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On the concentration phenomenon for φ-subgaussian random elements

Rita Giuliano Antonini[a,1], Tien-Chung Hu[b,2], Andrei Volodin[c,*,3]

[a] Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo 5, 56100 Pisa, Italy
[b] Department of Mathematics, National Tsing Hua University, Hsinchu 30043, Taiwan, Republic of China
[c] Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada S4S 0A2

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Abstract

We study the deviation probability $P\{|\|\mathbf{X}\| - E\|\mathbf{X}\|| > t\}$ where $\mathbf{X}$ is a φ-subgaussian random element taking values in the Hilbert space l_2 and $\varphi(x)$ is an N-function. It is shown that the order of this deviation is $\exp\{-\varphi^*(Ct)\}$, where C depends on the sum of φ-subgaussian standard of the coordinates of the random element $\mathbf{X}$ and $\varphi^*(x)$ is the Young–Fenchel transform of $\varphi(x)$. An application to the classically subgaussian random variables ($\varphi(x) = x^2/2$) is given.

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1. Introduction

The concentration of measure phenomenon roughly describes how a well-behaved function is almost a constant on almost all the space. We refer to the monographs Ledoux (2001), and Ledoux and Talagrand (1991, Chapter 1), where this phenomenon is discussed in detail and important examples are provided.

The purpose of this note is to further complete our understanding of the concentration phenomenon by obtaining the exponential estimate of the behaviour of the deviation probability with respect to the mean $P\{|\|\mathbf{X}\| - E\|\mathbf{X}\|| > t\}$ where $\mathbf{X}$ is a φ-subgaussian random element taking values in the Hilbert space l_2 and $\varphi(x)$ is an N-function. It is shown that the order of this deviation is $\exp\{-\varphi^*(Ct)\}$, where C depends on the φ-subgaussian standard of the coordinates of the random element $\mathbf{X}$ and $\varphi^*(x)$ is the Young–Fenchel transform of $\varphi(x)$. An application to the classically subgaussian random variables ($\varphi(x) = x^2/2$) is given.

*Corresponding author. Tel.: +1 306 585 4771; fax: +1 306 585 4020.
E-mail address: andrei@math.uregina.ca (A. Volodin).
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2. Definitions and a few technical lemmas

In this section we present definitions and a few technical results that we will use in the proof of the main result of the paper.

A continuous even convex function $\varphi(x),\ x\in\mathbf{R}$, is called an *N-function*, if

(a) $\varphi(0)=0$ and $\varphi(x)$ is monotone increasing for $x>0$;
(b) $\lim_{x\to 0}\frac{\varphi(x)}{x}=0$ and $\lim_{x\to\infty}\frac{\varphi(x)}{x}=\infty$.

In the following the notation $\varphi(x)$ always stands for an N-function. It is obvious that the function $\varphi(x)=|x|^p/p,\ p>1$ is an example of N-function.

The function $\varphi^*(x),\ x\in\mathbf{R}$, defined by $\varphi^*(x)=\sup_{y\in\mathbf{R}}(xy-\varphi(y))$ is called the *Young–Fenchel transform* of $\varphi(x)$. It is well known that $\varphi^*(x)$ is an N-function, too, and if $\varphi(x)=|x|^p/p,\ p>1$ for all x, then $\varphi^*(x)=|x|^q/q$ for all x, where $1/p+1/q=1$.

A random variable X is said to be *φ-subgaussian* if there exists a constant $a>0$ such that, for every $t\in\mathbf{R}$, we have $E\exp\{tX\}\leqslant\exp\{\varphi(at)\}$. The *$\varphi$-subgaussian standard* $\tau_\varphi(X)$ is defined as

$$\tau_\varphi(X)=\inf\{a>0: E\exp\{tX\}\leqslant\exp\{\varphi(at)\},\ t\in\mathbf{R}\}.$$

We refer to the monograph Buldygin and Kozachenko (2000) and the paper Giuliano Antonini et al. (2003) where this notion is discussed in detail and important examples are provided. In the case $\varphi(x)=x^2/2$ the notion of φ-subgaussian gives us the notion of classical *subgaussian* random variable, cf. for example in Hoffmann-Jørgensen (1994, Section 4.29).

