, m \geq 1 \tag{1.4}$$

and

$$\sum_{m=1}^{\infty} \frac{\rho_m}{m^{\varphi-1}} < \infty. \tag{1.5}$$

The main result of this paper is Theorem 3.1 which asserts that under (i) and (ii), the SLLN

$$\lim_{n\to\infty} \frac{S_n - ES_n}{n} = 0 \quad \text{almost surely} \tag{1.6}$$

holds. (*Almost sure* convergence is probabilistic jargon for saying in the language of general measure theory that the convergence takes place almost everywhere with respect to the probability measure P; "almost surely" will be abbreviated by "a.s.") We emphasize that we are not assuming that $\{X_n, n \geq 1\}$ is a weakly stationary sequence. However, under this stronger assumption, (i) is automatic and Gapoškin [5] proved the SLLN (1.6) under a condition weaker than (1.5) but with the same spirit.

The only results in the literature that we are aware of providing a SLLN for a sequence of correlated random variables satisfying (1.4) have the assumption that Var $X_n = \mathcal{O}(1)$, and the best of these results is that of Lyons [9]. While Lyons' [9] result has a condition which is indeed weaker than (1.5) (namely, $m^{\varphi-1}$ is replaced by m in the denominator), we allow in the current work for Var $X_n \to \infty$ (provided (1.3) holds). Other results on the SLLN for a sequence of correlated random variables have been obtained by Serfling [14], Móricz [10], Serfling [15], Móricz[11], and Chandra [1].

Our proof of Theorem 3.1 is more elementary than those of Gapoškin [5] and Lyons [9] discussed above. The proof of Theorem 3.1 is classical in nature in that it is based on the general "method of subsequences". This method was apparently developed initially by Rajchman [12] (see Chung [2], p. 103) and has since been used by numerous other authors. There does not seem to be any SLLNs for dependent random variables presented in the standard middle level probability textbooks and we believe that our result can find a place in such a course.

The laws of large numbers lie at the very foundation of statistical science and have a history going back almost three hundred years. In 1713, Jacob Bernoulli proved the first weak law of large numbers (WLLN) wherein the mode of convergence is "convergence in probability"; that is, "convergence in P-measure". Almost two hundred years later, the first SLLN was proved by Emile Borel in 1909. The WLLN and SLLN have been studied extensively in the case of independent summands.

The plan of the paper is as follows. In Section 2, some notation is indicated and two lemmas used in the proof of Theorem 3.1 are presented. Theorem 3.1 is proved in Section 3. Section 4 contains a discussion regarding the mysterious role that the Golden Ratio φ plays in Theorem 3.1. Finally, in Section 5, we present an application of Theorem 3.1 to first passage times.

2. PRELIMINARIES

Corresponding to the sequence $\{\rho_m, m \geq 1\}$, we consider the sequence of Cesàro means denoted by

$$c_n = \frac{\sum_{m=1}^{n} \rho_m}{n}, n \geq 1.$$

For $0 < x < \infty$, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x and let $\lceil x \rceil$ denote the smallest integer greater than or equal to x. Throughout, the symbol C will denote a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance.

The ensuing two lemmas play a key role in the proof of Theorem 3.1.

Lemma 2.1: For $n_2 \geq n_1 \geq 1$,

$$\mathrm{Var}(\sum_{i=n_1}^{n_2} X_i) \leq (n_2 - n_1 + 1)H(n_2) + 2(n_2 - n_1 + 1)^2 c_{n_2 - n_1 + 1}.$$

Proof: The lemma is obvious if $n_2 = n_1 \geq 1$. When $n_2 > n_1 \geq 1$,

$$\begin{aligned}
\mathrm{Var}(\sum_{i=n_1}^{n_2} X_i) &= \sum_{i=n_1}^{n_2} \mathrm{Var}\, X_i + 2\sum_{i=n_1}^{n_2-1}\sum_{j=i+1}^{n_2} \mathrm{Cov}(X_i, X_j) \\
&\leq (n_2 - n_1 + 1)H(n_2) + 2\sum_{i=n_1}^{n_2-1}\sum_{j=i+1}^{n_2} \rho_{j-i} \qquad \text{(by (1.1), (1.2), and (1.4))} \\
&= (n_2 - n_1 + 1)H(n_2) + 2\sum_{m=1}^{n_2-n_1} (n_2 - n_1 - m + 1)\rho_m \\
&\leq (n_2 - n_1 + 1)H(n_2) + 2(n_2 - n_1 + 1)\sum_{m=1}^{n_2-n_1+1} \rho_m \\
&= (n_2 - n_1 + 1)H(n_2) + 2(n_2 - n_1 + 1)^2 c_{n_2-n_1+1}. \qquad \square
\end{aligned}$$

Lemma 2.2: Let $\alpha = \varphi + 1$ and $\beta = \varphi - 1$. Then the three series

$$\sum_{n=1}^{\infty} c_{\lfloor n^\alpha \rfloor}, \sum_{n=1}^{\infty} \frac{c_n}{n^{1/(\alpha-1)}}, \text{ and} \sum_{n=1}^{\infty} \frac{\rho_n}{n^\beta} \tag{2.1}$$

either all converge or else all diverge.

