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ON THE STRONG CONVERGENCE OF WEIGHTED SUMS

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ABSTRACT

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ an array of constants. Some strong convergence results for the weighted sums $\sum_{i=1}^{n} a_{ni} X_i$ are obtained.

1. INTRODUCTION

The concept of complete convergence introduced by Hsu and Robbins (1) is as follows. A sequence $\{U_n, n \geq 1\}$ of random variables converges completely to the constant θ if $\sum_{n=1}^{\infty} P(|U_n - \theta| > \epsilon) < \infty$ for all $\epsilon > 0$. If $U_n \to \theta$ completely, then the Borel–Cantelli lemma implies that $U_n \to 0$ a.s. (almost sure). The converse is generally not true.

Recently, Wu (2) proved the equivalence of the a.s. and complete convergence of weighted sum $\sum_{i=1}^{n} X_i/((n+2-i)\log(n+2-i)\log\log n)$ of i.i.d. (independent and identically distributed) random variables. In this paper, we give some conditions on weights so that the weighted sum converges completely to zero. This result improves the theorem of Chow and Lai (3). By using this result, we extend Wu's (2) theorem to the more general weighted sums.

Throughout this paper, C denotes a positive constant which may be different in various places.

2. MAIN RESULTS

To prove the main results, we need the following lemmas. They provide some conditions under which some moments of the random variable exist.

Lemma 1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ an array of constants. Suppose that the following conditions hold.*

(i) $1/(|a_{nn}|\log n) = O(1)$,
(ii) $a_{nn}X_n \to 0$ *in probability,*
(iii) $\sum_{i=1}^{n} a_{ni}X_i \to 0$ a.s.

Then $E[e^{t|X_1|}] < \infty$ for all $t > 0$.

Proof. Let $Y_n = \sum_{i=1}^{n-1} a_{ni}X_i$ and $Z_n = a_{nn}X_n$. Then Y_n and $\{Z_n, Z_{n+1}, \ldots\}$ are independent, and hence $Z_n \to 0$ a.s. by Lemma 3.3.4 in Chow and Teicher (4). Since Z_n are independent, it follows by the Borel–Cantelli lemma that for $\epsilon > 0$ □

$$\infty > \sum_{n=1}^{\infty} P(|Z_n| > \epsilon) = \sum_{n=1}^{\infty} P(|a_{nn}X_1| > \epsilon) \geq \sum_{n=1}^{\infty} P(e^{\frac{C}{\epsilon}|X_1|} > n).$$

Hence $E[e^{t|X_1|}] < \infty$ for all $t > 0$. □

The following lemma is only a slight modification of Lemma 1.

Lemma 2. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables and $\{a_{ni}, 1 \leq i \leq m_n, n \geq 1\}$ an array of constants, where $\{m_n, n \geq 1\}$ is a strictly increasing sequence of positive integers. Let $\phi(n) = 1/\max_{1 \leq i \leq m_n}|a_{ni}|$. Suppose that the following conditions hold.*

(i) $\phi(n) \to \infty$ as $n \to \infty$,
(ii) $\sum_{i=1}^{m_n} a_{ni}X_i \to 0$ *completely.*

Then for any positive nondecreasing function ψ such that $\psi(\phi(n) \leq n$ for all $n \geq 1$, we have that $E\psi[t|X_1|] < \infty$ for all $t > 0$.

Proof. Let k_n be such that $|a_{nk_n}| = 1/\phi(n), n \geq 1$. Let $Y_n = \sum_{i=1}^{m_n} a_{ni}X_i - a_{nk_n}X_{k_n} - a_{nm_n}X_{m_n} + a_{nm_n}X_{k_n}$ and $Z_n = a_{nk_n}X_{m_n}$. Then $Y_n + Z_n$ and $\sum_{i=1}^{m_n} a_{ni}X_i$ have the same probability distribution, and Y_n and $\{Z_n, Z_{n+1}, \ldots\}$ are independent. The rest of proof is similar to that of Lemma 1 and omitted. □

Now, we state and prove one of our main results.