Let $\mathbf{X}=(X_1,X_2,\ldots)$ be an l_2-valued random element defined on a probability space $(\Omega,\mathscr{F},P)$, that is, for almost all $\omega\in\Omega$ the norm

$$\|\mathbf{X}(\omega)\|=\left(\sum_{k=1}^{\infty}X_k^2(\omega)\right)^{1/2}$$

is finite.

A random element $\mathbf{X}$ taking values in l_2 is said to be *scalarly φ-subgaussian* if each coordinate $X_k,\ k\geqslant 1$, is a φ-subgaussian random variable and

$$\tau_\varphi(\mathbf{X})=\sum_{k=1}^{\infty}\tau_\varphi(X_k)<\infty.$$

Now we will present a few technical lemmas which are important for the proof of the main result.

Lemma 1. *Let X and Y be two independent identically distributed φ-subgaussian random variables, and a and b be constants. If for some $p\geqslant 1$ the function $\varphi(|x|^{1/p})$ is convex, then*

$$\tau_\varphi(aX+bY)\leqslant(|a|^p+|b|^p)^{1/p}\tau_\varphi(X).$$

Proof. According to Buldygin and Kozachenko (2000, Chapter 2, Theorem 5.2), we can give the following estimate

$$\begin{aligned}\tau_\varphi^p(aX+bY)&\leqslant\tau_\varphi^p(aX)+\tau_\varphi^p(bY)\quad\text{since } X \text{ and } Y \text{ are independent}\\ &=|a|^p\tau_\varphi^p(X)+|b|^p\tau_\varphi^p(Y)=(|a|^p+|b|^p)\tau_\varphi^p(X)\end{aligned}$$

since X and Y are identically distributed. □

The second lemma is well known. We present its proof for the sake of completeness.

Lemma 2. *Let X and Y be two independent identically distributed random variables and $\psi:\mathbf{R}\to\mathbf{R}$ be a convex function. Then*

$$E[\psi(X-EX)]\leqslant E\psi(X-Y).$$

Proof. Let $F_X(x)$ and $F_Y(y)$ be the distribution functions of X and Y, respectively. Then their joint distribution function is $F_{X,Y}(x,y) = F_X(x)F_Y(y)$.

For each x we have that $x - EX = E(x - X)$, hence by Jensen's inequality $\psi(x - EX) \leqslant E\psi(x - X)$. Integrating this inequality, using the change of variables formula and the Fubini theorem, we obtain that

$$E[\psi(X - EX)] = \int_{-\infty}^{\infty} \psi(x - EX)\,\mathrm{d}F_X(x)$$
$$\leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(x - y)\,\mathrm{d}F_Y(y)\,\mathrm{d}F_X(x) = E\psi(X - Y). \qquad \square$$

Lemma 3. *Let $\mathbf{X} = (X_1, X_2, \ldots)$ be an l_2-valued scalarly φ-subgaussian random element. Then for all $t \in \mathbf{R}$*

$$E\,\exp\{t\|\mathbf{X}\|\} \leqslant 2\,\exp\{\varphi(t\tau_\varphi(\mathbf{X}))\}.$$

Proof. First of all, note that for any $t \in \mathbf{R}$ and $k \geqslant 1$

$$E\,\exp\{t|X_k|\} \leqslant E\,\exp\{|tX_k|\} = E\,\exp\{|t|X_k\}I\{X_k \geqslant 0\} + E\,\exp\{-|t|X_k\}I\{X_k < 0\}$$
$$\leqslant E\,\exp\{|t|X_k\} + E\,\exp\{-|t|X_k\} \leqslant 2\,\exp\{\varphi(t\tau_\varphi(X_k))\}. \tag{1}$$

