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On a Hsu–Robbins–Erdős law of large numbers

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We extend and generalize some recent results on complete convergence for arrays of rowwise independent random variables. In the main result, no assumptions are made concerning the existence of expected values or absolute moments of the random variables. Some well-known results from the literature are obtained easily as corollaries.

The concept of complete convergence was introduced by Hsu and Robbins [2] as follows. A sequence of random variables $\{U_n, n \geq 1\}$ is said to *converge completely* to a constant C if

$$\sum_{n=1}^{\infty} P\{|U_n - C| > \epsilon\} < \infty$$

for all $\epsilon > 0$.

In view of the Borel-Cantelli lemma, this implies that $U_n \to C$ almost surely (a.s.). The converse is true if the $\{U_n, n \geq 1\}$ are independent. Hsu and Robbins [2] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdös [1] proved the converse. The Hsu–Robbins–Erdös result may be formulated as follows.

Hsu–Robbins–Erdös Theorem. *If $\{X, X_n, n \geq 1\}$ are i.i.d. random variables, then $\frac{1}{n}\sum_{k=1}^{n} X_k$ converges completely to 0 if and only if $EX = 0$ and $EX^2 < \infty$.*

This result has been generalized and extended in several directions (cf. reference in [4]). Our work is devoted to an extension of the Hsu–Robbins–Erdös theorem to general arrays of rowwise independent but not necessarily identically distributed random variables. It unifies and extends the ideas in previously obtained results. Versions of those results can be obtained from our work as simple corollaries.

Theorem. *Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise independent*

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random variables. Suppose that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{k_n} P\{|X_{nk}| > \epsilon\} < \infty \text{ for all } \epsilon > 0,$$

there exist $p \geq 1$, $J \geq 2$, and $\delta > 0$ such that

$$\sum_{n=1}^{\infty}\left(E\left[\sum_{k=1}^{k_n}|X_{nk}I\{|X_{nk}| \geq \delta\}|^2\right]^p\right)^J < \infty,$$

and

$$\sum_{k=1}^{k_n} EX_{nk}I\{|X_{nk}| \leq \delta\} \to 0 \text{ as } n \to \infty.$$

Then

$$\sum_{n=1}^{\infty} P\left\{\left|\sum_{k=1}^{k_n} X_{nk}\right| > \epsilon\right\} < \infty \text{ for all } \epsilon > 0.$$

The Theorem is generalized to Banach space setting in [4]. Corresponding convergence rates are established in [3] and [5]. This is a part of my joint work with Professor Tien-Chung Hu (Tsing Hua University, Taiwan), Professor Andrew Rosalsky (University of Florida, U.S.A.), Professor Soo Hak Sung (Pai Chai University, South Korea) and Professor Dominik Szynal (Maria Curie-Sklodowska University, Poland).

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