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ON CONVERGENCE OF SERIES OF RANDOM ELEMENTS VIA MAXIMAL MOMENT RELATIONS WITH APPLICATIONS TO MARTINGALE CONVERGENCE AND TO CONVERGENCE OF SERIES WITH p-ORTHOGONAL SUMMANDS. CORRECTION

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Abstract. The result by Móricz, Su, and Taylor from *Acta Math. Hungar.* **65**(1994), 1–16, was misstated in authors' paper in *Georgian Math. J.* **8**(2001), No. 2, 377–388, where due to this misstatement the invalid formulation and proof of a corollary is given. In this correction note, the needed result is correctly stated and a corrected version of the invalid corollary is proved.

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The purpose of this note is to correct the error and its consequence in the authors' article [2]. Specifically, the authors misstated in Lemma 3 of [2] the result of Móricz, Su, and Taylor [1]. This misstatement made in the formulation and proof of Corollary 3 of [2] invalid. The authors are grateful to Le Van Thanh (Vinh University, Nghe An Province, Socialist Republic of Viet Nam) for pointing out these errors to them. Below we provide the correct statement for Lemma 3 of [2] and a corrected (but admittedly, slightly weaker) version of Corollary 3 of [2]. These will be called Lemma 3′ and Corollary 3′, respectively. All notation and terminology used in this note are those of [2]. As in [2], the symbol C denotes a generic constant ($0 < C < \infty$) which is not necessarily the same each time it appears.

Lemma 3 of [2] should read as follows.

Lemma 3′ (Móricz, Su, and Taylor [1]). *Let $\{V_n, n \geq 1\}$ be a p-orthogonal $(1 \leq p < \infty)$ sequence of random elements and suppose that there exists a sequence of nonnegative numbers $\{u_n, n \geq 1\}$ such that*

$$E\left|\left|\sum_{j=n}^{m} V_j\right|\right|^p \leq \sum_{j=n}^{m} u_j$$

for all $m \geq n \geq 1$. Then

$$E\left\{\max_{n\leq k\leq m}\left|\left|\sum_{j=n}^{k} V_j\right|\right|^p\right\} \leq (\mathrm{Log}(2(m-n+1)))^p \sum_{j=n}^{m} u_j, \quad m \geq n \geq 1,$$

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where Log denotes the logarithm to base 2.

We now present the corrected version of Corollary 3 of [2].

Corollary 3′. *Let $\{V_n, n \geq 1\}$ be a p-orthogonal sequence of random elements taking values in a real separable, Rademacher type p $(1 \leq p \leq 2)$ Banach space and let $\{b_n, n \geq 1\}$ be a sequence of positive constants.*

(i) *If $b_n = o(1)$ and*

$$\sum_{j=n}^{\infty} (\text{Log } j)^p E\|V_j\|^p = \mathcal{O}(b_n^p),$$

then the series $\sum_{n=1}^{\infty} V_n$ converges a.s. and the tail series $T_n \equiv \sum_{j=n}^{\infty} V_j$ satisfies the relation

$$\sup_{k \geq n} \|T_k\| = \mathcal{O}_P(b_n).$$

(ii) *If $b_n = \mathcal{O}(1)$ and*

$$\sum_{j=n}^{\infty} (\text{Log } j)^p E\|V_j\|^p = o(b_n^p),$$

then the series $\sum_{n=1}^{\infty} V_n$ converges a.s. and the tail series T_n obeys the limit law

$$\frac{\sup_{k \geq n} \|T_k\|}{b_n} \xrightarrow{P} 0.$$

Proof. By Lemma 2 of [2] and Lemma 3′, we have

$$E\left\{\max_{n \leq k \leq m} \left\|\sum_{j=n}^{k} V_j\right\|^p\right\} \leq C\,(\text{Log } (2(m-n+1)))^p \sum_{j=n}^{m} E\|V_j\|^p \tag{0.1}$$

for all $m \geq n \geq 1$. For $n \geq 2$, set $r(n) = \min\{i \geq 1 : 2^i \geq n\}$. Then for $n \geq 2$,

$$\lim_{m \to \infty} E\left\{\max_{n \leq k \leq m} \left\|\sum_{j=n}^{k} V_j\right\|^p\right\} = E\left\{\sup_{k \geq n} \left\|\sum_{j=n}^{k} V_j\right\|^p\right\}$$