Next, since

$$\|\mathbf{X}\| = \left(\sum_{k=1}^{\infty} X_k^2\right)^{1/2} \leqslant \sum_{k=1}^{\infty} |X_k|$$

and from the generalized Holder inequality, for any sequence of positive numbers $\{p_k,\ k \geqslant 1\}$ such that $\sum_{k=1}^{\infty} 1/p_k = 1$, we get

$$E\,\exp\{t\|\mathbf{X}\|\} \leqslant E\,\exp\left\{t\sum_{k=1}^{\infty}|X_k|\right\}$$
$$\leqslant \prod_{k=1}^{\infty}[E(\exp\{t|X_k|\})^{p_k}]^{1/p_k} = \prod_{k=1}^{\infty}[E\,\exp\{tp_k|X_k|\}]^{1/p_k}$$
$$\leqslant \prod_{k=1}^{\infty}[2\,\exp\{\varphi(tp_k\tau_\varphi(X_k))\}]^{1/p_k} \quad \text{by (1)}$$
$$= 2\,\exp\left\{\sum_{k=1}^{\infty}\frac{\varphi(tp_k\tau_\varphi(X_k))}{p_k}\right\}.$$

Taking $p_k = (\sum_{i=1}^{\infty}\tau_\varphi(X_i))/\tau_\varphi(X_k)$ we obtain the result. $\square$

The next lemma is only a slight modification of Buldygin and Kozachenko (2000, Chapter 2, Lemma 4.3). We again give a proof for the sake of completeness.

Lemma 4. *Let X be a random variable such that for each $t \in \mathbf{R}$ we have that $E\,\exp\{t|X|\} \leqslant A\,\exp\{\varphi(tB)\}$, where A and B are positive constants. Then for any t*

$$P\{|X| > t\} \leqslant 2A\,\exp\left\{-\varphi^*\left(\frac{t}{B}\right)\right\}.$$

Proof. By Markov's inequality, for any $\lambda > 0$ and t

$$P\{X > t\} \leqslant \exp\{-\lambda t\}E\,\exp\{\lambda X\} \leqslant A\,\exp\{\varphi(\lambda B) - \lambda t\}.$$

Next,

$$\inf_{\lambda>0}[\varphi(\lambda B)-\lambda t] = -\sup_{\lambda>0}[\lambda t-\varphi(\lambda B)]$$
$$= -\sup_{\mu>0}\left[\frac{t}{B}\,\mu-\varphi(\mu)\right] \quad \text{where } \mu=\lambda B$$
$$= -\varphi^*\left(\frac{t}{B}\right).$$

The same argument shows that

$$P\{X<-t\}\leqslant A\,\exp\left\{-\varphi^*\left(\frac{t}{B}\right)\right\}. \qquad \square$$

3. The main result

With the preliminaries accounted for, we can now state and prove the main results of the paper.

Theorem 1. *Assume that for some* $p\geqslant 1$ *the function* $\varphi(|x|^{1/p})$ *is convex. Let* $\mathbf{X}=(X_1,X_2,\ldots)$ *be an* l_2*-valued scalarly* φ*-subgaussian random element. Then for any t*

$$P\{|\|\mathbf{X}\|-E\|\mathbf{X}\||>t\}\leqslant 4\,\exp\left\{-\varphi^*\left(\frac{2t}{\pi C_p\tau_\varphi(\mathbf{X})}\right)\right\},$$

where $C_p=\max(1,2^{1/p\,-1/2})$.

Proof. Let $\mathbf{Y}=(Y_1,Y_2,\ldots)$ be an independent copy of $\mathbf{X}$ and denote

$$\mathbf{X}(s)=\mathbf{X}\,\sin(s)+\mathbf{Y}\,\cos(s),\quad 0\leqslant s\leqslant\pi/2.$$

Then $\mathbf{X}(0)=\mathbf{Y}$ and $\mathbf{X}(\pi/2)=\mathbf{X}$. Moreover,

$$\mathbf{X}'(s)=\mathbf{X}\,\cos(s)-\mathbf{Y}\,\sin(s).$$

Note that by Lemma 1

$$\tau_\varphi(\mathbf{X}'(\mathbf{s}))\leqslant(|\cos(s)|^p+|\sin(s)|^p)^{1/p}\tau_\varphi(\mathbf{X})\leqslant C_p\tau_\varphi(\mathbf{X}). \tag{2}$$

