Convergence of weighted sums of random elements in Banach spaces of type p

A.I. Volodin

1 Introduction

In the present article the notion of $f\,$sup-convergence of random elements, which is stronger than almost sure convergence, is introduced, the laws of large numbers with respect to this convergence for weighted sums of random elements with values in the Banach spaces of type p are studied.

A convergence, more strong than a. s. one, is defined as follows: for a continuous increasing function $f : \mathbf{R}^+ \to \mathbf{R}^+$, $f(0) = 0$, $f(\infty) = \infty$ and random elements $(T_n)_1^\infty$ which take values in the Banach space E the sequence $T_n \to 0$ in sense $f\,$sup if $\mathbf{Ef}(\sup\limits_{\mathbf{k}\geq\mathbf{n}} \|\mathbf{T_k}\|) \to \mathbf{0}$ as $n \to \infty$. Obviouly, the $f\,$sup-convergence is a more strong one than both convergence a. s. and convergence in Orlicz space $L_f(E)$ (if only f satisfies the Δ_2-condition, see [1], theorem 9.4).

In the present article we shall find the moment conditions on the norms of random elements with their values in a space of type p by making the weighted sums of these elements to converge to zero in the sense $f\,$sup.

Let us introduce notation. In the sequel, E is always a real separable Banach space, $(X_k)_1^\infty$ is a sequence of independent symmetric random elements with values in E, and the symbol C stands for a finite positive constant which is not necessarily the same in each appearance. A sequence of random elements (X_k) is *stochastically dominated* by a positive random variable ξ (we write $(X_k) \prec \xi$) if there exists $C > 0$ such that for all $t > 0$

$$\sup_{k\geq 1} \mathbf{P}\{\|\mathbf{X_k}\| > \mathbf{t}\} \leq \mathbf{CP}\{\xi > \mathbf{t}\}.$$

The notions of both Rademacher and stable types of Banach spaces, used in the sequel, can be found in [2] (Chap. 3–4). We recall also

Kwapien's inequality (see [3]). *Let* $(X_k)_1^n$ *be a sequence of independent symmetric random elements and* $(s_k)_1^n \subset \mathbf{R}$*. Then, for all* $t \geq 0$*, the probability satisfies the relation:*

$$\mathbf{P}\Big\{\Big\|\sum_{\mathbf{k=1}}^{\mathbf{n}} \mathbf{s_k X_k}\Big\| > \mathbf{t}\Big\} \leq \mathbf{2P}\Big\{\max_{\mathbf{1\leq k\leq n}} |\mathbf{s_k}|\Big\|\sum_{\mathbf{k=1}}^{\mathbf{n}} \mathbf{X_k}\Big\| > \mathbf{t}\Big\}.$$

As for the history of the question, the study of the convergence of weighted sums is a natural extension of the studies of the Marcinkiewicz law of large numbers. The Marcinkiewicz law of large numbers was examined by T.A. Azlarov and N.A. Volodin [4], A. de Acosta [5] in Banach spaces of Rademacher type, by M. Marcus and W.A. Woyczynsky [6] in Banach spaces of stable type. Generalizations of these results to weighed sums of random elements were studied by A. Adler, A. Rosalsky and R.L. Taylor [7], Th. Mikosch and R. Norvaisa [8], R.L. Taylor [9]. The present article continues these investigations. Some results were announced in [10] and [11].

2 Formulation of the main result and preliminary lemmas.

Let (X_k) be a sequence of independent symmetric random elements with values in the Banach space E. Let $\mathbf{a} = \{a_k(n),\ n \in \mathbf{N},\ \mathbf{1} \leq \mathbf{k} \leq \mathbf{n}\}$ be a triangular array of constants which will be called the *weight* (of course, not all of $a_k(n)$ equal 0). We shall study the f sup-convergence to zero of the weighted sums of the form

$$T_n = \sum_{k=1}^{n} a_k(n) X_k.$$

Write $\varphi(n) = 1/\max_{1\leq k\leq n} |a_k(n)|$, $n \in \mathbf{N}$, and require that the sequence $\varphi = (\varphi(n))$ to increase with $\lim_{n\to\infty} \varphi(n) = \infty$. Since we study the law of large numbers in spaces of type p, it becomes apparent that certain conditions must be imposed on both the function f (from the definition of f sup-convergence) and the sequence φ (corresponding to the weight $\mathbf{a}$), connected with value p. Next, we introduce the following classes of continuous functions:

