CONVERGENCE OF WEIGHTED SUMS OF RANDOM ELEMENTS IN SPACES OF TYPE p

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§1. INTRODUCTION

Hoffmann-Jorgensen and Pisier [1] have characterized Banach spaces of type p with the aid of strong law of large numbers (see also [2]). We continue their analysis by establishing the necessary and sufficient conditions for type p in terms of a special L_f^{sup}-convergence of weighted sums of independent random elements with values in that Banach space.

We define this convergence as follows: for an increasing function $f : R^+ \to R^+, f(0) = 0$; random elements $\{X_n\}_1^\infty$ and X with values in the Banach space E, $X_n \to X$ in $L_f^{sup}(E)$ if $Ef\ (\sup_{n \geq k} \|X_n - X\|) \to 0$ as $k \to \infty$.

Note that $L_f^{sup}(E)$-convergence can be interpreted as (o)-convergence in an ordered lattice (see [3]). Besides, $L_f^{sup}(E)$-convergence is stronger both than the almost-certain convergence and the convergence in the Orlich space $L_f(E)$ if f is an Orlich function that satisfies Δ_2-condition (see [4, Theorem 9.4]).

As in [2], we establish that for spaces of type p, weighted sums of independent random elements with identically distributed norms are convergent to zero in the sense of $L_f^{sup}(E)$ if and only if a certain moment of the norm of the random element is bounded.

Let A be a Banach space; $\{x_k\}_1^\infty$, a certain sequence of vectors from E; $\{X_k\}_1^\infty$, a sequence of independent symmetric random elements with values in E. We will investigate the convergence to zero of weighted sums of the form

$$S_n = \frac{1}{A_n} \sum_{i=1}^{n} a_i X_i ,$$

where the sequences $\{a_i\}_1^\infty$ and $\{A_n\}_1^\infty$ are defined by monotonic functions $a(t)$ and $A(t)$ from R^+ into R^+, which satisfy the following conditions: $A(0) = 0, A(+\infty) = +\infty, A(\cdot)$ increases, $a(\cdot)$ does not increase. We set $A_n = A(n), a_i = a(i)$, $\varphi(t) = A(t)/a(t)$ and let ψ be a function that is inverse of $\varphi: \psi(\varphi(t)) = t$. Obviously, the function ϕ is increasing.

It will be recalled that E is of a (Rademacher) type p, $1 \leq p \leq 2$ if there exists C > 0 such that for any x_1,

..., x_n

$$E\|\sum_{i=1}^{n} \varepsilon_i x_i\|^p < C\sum_{i=1}^{n}\|x_i\|^p,$$

where (ε_i) are independent random quantities with $P(\varepsilon_i = \pm 1) = 1/2$.

§2. CHARACTERIZATION OF SPACES OF TYPE p

We will characterize type p in terms of the convergence of weighted sums of random elements. first consider convergence a.c. and obtain some generalization of the result of Hoffmann-Jorgensen and P Theorem 2.1].

Theorem 1. Let a function ϕ satisfy Δ_2-condition. The following statements are equivalent:

(i) E is of type p;

(ii) if $\sum_{k=1}^{\infty} E\|X_k\|^p/\varphi^p(k) < \infty$, then $S_n \to 0$ a.c.;

(iii) if $\sum_{k=1}^{\infty}\|x_k\|^p/\varphi^p(k) < \infty$, then $\frac{1}{A_n}\sum_{k=1}^{n} a_k\varepsilon_k x_k \to 0$ by probability.

We will now consider the characterization of spaces of type p in terms of $L_f^{\sup}$-convergence.

Theorem 2. Let ϕ satisfy j_2-condition and

$$\int_1^{\infty} f(t)/t^{p+1}\, dt < \infty.$$

The following statements are equivalent:

(i) E is of type p;

(ii) if $\sum_{k=1}^{\infty} E\|X_k\|^p/\varphi^p(k) < \infty$, then $S_n \to 0$ in $L_f^{\sup}(E)$;

(iii) if $\sum_{k=1}^{\infty}\|x_k\|^p/\varphi^p(k) < \infty$ then $\frac{1}{A_n}\sum_{1}^{n} a_i x_i \varepsilon_i \to 0$ in $L_f^{\sup}(E)$.