Theorem 1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $EX_1 = 0$ and $E[e^{t|X_1|}] < \infty$ for all $t > 0$. Let $\{a_{ni}, 1 \leq i \leq m_n, n \geq 1\}$ be an array of constants satisfying the following conditions, where $\{m_n, n \geq 1\}$ is a sequence of positive integers.*

(i) $\max_{1\leq i\leq m_n}|a_{ni}|\log n = O(1)$,
(ii) $\sum_{i=1}^{m_n} a_{ni}^2 \log n = o(1)$.

Then $\sum_{i=1}^{m_n} a_{ni}X_i$ converges completely to zero.

Proof. From an equality $e^x \leq 1 + x + 1/2x^2e^{|x|}$ for all $x \in R$, we have

$$E[e^{ta_{ni}X_i}] \leq 1 + E\left[\frac{1}{2}t^2a_{ni}^2X_1^2e^{t|a_{ni}||X_1|}\right]$$

for any $t > 0$. Let $\epsilon > 0$ be given. By putting $t = 2\log n/\epsilon$, we obtain

$$\begin{aligned} E[e^{ta_{ni}X_i}] &\leq 1 + \frac{1}{2}\left(\frac{2}{\epsilon}\right)^2 \log^2 n a_{ni}^2 E[X_1^2 e^{\frac{2}{\epsilon}\log n|a_{ni}||X_1|}] \\ &\leq 1 + \frac{1}{2}\left(\frac{2}{\epsilon}\right)^2 \log^2 n a_{ni}^2 E[X_1^2 e^{C|X_1|}] \\ &\leq 1 + \frac{1}{2}\left(\frac{2}{\epsilon}\right)^2 \log^2 n a_{ni}^2 E[e^{(1+C)|X_1|}], \end{aligned}$$

since $x^2 \leq e^{|x|}$ for all $x \in R$. It follows that

$$\begin{aligned} P\left(\sum_{i=1}^{m_n} a_{ni}X_i > \epsilon\right) &\leq e^{-t\epsilon}E\left[e^{t\sum_{i=1}^{m_n} a_{ni}X_i}\right] \\ &\leq e^{-2\log n}\prod_{i=1}^{m_n}(1 + Ca_{ni}^2\log^2 n) \\ &\leq e^{-2\log n}\prod_{i=1}^{m_n} e^{Ca_{ni}^2\log^2 n} \leq e^{-\frac{3}{2}\log n} \end{aligned} \tag{1}$$

for all sufficiently large n. By replacing X_i by $-X_i$ from the above statement, we

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obtain

$$P\left(\sum_{i=1}^{m_n} a_{ni}X_i < -\epsilon\right) \leq e^{-\frac{3}{2}\log n} \tag{2}$$

for all sufficiently large n. Hence the result follows by (1) and (2). □

Corollary 1. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $EX_1 = 0$. Then the following statements are equivalent.*

(i) $E[e^{t|X_1|}] < \infty$ *for all t>0,*
(ii) *for every array $\{c_{ni}\}$ of constants such that* $\limsup_{n\to\infty}\sum_{i=1}^{n} c_{ni}^2 < \infty$, $\sum_{i=1}^{n} c_{ni}X_i/\log n$ *converges completely to zero.*

Proof. Let $a_{ni} = c_{ni}/\log n$. The implication (i)⇒(ii) is easily proved by Theorem 1. To prove the converse using Lemma 2, let $c_{ni} = 1/(n+1-i)$. Then $\phi(n) = \log n$, and $\limsup_{n\to\infty}\sum_{i=1}^{n} c_{ni}^2 = \limsup_{n\to\infty}\sum_{i=1}^{n} 1/i^2 < \infty$. If $\sum_{i=1}^{n} c_{ni}X_i/\log n \to 0$ completely, then $E[e^{t|X_1|}] < \infty$ for all $t > 0$ by Lemma 2 with $\psi(x) = e^x$. □

Remark 1. Corollary 1 was proved by Chow and Lai (3). Let $c_{ni} = 1/\sqrt{(i+1)\log(i+1)}$ and $a_{ni} = c_{ni}/\log n$. Then $\limsup_{n\to\infty}\sum_{i=1}^{n} c_{ni}^2 = \infty$. However, $\{a_{ni}\}$ satisfies the conditions (i) and (ii) of Theorem 1. Hence Theorem 1 improves the result of Chow and Lai (3).