$$\mathbf{I}_p = \{f : \mathbf{R}^+ \to \mathbf{R}^+, \mathbf{f(0)} = \mathbf{0}, \mathbf{f}(\infty) = \infty,$$

$$f \text{ increases and } \int_1^\infty f(t) t^{-p-1} dt < \infty\}$$

and $\mathbf{G}_p$ is the class of semiadditive functions f such that there exists $q < p$ with which $f \in \mathbf{I}_q$. Recall that the semiadditivity of a function f means $f(t+s) \le C(f(t) + f(s))$ for a certain constant C depending only on f and for all $t > 0$, $s > 0$.

We say that a weight $\mathbf{a} \in F_p$ if the corresponding sequence φ is a regular varying sequence with the exponent $1/p$, $1 < p < 2$, i. e., $\varphi(n) = n^{1/p} L(n)$, where $\lim\limits_{n\to\infty} L(mn)/L(n) = 1$ for all $m \in \mathbf{N}$ (see [12]).

The main result of the present article is

Theorem 1. *Let E be a Banach space of the stable type p, $1 < p < 2$. If a weight $\mathbf{a} \in F_p$, a function $f \in \mathbf{G}_p$, and (X_k) are independent symmetric random elements with $(X_k) \prec \xi$, then $\sum\limits_{k=1}^{\infty} \mathbf{P}\{\xi > \varphi(\mathbf{k})\} < \infty$ implies that $T_n \to 0$ in $f \sup$ as $n \to \infty$.*

To prove Theorem we shall need several lemmas. The first lemma is a slight generalization of a classical result (see [13], p. 127–128; [14], lemma 2.2) and hence its proof is omitted.

Lemma 1 *Let $(X_k) \prec \xi$ and $\sum\limits_{k=1}^{\infty} \mathbf{P}\{\xi > \varphi(\mathbf{k})\} < \infty$.*

1) *If for a certain $r > 0$ we have $\sum\limits_{k=n}^{\infty} \varphi^{-r}(k) = O(n\varphi^{-r}(n))$, then*

$$\sum_{n=1}^{\infty} \varphi^{-r}(n) \mathbf{E}\|\mathbf{X_n}\|^{\mathbf{r}} \mathbf{I}\{\|\mathbf{X_n}\| < \varphi(\mathbf{n})\} < \infty.$$

2) *If for a certain $q > 0$ there holds $\sum\limits_{k=1}^{n} \varphi^{-q}(k) = O(n\varphi^{-q}(n))$ and there exists $C > 0$ such that $\varphi(k+1) \le C\varphi(k)$ for all $k \in \mathbf{N}$, then*

$$\sum_{n=1}^{\infty} \varphi^{-q}(n) \mathbf{E}\|\mathbf{X_n}\|^{\mathbf{q}} \mathbf{I}\{\|\mathbf{X_n}\| \ge \varphi(\mathbf{n})\} < \infty.$$

Lemma 2. *If (X_k) are independent symmetric random elements, then*

$$\mathbf{Ef}(\sup_{\mathbf{n}\geq\mathbf{k}}\|\mathbf{T_n}\|) \leq \mathbf{4Ef}(\sup_{\mathbf{n}\geq\mathbf{k}}\|\mathbf{S_n}\|/\varphi(\mathbf{n}))$$

for any $k \geq 1$; $S_n = \sum\limits_{k=1}^{n} X_k$.

Proof. By Levy's inequality we have for all $k \leq m$ and $\varepsilon > 0$

$$\mathbf{Q} = \mathbf{P}\{\sup_{\mathbf{k}\leq\mathbf{n}\leq\mathbf{m}}\|\mathbf{T_n}\| > \varepsilon\} \leq \mathbf{2P}\{\|\mathbf{T_m}\| > \varepsilon\}.$$

Then, by Kwapien's inequality,

$$\mathbf{Q} \leq \mathbf{4P}\{\|\mathbf{S_m}\|/\varphi(\mathbf{m}) > \varepsilon\} \leq \mathbf{4P}\{\sup_{\mathbf{k}\leq\mathbf{n}\leq\mathbf{m}}\|\mathbf{S_n}\|/\varphi(\mathbf{n}) > \varepsilon\}.$$