(by the Lebesgue monotone convergence theorem)

$$\leq E\left\{\max_{n \leq k \leq 2^{r(n)}} \left\|\sum_{j=n}^{k} V_j\right\|^p\right\} + \sum_{i=r(n)}^{\infty} E\left\{\max_{2^i+1 \leq k \leq 2^{i+1}} \left\|\sum_{j=n}^{k} V_j\right\|^p\right\}$$

$$\le C(\mathrm{Log}(2(2^{r(n)} - n + 1)))^p \sum_{j=n}^{2^{r(n)}} E\big|\big|V_j\big|\big|^p$$

$$+ \sum_{i=r(n)}^{\infty} C(\mathrm{Log}(2(2^{i+1} - 2^i)))^p \sum_{j=2^i+1}^{2^{i+1}} E\big|\big|V_j\big|\big|^p \qquad \text{(by (0.1))}$$

$$\le C(r(n)+1)^p \sum_{j=n}^{2^{r(n)}} E\big|\big|V_j\big|\big|^p + \sum_{i=r(n)}^{\infty} C(i+1)^p \sum_{j=2^i+1}^{2^{i+1}} E\big|\big|V_j\big|\big|^p$$

$$\le C \sum_{j=n}^{2^{r(n)}} (2 + \mathrm{Log}\ n)^p E\big|\big|V_j\big|\big|^p + C \sum_{i=r(n)}^{\infty} \sum_{j=2^i+1}^{2^{i+1}} (1 + \mathrm{Log}\ j)^p E\big|\big|V_j\big|\big|^p$$

$$\text{(since } 2^{r(n)-1} < n\text{)}$$

$$\le C \sum_{j=n}^{\infty} (\mathrm{Log}\ j)^p E\big|\big|V_j\big|\big|^p. \tag{0.2}$$

It follows from (0.2) that the conditions of part (i) ensure

$$\lim_{m\to\infty} E\bigg\{ \max_{n\le k\le m} \bigg|\bigg| \sum_{j=n}^{k} V_j \bigg|\bigg|^p \bigg\} = \mathcal{O}(b_n^p) = o(1) \text{ as } n \to \infty$$

and that the conditions of part (ii) ensure

$$\lim_{m\to\infty} E\bigg\{ \max_{n\le k\le m} \bigg|\bigg| \sum_{j=n}^{k} V_j \bigg|\bigg|^p \bigg\} = o(b_n^p) = o(1) \text{ as } n \to \infty.$$

The Corollary follows immediately from Theorem 1 of [2]. □

Remarks.

(i) The inequality (0.1) (rather than (3.5) of [2] as was asserted in [2]) is a generalization of the famous Menchoff fundamental maximal inequality for sums of orthogonal random variables (see, e.g., Stout [3, p. 18]).

(ii) The *proof* of Corollary 3′(ii) shows that the hypotheses indeed entail the limit law

$$\frac{\sup_{k\ge n} \|T_k\|}{b_n} \xrightarrow{\mathcal{L}_p} 0\,.$$

(iii) Suppose $E\|V_n\|^p \sim n^{-\alpha}$ for some $\alpha > 1$. It is well known that

$$\sum_{j=n}^{\infty} (\mathrm{Log}\ j)^p j^{-\alpha} \sim (\mathrm{Log}\ n)^p \sum_{j=n}^{\infty} j^{-\alpha}$$

and so the *statement* of Corollary 3 of [2] is correct when $E\|V_n\|^p \sim n^{-\alpha}$ for some $\alpha > 1$.

References

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