The following example shows that Theorem 1 does not hold if the condition (ii) is replaced by the weaker condition (ii)′ $\sum_{i=1}^{n} a_{ni}^2 \log n = O(1)$.

Example 1. Let $X_1, X_2, \ldots$ be i.i.d. N (0, 1) random variables. Define a_{ni} by

$$a_{ni} = \begin{cases} 1/\lfloor \log n \rfloor & \text{if } 1 \leq i \leq \lfloor \log n \rfloor \\ 0 & \text{if } \lfloor \log n \rfloor + 1 \leq i \leq n \end{cases},$$

where $\lfloor a \rfloor$ denotes the integer part of a. Then the condition (i) of Theorem 1 and the above condition (ii)′ are all easily satisfied. Noting $X_1 \sim N(0,1)$, it follows that $E[e^{t|X_1|}] \leq 2e^{t^2/2}$ for all $t > 0$. Since $\sum_{i=1}^{\lfloor \log n \rfloor} X_i/\sqrt{\lfloor \log n \rfloor} \sim N(0,1)$, we have by Lemma 5.1.1 in Stout (5) that

$$P\left(|\sum_{i=1}^{n} a_{ni}X_i| > 1\right) = P\left(|\sum_{i=1}^{\lfloor \log n \rfloor} X_i/\sqrt{\lfloor \log n \rfloor}| > \sqrt{\lfloor \log n \rfloor}\right)$$
$$\geq 2\exp(-\lfloor \log n \rfloor) \geq \frac{2}{n}$$

for all sufficiently large n, and so $\sum_{i=1}^{n} a_{ni}X_i$ does not converge completely to zero.

Recently, Wu (2) proved the equivalence of the a.s. and complete convergence of weighted sum $\sum_{i=1}^{n} X_i/((n+2-i)\log(n+2-i)\log\log n)$ of i.i.d. random variables. From the following corollary, we also have the equivalence of the a.s and complete convergence of some weighted sums.

Corollary 2. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $EX_1 = 0$. Let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying the following conditions.*

(i) $\max_{1\leq i\leq n}|a_{ni}|\log n = O(1)$,
(ii) $\sum_{i=1}^{n} a_{ni}^2 \log n = o(1)$,
(iii) $1/(|a_{nn}|\log n) = O(1)$.

Then $\sum_{i=1}^{n} a_{ni}X_i \to 0$ a.s. if and only if $\sum_{i=1}^{n} a_{ni}X_i \to 0$ completely.

Proof. If $\sum_{i=1}^{n} a_{ni}X_i \to 0$ completely, then the Borel–Cantelli lemma trivially implies that $\sum_{i=1}^{n} a_{ni}X_i \to 0$ a.s. Now we assume that $\sum_{i=1}^{n} a_{ni}X_i \to 0$ a.s. Since $E|a_{nn}X_n| \leq \max_{1\leq i\leq n}|a_{ni}|E|X_1| \leq C/\log n$, it follows that $a_{nn}X_n \to 0$ in probability, and so $E[e^{t|X_1|}] < \infty$ for all $t > 0$ by Lemma 1. Thus, it follows by Theorem 1 that $\sum_{i=1}^{n} a_{ni}X_i \to 0$ completely. □

We can find some arrays $\{a_{ni}\}$ satisfying the conditions of Corollary 2. For example,

$$a_{ni} = 1/(n+1-i)\log n) \text{ or } a_{ni} = 1/((\sqrt{(n+2-i)\log(n+2-i)}\log n).$$

However, the result of Wu (2) does not follow from Corollary 2. The following theorem extends the result of Wu to the more general weighted sums. The proof is similar to that of Wu.

Theorem 2. *Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with $EX_1 = 0$. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constants satisfying the following conditions.*

(a) $|a_n| \downarrow$ *as* $n \uparrow \infty$,
(b) $0 < b_n \to \infty$,
(c) $b_n/\log n = O(1)$.

Furthermore, assume that there exists a non-decreasing sequence $\{k_n, n \geq 1\}$ satisfying the following conditions.