Consequently, $\mathbf{P}\{\sup\limits_{n\geq k}\|T_n\| > \varepsilon\} \leq 4\mathbf{P}\{\sup\limits_{n\geq k}\|S_n\|/\varphi(n) > \varepsilon\}$. Let $g(t)$ be the inverse function of $f(t)$: $f(g(t)) = t$. Then

$$\mathbf{E}f(\sup_{n\geq k}\|T_n\|) = \int_0^\infty \mathbf{P}\{f(\sup_{n\geq 1}\|T_n\|) > t\}dt = \int_0^\infty \mathbf{P}\{\sup_{n\geq 1}\|T_n\| > g(t)\}dt$$

$$\leq 4\int_0^\infty \mathbf{P}\{\sup_{n\geq 1}\|S_n\|/\varphi(n) > g(t)\}dt = 4\mathbf{E}f(\sup_{n\geq k}\|S_n\|/\varphi(n)). \quad \square$$

The third Lemma is a generalization of a result by B.D. Choi–S.H. Sung (see [14], theorem 2.1). Essentially new here is that we consider the case of both general Banach space and a general normalization there.

Lemma 3. *Let E be a Banach space of the Rademacher type p, $1 < p < 2$. If (X_k) are independent symmetric random elements such that*

$$\sum_{k=1}^{\infty} \mathbf{E}\|X_k\|^p/\varphi^p(k) < \infty,$$

then $\mathbf{E}f(\sup\limits_{n\geq 1}\|S_n\|/\varphi(n)) < \infty$ for any function $f \in \mathbf{I}_p$.

Proof. Let $g(t)$ be the inverse function of $f(t)$, i. e., $f(g(t)) = t$, $t \geq 0$. For any integer $k \geq 0$, we set $m_k = \min\{n : \varphi(n) \geq 2^k\}$ and $N_k = \{n : m_k \leq n < m_{k+1}\}$. Then by the choice of N and Levy's inequality we have

$$\mathbf{Q} = \mathbf{P}\{f(\sup_{n\geq 1}\|S_n\|/\varphi(n)) > t\} = \mathbf{P}\{\sup_{k\geq 0}\max_{n\in N_k}\|S_n\|/\varphi(n) > g(t)\} \leq$$

$$\leq \sum_{k=0}^{\infty} \mathbf{P}\{\max_{n\in N_k} \|S_n\|/\varphi(n) > g(t)\} \leq \sum_{k=0}^{\infty} \mathbf{P}\{\max_{n\in N_k} \|S_n\| > 2^k g(t)\}$$

$$\leq 2\sum_{k=0}^{\infty} \mathbf{P}\{\|S_{m_{k+1}-1}\| > 2^k g(t)\}.$$

Applying the Chebyshev's inequality and taking advantage that the Banach space E is of type p (see [2], Chap. 3), we have that

$$\mathbf{Q} \leq 8g^{-p}(t)\sum_{k=0}^{\infty} 2^{-kp}\mathbf{E}\|S_{m_{k+1}-1}\|^p \leq Cg^{-p}(t)\sum_{k=0}^{\infty} 2^{-kp}\sum_{i=1}^{m_{k+1}-1}\mathbf{E}\|X_i\|^p.$$

By changing the order of summation, we get

$$\mathbf{Q} \leq Cg^{-p}(t)\sum_{i=1}^{\infty}\mathbf{E}\|X_i\|\sum_{k\geq k_i} 2^{-kp},$$

where $k_i = \min\{k : m_{k+1}-1 \geq i\}$. Note that $m_{k_i+1}-1 \geq i$ and $\varphi(m_{k_i+1}-1) > 2^{k_i+1}$. Hence

$$\sum_{k\geq k_i} 2^{-kp} = \frac{2^p}{1-2^{-p}}2^{-(k_i+1)p} \leq C(\varphi(m_{k_i+1}-1))^{-p} \leq C\varphi^{-p}(i).$$

Thus, $\mathbf{Q} \leq Cg^{-p}(t)\sum_{i=1}^{\infty}\mathbf{E}\|X_i\|^p/\varphi^p(i)$.