Proof: Note at the outset that $\frac{1}{\alpha-1} = \frac{\alpha-1}{\alpha} = \beta$. Now for some sequence $\varepsilon_m = o(1)$,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{c_n}{n^{1/(\alpha-1)}} &= \sum_{n=1}^{\infty}\frac{c_n}{n^{\beta}}\\ &= \sum_{n=1}^{\infty}\frac{1}{n^{\beta+1}}\sum_{m=1}^{n}\rho_m\\ &= \sum_{m=1}^{\infty}\rho_m\sum_{n=m}^{\infty}\frac{1}{n^{\beta+1}}\\ &= \sum_{m=1}^{\infty}\frac{\rho_m}{m^{\beta}}C(1+\varepsilon_m)\end{aligned}$$

and hence the second and third series of (2.1) converge or diverge together.

Next, let $n(m) = \lceil m^{1/\alpha}\rceil, m \geq 1$. Then $n(m) \geq m^{1/\alpha} > n(m) - 1, m \geq 1$ whence

$$n(m) \sim m^{1/\alpha} \tag{2.2}$$

Now for some sequence $\delta_m = o(1)$

$$\begin{aligned}\sum_{n=1}^{\infty} c_{\lfloor n^{\alpha}\rfloor} &= \sum_{n=1}^{\infty}\frac{1}{\lfloor n^{\alpha}\rfloor}\sum_{m=1}^{\lfloor n^{\alpha}\rfloor}\rho_m\\ &= \sum_{m=1}^{\infty}\rho_m\sum_{n=n(m)}^{\infty}\frac{1}{\lfloor n^{\alpha}\rfloor}\\ &\geq \sum_{m=1}^{\infty}\rho_m\sum_{n=n(m)}^{\infty}\frac{1}{n^{\alpha}}\\ &= \sum_{m=1}^{\infty}\rho_m\frac{C(1+\delta_m)}{(n(m))^{\alpha-1}}\\ &\geq \sum_{m=1}^{\infty}\rho_m\frac{C(1+\delta_m)}{m^{\beta}} \qquad \text{(by (2.2))}\end{aligned}$$

and, similarly,

$$\sum_{n=1}^{\infty} c_{\lfloor n^{\alpha}\rfloor} \leq \sum_{m=1}^{\infty}\rho_m\frac{C(1+\delta_m)}{m^{\beta}}.$$

It follows that the first and third series of (2.1) converge or diverge together. □

3. THE MAIN RESULT

With the preliminaries accounted for, the main result may be established.
Theorem 3.1: For a sequence of random variables $\{X_n, n \geq 1\}$ satisfying the hypotheses (i) and (ii) indicated in Section 1, the SLLN

$$\lim_{n\to\infty} \frac{S_n - ES_n}{n} = 0 \quad \text{a.s.}$$

obtains.
Proof: Without loss of generality, we may and will assume that $EX_n = 0, n \geq 1$. Let $\alpha = \varphi + 1$ and $\beta = \varphi - 1$. Set

$$D_k = \max_{\lfloor k^\alpha \rfloor \leq j < \lfloor (k+1)^\alpha \rfloor} |S_j - S_{\lfloor k^\alpha \rfloor}|, k \geq 1.$$

For $n \geq 1$, let k_n be such that $\lfloor k_n^\alpha \rfloor \leq n < \lfloor (k_n+1)^\alpha \rfloor$ and note that

$$\frac{|S_n|}{n} \leq \frac{|S_{\lfloor k_n^\alpha \rfloor}| + |S_n - S_{\lfloor k_n^\alpha \rfloor}|}{\lfloor k_n^\alpha \rfloor} \leq \frac{S_{\lfloor k_n^\alpha \rfloor}}{\lfloor k_n^\alpha \rfloor} + \frac{D_{k_n}}{\lfloor k_n^\alpha \rfloor}.$$