Next, if we denote by $\langle\mathbf{x},\mathbf{y}\rangle$ the inner product of two elements $\mathbf{x},\mathbf{y}\in l_2$, then

$$\|\mathbf{X}\|-\|\mathbf{Y}\| = \int_0^{\pi/2}\frac{\mathrm{d}}{\mathrm{d}s}\,\|\mathbf{X}(s)\|\,\mathrm{d}s=\int_0^{\pi/2}\left\langle\frac{\mathbf{X}(s)}{\|\mathbf{X}(s)\|},\mathbf{X}'(s)\right\rangle\mathrm{d}s$$
$$\leqslant\int_0^{\pi/2}\frac{\pi}{2}\,\|\mathbf{X}'(s)\|\,\frac{\mathrm{d}s}{\pi/2}.$$

Fix any $t\in\mathbf{R}$ and let $\psi(u)=\exp\{tu\}$. Then ψ is convex and by Jensen inequality

$$\psi(\|\mathbf{X}\|-\|\mathbf{Y}\|)\leqslant\int_0^{\pi/2}\psi\left(\frac{\pi}{2}\,\|\mathbf{X}'(s)\|\right)\frac{\mathrm{d}s}{\pi/2}$$
$$=\frac{2}{\pi}\int_0^{\pi/2}\psi\left(\frac{\pi}{2}\,\|\mathbf{X}'(s)\|\right)\mathrm{d}s.$$

According to Lemma 2 we can write

$$E\,\exp\{t(\|\mathbf{X}\|-E\|\mathbf{X}\|)\}=E\psi(\|\mathbf{X}\|-E\|\mathbf{X}\|)$$
$$\leqslant E\psi(\|\mathbf{X}\|-\|\mathbf{Y}\|)$$
$$\leqslant\frac{2}{\pi}\int_0^{\pi/2}E\psi\left(\frac{\pi}{2}\,\|\mathbf{X}'(s)\|\right)\mathrm{d}s.$$

Hence by Lemma 3 and (2)

$$= \frac{2}{\pi}\int_0^{\pi/2} E\,\exp\left\{t\left(\frac{\pi}{2}\,\|\mathbf{X}'(s)\|\right)\right\}\mathrm{d}s \leqslant \frac{2}{\pi}\int_0^{\pi/2} 2\,\exp\left\{\varphi\left(t\,\frac{\pi}{2}\,C_p\tau_\varphi(\mathbf{X})\right)\right\}\mathrm{d}s$$
$$= 2\,\exp\left\{\varphi\left(t\,\frac{\pi}{2}\,C_p\tau_\varphi(\mathbf{X})\right)\right\}.$$

The conclusion of the theorem now follows from Lemma 4. □

Remark 1. Note that in the proof of the theorem we established the following inequality that may be of an independent interest:

$$E\,\exp\{t(\|\mathbf{X}\| - E\|\mathbf{X}\|)\} \leqslant 2\,\exp\left\{\varphi\left(t\,\frac{\pi}{2}\,C_p\tau_\varphi(\mathbf{X})\right)\right\}.$$

As a simple corollary of the theorem we can present the following application to the classically subgaussian random elements. In the case $\varphi(x) = x^2/2$, $x \in \mathbf{R}$, we will say that $\mathbf{X} = (X_1, X_2, \ldots)$ is an l_2-valued scalarly *classically subgaussian random element* if

$$\tau_{x^2/2}(\mathbf{X}) = \sum_{k=1}^{\infty} \tau_{x^2/2}(X_k) < \infty.$$

Corollary 1. *Let* $\mathbf{X} = (X_1, X_2, \ldots)$ *be an* l_2*-valued scalarly classically subgaussian random element. Then*

$$P\{|\|\mathbf{X}\| - E\|\mathbf{X}\|| > t\} \leqslant 4\,\exp\left\{-\frac{2t^2}{\pi^2\tau^2_{x^2/2}(\mathbf{X})}\right\}.$$

Proof. Note that in this case $\varphi^*(x) = x^2/2$ and we can apply theorem with $p = 2$, hence $C_p = 1$. □

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