By changing the variables $t = f(s)$ in the integral $\int_1^{\infty} f(s)ds^{-p}$, which converges by hypothesis, and integrating by parts, we see that $\int_1^{\infty} g^{-p}(t)dt < \infty$. So

$$\mathbf{E}f(\sup_{n\geq 1}\|S_n\|/\varphi(n)) \leq 1 + \int_1^{\infty}\mathbf{P}\{f(\sup_{n\geq 1}\|S_n\|/\varphi(n) > t\}dt \leq$$

$$\leq 1 + C\sum_{i=1}^{\infty}\mathbf{E}\|X_i\|^p/\varphi^p\int_1^{\infty} g^{-p}(t)dt < \infty. \quad \square$$

3 The main results

The next proposition can be considered as Chung's law of large numbers (see [15], theorem 3.1) for weighted sums with respect to f sup-convergence.

Proposition. *Let there be given a weight* **a** *and the corresponding sequence* φ. *Let* E *be a Banach space of the Rademacher type* p, $1 < p < 2$. *If* (X_k) *are independent symmetric random elements such that* $\sum_{k=1}^{\infty} \mathbf{E}\|X_k\|^p/\varphi^p(k) < \infty$, *then for any* $f \in \mathbf{I}_p$ *the sequence* T_n *tends to* 0 *in the sense* $f\,\mathrm{sup}$.

Proof. Since E is of the type p (see [2], Chap. 3), it follows that

$$\mathbf{E}\Big\|\sum_{k=1}^{\infty} X_k/\varphi(k)\Big\|^p \le C\sum_{k=1}^{\infty} \mathbf{E}\|X_k\|^p/\varphi^p(k).$$

This means that the series $\sum_{k=1}^{\infty} X_k/\varphi(k)$ converges in $L^p(E)$. Hence, it converges a. s. (see [9], § 4.5). By Kronecker's lemma, $\frac{1}{\varphi(n)}\sum_{k=1}^{n} X_k \to 0$ a. s. as $n \to \infty$.

By Kwapien's inequality,

$$\mathbf{P}\{\|\mathbf{T_m}\| > \mathbf{t}\} \le \mathbf{2P}\Big\{\frac{\mathbf{1}}{\varphi(\mathbf{n})}\Big\|\sum_{\mathbf{k=1}}^{\mathbf{n}} \mathbf{X_k}\Big\| > \mathbf{t}\Big\} \to \mathbf{0},$$

i.e., $T_n \to 0$ in probability as $n \to \infty$. Hence there exists a subsequence $(n_m)_{m=1}^{\infty}$ such that $T_{n_m} \to 0$ a. s. as $m \to \infty$. By Lemma 2 the expectation $\mathbf{E}f(\sup_{n\ge1}\|T_n\|) < \infty$. By the Lebesgue convergence theorem, $\mathbf{E}f(\sup_{k\ge n_m}\|T_k\|) \to 0$ as $m \to \infty$. Hence $\mathbf{E}f(\sup_{k\ge n}\|T_k\|) \to 0$ as $n \to \infty$.

Corollary 1. *Let there be given a weight* **a** *and the corresponding sequence* φ *and assume that* E *is a Banach space of the Rademacher type* p, $1 < p < 2$. *If* (X_k) *are independent symmetric random elements such that* $\sum_{k=1}^{\infty} \mathbf{E}\|X_k\|^p/\varphi^p(k) < \infty$, *then* $T_n \to 0$ *a. s.*

With the use of Lemmas 1 and 3 and Proposition we can now carry out the

Proof of Theorem 1 Since φ is a regular varying sequence with the exponent $1/p$, the hypotheses of Lemma 1 are satisfied for all $r > p$ and $q < p$ (see [12], exercise 2.1 and theorem 2.8). Write

$$X_k' = X_k I\{\|X_k\| < \varphi(k)\}, \quad X_k'' = X_k I\{\|X_k\| \ge \varphi(k)\},$$

$$T_n' = \sum_{k=1}^{n} a_k(n) X_k', \quad T_n'' = \sum_{k=1}^{n} a_k(n) X_k''.$$

Then $X_k = X'_k + X''_k$ and

$$\mathbf{E}f(\sup_{n\ge k}\|T_n\|) = \mathbf{E}f(\sup_{n\ge k}\|T'_n + T''_n\|) \le C\mathbf{E}f(\sup_{n\ge k}\|T'_n\|) + C\mathbf{E}f(\sup_{n\ge k}\|T''_n\|),$$

since f is semiadditive.