Hence it suffices to show that

$$\lim_{k\to\infty} \frac{S_{\lfloor k^\alpha \rfloor}}{k^\alpha} = 0 \quad \text{a.s.} \tag{3.1}$$

and

$$\lim_{k\to\infty} \frac{D_k}{k^\alpha} = 0 \quad \text{a.s.} \tag{3.2}$$

To prove (3.1), let $\varepsilon > 0$ be arbitrary. Then

$$\begin{aligned}
\sum_{k=1}^{\infty} P\left\{\frac{|S_{\lfloor k^\alpha \rfloor}|}{k^\alpha} > \varepsilon\right\} &\leq \sum_{k=1}^{\infty} \frac{\text{Var}\, S_{\lfloor k^\alpha \rfloor}}{k^{2\alpha}\varepsilon^2} \qquad \text{(by Chebyshev's inequality)} \\
&\leq C \sum_{k=1}^{\infty} \frac{\lfloor k^\alpha \rfloor H(\lfloor k^\alpha \rfloor) + 2(\lfloor k^\alpha \rfloor)^2 c_{\lfloor k^\alpha \rfloor}}{k^{2\alpha}} \qquad \text{(by Lemma 2.1)} \\
&\leq C \sum_{k=1}^{\infty} \frac{H(k^\alpha)}{k^\alpha} + C \sum_{k=1}^{\infty} c_{\lfloor k^\alpha \rfloor} \qquad \text{(by (1.1))} \\
&< \infty
\end{aligned}$$

by (1.3), $\alpha > 2$, (1.5), and Lemma 2.2. It then follows from the Borel-Cantelli lemma and the arbitrariness of $\varepsilon > 0$ that (3.1) holds.

Next, to prove (3.2), again let $\varepsilon > 0$ be arbitrary. It is easy to verify that

$$\lfloor (k+1)^\alpha \rfloor - \lfloor k^\alpha \rfloor - 1 \leq \lfloor \alpha (2k)^{\alpha - 1} \rfloor, k \geq 1. \tag{3.3}$$

Then

$$\sum_{k=1}^{\infty} P\left\{\frac{D_k}{k^\alpha} > \varepsilon\right\} \le \sum_{k=1}^{\infty}(k^\alpha\varepsilon)^{-2}E\left(\max_{\lfloor k^\alpha\rfloor < n < \lfloor (k+1)\rfloor^\alpha}\left(\sum_{i=\lfloor k^\alpha\rfloor+1}^{n} X_i\right)^2\right)$$

(by the Markov inequality)

$$\le C\sum_{k=1}^{\infty} k^{-2\alpha}\sum_{n=\lfloor k^\alpha\rfloor+1}^{\lfloor (k+1)^\alpha\rfloor-1}\left((n-\lfloor k^\alpha\rfloor)H((k+1)^\alpha)+2(n-\lfloor k^\alpha\rfloor)^2 c_{n-\lfloor k^\alpha\rfloor}\right)$$

(by Lemma 2.1 and (1.1))

$$\le C\sum_{k=1}^{\infty}\frac{H((k+1)^\alpha)}{k^{2\alpha}}\sum_{n=1}^{\lfloor \alpha(2k)^{\alpha-1}\rfloor} n \quad + C\sum_{k=1}^{\infty}k^{-2\alpha}\sum_{n=1}^{\lfloor \alpha(2k)^{\alpha-1}\rfloor} n^2 c_n$$

(by (3.3))

$$\le C\sum_{k=1}^{\infty}\frac{H((k+1)^\alpha)}{(k+1)^2} \quad + C\sum_{n=1}^{\infty} n^2 c_n \sum_{k=\lceil \frac{1}{2}(\frac{n}{\alpha})^{\frac{1}{\alpha-1}}\rceil}^{\infty} k^{-2\alpha}$$

$$\le C + C\sum_{n=1}^{\infty}\frac{c_n}{n^{1/(\alpha-1)}} \qquad \text{(by (1.3))}$$

$$< \infty \qquad \text{(by (1.5) and Lemma 2.2).}$$

Again using the Borel-Cantelli lemma, (3.2) follows since $\varepsilon > 0$ is arbitrary. □

4. WHY THE GOLDEN RATIO φ?

A discussion is in order as to why the Golden Ratio φ plays such a mysterious and prominent role in Theorem 3.1. The answer rests with Lemma 2.2. Let $\varphi_1 > 0$ and set $\alpha_1 = \varphi_1 + 1$ and $\beta_1 = \varphi_1 - 1$. As in Lemma 2.1, set $\alpha = \varphi + 1$ and $\beta = \varphi - 1$. Consider the three series

$$\sum_{n=1}^{\infty} c_{\lfloor n^{\alpha_1}\rfloor}, \sum_{n=1}^{\infty}\frac{c_n}{n^{1/(\alpha_1-1)}}, \text{ and} \sum_{n=1}^{\infty}\frac{\rho_n}{n^{\beta_1}}. \tag{4.1}$$

A perusal of the argument in Lemma 2.2 reveals that the *only* value of φ_1 for which the three series in (4.1) will always converge or diverge together satisfies the equation $\varphi_1 = \varphi_1^2 - 1$ whose only positive solution is $\varphi_1 = \varphi$. Thus, we formulated Theorem 3.1 using the Golden Ratio φ.