(d) $k_{2nk_n} \leq 2k_n \leq n$ *for all sufficiently large n,*
(e) $|a_{k_n}|\log n/b_n = O(1)$,

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(f) $|a_{k_n}|\sum_{i=1}^{n}|a_i|\log n/b_n^2 = o(1)$,
(g) $C_1 b_n \leq b_{2nk_n} \leq C_2 b_n$ *for some constants* $C_1 > 0$ *and* $C_2 > 0$.

Then $\sum_{i=1}^{n} a_{n+1-i}X_i/b_n \to 0$ *a.s. if and only if* $\sum_{i=1}^{n} a_{n+1-i}X_i/b_n \to 0$ *completely.*

Proof. Note that complete convergence implies a.s. convergence by the Borel–Cantelli lemma. Now we assume that $\sum_{i=1}^{n} a_{n+1-i}X_i/b_n \to 0$ a.s. Let $a_{ni} = a_{n+1-i}/b_n$. Then we have $1/(|a_{nn}|\log n) = O(1)$ by (c). We also have $a_{nn}X_n \to 0$ in probability, since $E|a_{nn}X_n| = |a_1|E|X_1|/b_n \to 0$ by (b). Hence, it follows by Lemma 1 that $E[e^{t|X_1|}] < \infty$ for all $t > 0$. We rewrite

$$\frac{1}{b_n}\sum_{i=1}^{n} a_{n+1-i}X_i = \frac{1}{b_n}\sum_{i=1}^{n-k_n} a_{n+1-i}X_i + \frac{1}{b_n}\sum_{i=n-k_n+1}^{n} a_{n+1-i}X_i. \tag{3}$$

By (a) and (e), we obtain that

$$\max_{1\leq i\leq n-k_n}\frac{|a_{n+1-i}|}{b_n}\log n \leq \frac{|a_{k_n}|\log n}{b_n} = O(1). \tag{4}$$

We also obtain by (a) and (f) that

$$\sum_{i=1}^{n-k_n}\left(\frac{a_{n+1-i}}{b_n}\right)^2 \log n \leq \frac{|a_{k_n}|\sum_{i=1}^{n}|a_i|\log n}{b_n^2} = o(1). \tag{5}$$

From (4) and (5), it follows by Theorem 1 that the first term on the right hand side of (3) converges completely to zero. Hence, it remains to show that the second term converges completely to zero. To do this, we let $n(m) = 2mk_m$, and S_n be the second term, i.e.,

$$S_n = \frac{1}{b_n}\sum_{i=1}^{k_n} a_i X_{n+1-i}.$$

Since $n(m+1) - k_{n(m+1)} + 1 > n(m)$ for $m \geq m_0$ by (d), $\{S_{n(m)}, m \geq m_0\}$ are independent. It follows by the Borel–Cantelli lemma that $S_{n(m)}$ converges completely to zero, since $S_n \to 0$ a.s. Hence, we have by (g) that

$$\frac{1}{b_m}\sum_{i=1}^{k_{n(m)}} a_i X_i \to 0 \quad \text{completely}. \tag{6}$$

Note that $\sum_{i=k_m+1}^{k_{n(m)}} a_i X_i/b_m$ converges completely to zero. The proof is similar to that of the first term on the right hand side of (3). Hence S_n converges completely to zero by (6), so the proof is complete. □

Remark 2. When $k_n = \lfloor \log n \rfloor$, $\{k_n\}$ satisfies the condition (d) of Theorem 2. When $b_n \sim C\sum_{i=1}^{n}|a_i|$ for some constant $C > 0$, the both conditions (e) and (f) are equivalent to $|a_{k_n}|\log n/b_n = o(1)$.

Finally, we give some examples satisfying the conditions of Theorem 2. It can be easily checked by using Remark 2.

Example 2

(1) $a_n = 1/((n+1)(\log(n+1))^{\alpha})(0 \leq \alpha < 1)$, $b_n = (\log n)^{1-\alpha}$, and $k_n = \lfloor \log n \rfloor$.
(2) $a_n = 1/((n+1)\log(n+1))$, $b_n = \log\log n$, and $k_n = \lfloor \log n \rfloor$.
(3) $a_n = 1/((n+1) + \log(n+1))$, $b_n = \log n$, and $k_n = \lfloor \log n \rfloor$.

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