In proving that (i) $\Rightarrow$ (ii) we first show that $Ef(\sup_{n\geq 1}\|S_n\|) < \infty$. Further, by virtue of Theorem 1 $S_n \to$

Whence, by virtue of Lebegue's theorem of majorant convergence, $Ef(\sup_{n\geq k}\|S_n\|) \to 0$ as $k \to \infty$.

The implication (ii)$\Rightarrow$(iii) is obvious. Finally, (iii) implies (i) because if (iii) takes place,

$$\frac{1}{A_n}\sum_{k=1}^{n} a_k\varepsilon_k x_k \to 0$$

by probability and we can apply Theorem 1.

Note. The proofs of the implications (i)$\Rightarrow$(ii)$\Rightarrow$(iii) in Theorems 1 and 2 do not make use of the fac the function ϕ satisfies Δ_2-condition.

§3. CONVERGENCE IN SPACES OF TYPE p

Shangua [2] has demonstrated that the strong law of large numbers holds in spaces of type p for a seque[…] of independent random elements with identically distributed norms. This result can be viewed as a characterizati[…] of spaces of type p in terms of the strong law of large numbers. We will examine a similar problem for weight[…] sums of random elements.

We impose the following constraints upon the functions f and ϕ:

(a) there exists $q < p$ such that $\int_1^{\infty} f(t)/t^{1+q}\, dt < \infty$;

(b) the function f is semiadditive, i.e., there exists $B > 0$ such that for any $t_1, t_2 > 0 : f(t_1 + t_2) \leq B(f(t_1)$[…] $f(t_2))$;

(c) $\sum_{k=1}^{\infty} 1/\varphi^p(k) < \infty$;

(d) for $q < p$, condition (a) and a certain δ, $0 < \delta < 1$ imply that the function $g(t) = \varphi^q(t)/t^{\delta}$ is monoto[…] and not increasing;

(e) the function $1/\phi$ satisfies Δ'-condition, i.e., there exists $A > 0$ such that for any $t_1, t_2 > 0 : 1/\varphi(t_1$[…] $\leq A/(\varphi(t_1)\varphi(t_2))$.

Note that if f satisfies Δ_2-condition, then it is semiadditive (condition (b)). An elementary example [...] functions that satisfy conditions (a)-(e) are power functions $f(t) = t^{\alpha}$ and $\varphi(t) = t^{\beta}$ with $\alpha < q$ and $1/p < \beta < 1/q$. [...] more interesting example is $\varphi(t) = t^{\beta}/(|\ln t| + 1)$.

The case of convergence a.c. of S_n to zero has been investigated by Mikosch and Norvaisa [5, Theorem 3.2[…] see also by Adler and Rosalsky [6].

Theorem 3. Let the norms X_i, $i = 1, 2, \ldots$ be identically distributed, E be of type p. The functions f and [...] satisfy conditions (a)-(e). The following statements are equivalent:

$$\text{(i)}\ E\psi(\|X_1\|) < \infty;$$
$$\text{(ii)}\ S_n \to 0\ \text{в}\ L_f^{\sup}(E);$$
$$\text{(iii)}\ Ef(\sup_{n\geq 1}\|S_n\|) < \infty;$$
$$\text{(iv)}\ Ef(\sup_{n\geq 1}\|X_n\|/\varphi(n)) < \infty.$$

Note. In the proof that (ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(i), in Theorem 3, the fact that the Banach space is of type p doe[…] not have a significant role. Besides, in proving that (iii)$\Rightarrow$(iv) only condition (b) is used; for implication (iv) $\Rightarrow$ ([…] condition (e) is sufficient. The implication (ii)$\Rightarrow$(iii) is trivial.

The following problem is also interesting: What constraints should be imposed on the function ϕ so tha[…] in the framework of Theorem 3 or 4 the equivalence of conditions (i)-(iv) would imply that the Banach space E i[…]

of type p?

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