The space E is of a stable type $p<2$, hence it is of the Rademacher type r for some $r>p$ (see [2], Chap. 4), and by Lemma 1 (1)

$$\sum_{k=1}^{\infty}\mathbf{E}\|X'_k\|^r/\varphi^r(k) < \infty.$$

Hence, by Proposition, $T'_n \to 0$ in $f\sup$.

Since $f \in \mathbf{G}_p$, there exists $q<p$ such that $f \in \mathbf{I}_q$. By Lemma 1 (2) $\sum_{k=1}^{\infty}\mathbf{E}\|X''_n\|^q/\varphi^q(k) < \infty$, whence, by Proposition, $T_n \to 0$ in $f\sup$.

4 Applications

Theorem 1 is connected with some problems associated with the planning of optimal stopping rules. In this connection there arises the question on the finiteness of $\mathbf{E}\sup_{n\ge 1}\Big(\Big|\sum_{k=1}^{n}\xi_k\Big|/\varphi(n)\Big)^r$ for some $\varphi(n)$ and $r>0$ (for example $\varphi(n)=n^q$ in [16], [14]) where ξ_k are i. i. d. centered random variables. We establish a result of this type for the Banach space valued random elements.

Theorem 2. *Let E be a Banach space of the stable type p, $1<p<2$, the sequence $\varphi=(\varphi(n))$ be regularly varying with the exponent $1/p$, function $f \in \mathbf{G}_p$, and (X_k) be a sequence of independent random elements with identically distributed norms. The following statements are equivalent*:

(1) $\sum_{k=1}^{\infty}\mathbf{P}\{\|X_1\| > \varphi(k)\} < \infty,$

(2) $S_n/\varphi(n) \to 0$ *in* $f\sup$ *as* $n\to\infty$,

(3) $\mathbf{E}f(\sup_{n\ge 1}\|S_n\|/\varphi(n)) < \infty,$

(4) $\mathbf{E}f(\sup_{n\ge 1}\|X_n\|/\varphi(n)) < \infty.$

Proof. The implication (1)$\Longrightarrow$(2) has been proved in Theorem 1 and (2)$\Longrightarrow$(3) is trivial. Next let us show that (3)$\Longrightarrow$(4). Since the function f is semiadditive and $\varphi(n)$ increases, we have

$$\mathbf{E}f(\sup_{n\geq 1}\|X_n\|/\varphi(n)) = \mathbf{E}f(\sup_{n\geq 1}\Big\|S_n/\varphi(n) - \frac{\varphi(n)}{\varphi(n-1)}S_{n-1}/\varphi(n)\Big\| \leq$$

$$\leq C\mathbf{E}f(\sup_{n\geq 1}\|S_n\|/\varphi(n)) + C\mathbf{E}f(\sup_{n\geq 1}\|S_{n-1}\|/\varphi(n-1))$$

$$\leq 2C\mathbf{E}f(\sup_{n\geq 1}\|S_n\|/\varphi(n)).$$

Finally, we prove by contradiction that (4)$\Longrightarrow$(1). Assume that

$$\sum_{k=1}^{\infty}\mathbf{P}\{\|X_1\| > \varphi(k)\} = \infty.$$

Then for all $M \in \mathbf{N}$

$$\infty = \sum_{k=1}^{\infty}\mathbf{P}\{\|X_1\| > \varphi(Mk)\} = \sum_{k=1}^{\infty}\mathbf{P}\{\|X_1\| > C\varphi(M)\varphi(k)\}$$

$$\leq \sum_{k=1}^{\infty}\mathbf{P}\{\|X_1\|/\varphi(k) > C\varphi(M)\},$$

where we use the condition of regular varying of φ in the penultimate inequality. The Borel-Cantelli lemma implies that $\sup_{n\geq 1}\|X_n\|/\varphi(n) > \varphi(M)$ a. s., i. e., $\sup_{n\geq 1}\|X_n\|/\varphi(n) = \infty$ a. s. Hence $\mathbf{E}f(\sup_{n\geq 1}\|X_n\|/\varphi(n)) = \infty$, which contradicts (4).

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