Now as far as Lemma 2.2 pertains to the proof of Theorem 3.1, we of course only care about the two implications

$$\sum_{n=1}^{\infty}\frac{\rho_n}{n^\beta} < \infty \Rightarrow \sum_{n=1}^{\infty} c_{\lfloor n^\alpha\rfloor} < \infty$$

and

$$\sum_{n=1}^{\infty}\frac{\rho_n}{n^\beta} < \infty \Rightarrow \sum_{n=1}^{\infty}\frac{c_n}{n^{1/(\alpha-1)}} < \infty.$$

Furthermore, a perusal of the argument in Lemma 2.2 reveals that the two implications

$$\sum_{n=1}^{\infty} \frac{\rho_n}{n^{\beta_1}} < \infty \Rightarrow \sum_{n=1}^{\infty} c_{\lfloor n^{\alpha_1} \rfloor} < \infty \tag{4.2}$$

and

$$\sum_{n=1}^{\infty} \frac{\rho_n}{n^{\beta_1}} < \infty \Rightarrow \sum_{n=1}^{\infty} \frac{c_n}{n^{1/(\alpha_1-1)}} < \infty \tag{4.3}$$

both hold for $\varphi_1^2 - \varphi_1 - 1 \leq 0$ (that is, for $0 < \varphi_1 \leq \varphi$) and *both fail* for $\varphi_1 > \varphi$. Thus we see that the Golden Ratio φ, which was involved as a parameter in our conditions (i) and (ii) of Section 1, is in fact on the boundary between the validity and failure of both implications (4.2) and (4.3). Let $0 < \varphi_1 \leq \varphi$ and consider the conditions (ia) and (iia) where (ia) and (iia) are as in (i) and (ii), respectively, except that the condition

$$\sum_{n=1}^{\infty} \frac{H(n^{\varphi_1+1})}{n^2} < \infty \tag{4.4}$$

replaces (1.3) in (i) and the condition

$$\sum_{m=1}^{\infty} \frac{\rho_m}{m^{\varphi_1-1}} < \infty \tag{4.5}$$

replaces (1.5) in (ii). Now a perusal of the argument in Theorem 3.1 reveals that the theorem indeed holds with (i) and (ii) replaced, respectively, by (ia) and (iia). It is clear that when $0 < \varphi_1 < \varphi$, (4.4) is weaker than (1.3) whereas (4.5) is stronger than (1.5) and, moreover, there is a trade-off involving φ_1 in the conditions (4.4) and (4.5); the smaller φ_1 is taken, the condition (4.4) becomes weaker and the condition (4.5) becomes stronger.

Finally, we note that there does not always exist a function $H(x)$ and a sequence $\{\rho_m, m \geq 1\}$ satisfying (i) and (ii), and if they do exist they are not unique. For example, if $\operatorname{Var} X_n = n^\gamma, n \geq 1$ where $\gamma \geq (\varphi+1)^{-1}$, then for any positive nondecreasing function $H(x)$ on $(0, \infty)$ with $\operatorname{Var} X_n \leq H(n)$, $n \geq 1$ we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H(n^{\varphi+1})}{n^2} &\geq \sum_{n=1}^{\infty} \frac{H(n^{1/\gamma})}{n^2} \\ &\geq \sum_{n=1}^{\infty} \frac{H(\lfloor n^{1/\gamma} \rfloor)}{n^2} \\ &\geq \sum_{n=1}^{\infty} \frac{\operatorname{Var} X_{\lfloor n^{1/\gamma} \rfloor}}{n^2} \\ &= \sum_{n=1}^{\infty} \frac{\lfloor n^{1/\gamma} \rfloor^\gamma}{n^2} \\ &= \infty. \end{aligned}$$

Thus (1.3) cannot be satisfied. On the other hand, if Var $X_n = n^\gamma, n \geq 1$ where $0 < \gamma < (\varphi+1)^{-1}$, then taking $H(x) = Cx^{\gamma_1}$, $x > 0$ where $C \geq 1$ and $\gamma \leq \gamma_1 < (\varphi+1)^{-1}$, we have that (1.2) holds and, moreover,

$$\sum_{n=1}^{\infty} \frac{H(n^{\varphi+1})}{n^2} = \sum_{n=1}^{\infty} \frac{Cn^{\gamma_1(\varphi+1)}}{n^2} < \infty$$

since $\gamma_1(\varphi+1) < 1$ thereby verifying (1.3). In a similar manner, examples can be constructed apropos of (ii).

5. THE FIRST PASSAGE PROBLEM

In this section, $\{X_n, n \geq 1\}$ is a sequence of random variables again satisfying the assumptions (i) and (ii) spelled out in Section 1 plus the additional condition that the $\{X_n, n \geq 1\}$ all have the same mean $\mu > 0$. Let us emphasize that the random variables $\{X_n, n \geq 1\}$ can be independent, uncorrelated, or correlated. By Theorem 3.1, $S_n/n \to \mu$ a.s. and so $S_n \to \infty$ a.s. Consequently, for all $c > 0$,

$$N(c) \equiv \min\{n \geq 1 : S_n \geq c\}$$

is a well-defined random variable. The function $R(c) = EN(c)$, $c > 0$ is called the *renewal function* and is important in renewal theory. Much research work has been done on the asymptotic behavior of $R(c)$ as $c \to \infty$ particularly when the $\{X_n, n \geq 1\}$ are independent and identically distributed (i.i.d.).

Note that whenever $c_2 > c_1 > 0$ we have

$$\{n \geq 1 : S_n \geq c_2\} \subseteq \{n \geq 1 : S_n \geq c_1\}$$

and hence $N(c_2) \geq N(c_1)$ a.s. Since $S_{N(c)} \geq c$,

$$S_{N(c)} \to \infty \quad \text{a.s.} \quad \text{as} \quad c \to \infty$$

and therefore $N(c) \to \infty$ a.s. Thus $N(c) \uparrow \infty$ a.s.

In Theorem 5.1 below, we prove that

$$\lim_{c\to\infty} \frac{N(c)}{c} = \frac{1}{\mu} \quad \text{a.s.} \tag{5.1}$$

For a sequence of i.i.d. nonnegative random variables, (5.1) is a classical result due to Doob [3] and it was also obtained by Heyde [6] in the i.i.d. case without the nonnegativity proviso. Our proof of Theorem 5.1 follows along the same lines of those of Doob [3] and Heyde [6]; we apply our Theorem 3.1 whereas Doob [3] and Heyde [6] applied the Kolmogorov SLLN. A valiant attempt to obtain a version of (5.1) for a sequence of correlated random variables was made by Srinivasan, Nillaswamy, and Sridharan [16]. However, unfortunately their proof is not valid as was pointed out by Rosalsky [13] in his review of their article.

Theorem 5.1: For a sequence of random variables $\{X_n, n \geq 1\}$ all having the same mean $\mu > 0$ and satisfying the hypotheses (i) and (ii) indicated in Section 1, the relation (5.1) holds.

Proof: Set $S_0 = 0$. Let ε be an arbitrary number in $(0, \mu)$. By Theorem 3.1, with probability 1, for all large n

$$-\varepsilon \le \frac{S_n}{n} - \mu \le \varepsilon$$

and hence

$$(\mu - \varepsilon)n \le S_n \le (\mu + \varepsilon)n. \tag{5.2}$$

It follows from the first inequality of (5.2) and the definition of $N(c)$ that with probability 1, for all large c

$$(\mu - \varepsilon)(N(c) - 1) \le S_{N(c)-1} < c. \tag{5.3}$$

Similarly, the second inequality of (5.2) and the definition of $N(c)$ ensure that with probability 1, for all large c

$$c \le S_{N(c)} \le (\mu + \varepsilon)N(c). \tag{5.4}$$

It thus follows from (5.3) and (5.4) that with probability 1, for all large c

$$\frac{1}{\mu + \varepsilon} \le \frac{N(c)}{c} = \frac{N(c) - 1}{c} + \frac{1}{c} < \frac{1}{\mu - \varepsilon} + \frac{1}{c}$$

whence

$$\frac{1}{\mu + \varepsilon} \le \liminf_{c\to\infty} \frac{N(c)}{c} \le \limsup_{c\to\infty} \frac{N(c)}{c} \le \frac{1}{\mu - \varepsilon} \quad \text{a.s.}$$

The conclusion (5.1) follows since ε is an arbitrary number in $(0, \mu)$. $\square$

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