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ON THE WEAK LAWS WITH RANDOM INDICES FOR PARTIAL SUMS FOR ARRAYS OF RANDOM ELEMENTS IN MARTINGALE TYPE p BANACH SPACES

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Abstract. Sung et al. [13] obtained a WLLN (weak law of large numbers) for the array $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ of random variables under a Cesàro type condition, where $\{u_n \geq -\infty, n \geq 1\}$ and $\{v_n \leq +\infty, n \geq 1\}$ are two sequences of integers. In this paper, we extend the result of Sung et al. [13] to a martingale type p Banach space.

1. Introduction

The classical weak law of large numbers (WLLN) says that if $\{X_n, n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables satisfying $nP(|X_1| > n) = o(1)$, then $\sum_{i=1}^{n}(X_i - EX_1I(|X_1| \leq n))/n \to 0$ in probability as $n \to \infty$. The WLLN has been extended to the arrays of random variables or random elements (for random variables, see Hong and Lee [5], Hong and Oh [6], and Sung [12], and for random elements, see Adler et al. [1], Ahmed et al. [2], and Hong et al. [7]).

Recently, Sung et al. [13] obtained a WLLN for the array $\{X_{ni}, u_n \leq i \leq v_n, n \geq 1\}$ of a random variables under a Cesàro type condition, where $\{u_n \geq -\infty, n \geq 1\}$ and $\{v_n \leq +\infty, n \geq 1\}$ are two sequences of

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integers. In this paper, we extend the result of Sung et al. [13] to a martingale type p Banach space.

2. Preliminary definitions

Technical definitions relevant to the current work will be discussed in this section. Scalora [11] introduced the idea of the conditional expectation of a random element in a Banach space. For a random element V and sub σ-algebra $\mathcal{G}$ of $\mathcal{F}$, the conditional expectation $E(V|\mathcal{G})$ is defined analogously to that in the random variable case and enjoys similar properties. See Scalora [11] for a complete development, as well as for a development of Banach space valued martingales including martingale convergence theorems.

A real separable Banach space $\mathcal{X}$ is said to be of martingale type p $(1 \leq p \leq 2)$ if there exists a finite constant C such that for all martingales $\{S_n, n \geq 1\}$ with values in $\mathcal{X}$,

$$\sup_{n\geq 1} E||S_n||^p \leq C \sum_{n=1}^{\infty} E||S_n - S_{n-1}||^p,$$

where $S_0 \equiv 0$. It can be shown using classical methods from martingale theory that if $\mathcal{X}$ is of martingale type p, then for all $1 \leq r < \infty$ there exists a finite constant C' such that for all $\mathcal{X}$-valued martingales $\{S_n, n \geq 1\}$

$$E \sup_{n\geq 1} ||S_n||^r \leq C' E(\sum_{n=1}^{\infty} ||S_n - S_{n-1}||^p)^{r/p}.$$

Clearly every real separable Banach space is of martingale type 1 and the real line (the same as any Hilbert space) is of martingale type 2. It follows from the Hoffmann-Jϕrgensen and Pisier [4] characterization of Rademacher type p Banach spaces that if a Banach space is of martingale type p, then it is of Rademacher type p. But the notion of martingale type p is only superficially similar to that of Rademacher type p and has a geometric characterization in terms of smoothness. For proofs and more details, the reader may refer to Pisier [9, 10].

We say that a sequence $\{X_n, n \geq 1\}$ of random elements is uniformly bounded by a random variable X if there exists a constant $C > 0$ such that for all $n \geq 1$ and all $t > 0$:

$$P(||X_n|| > t) \leq CP(|X| > Ct).$$

Without loss of generality we assume that $C = 1$.

3. Main results

Throughout this section, let $\{X_{ni}, -\infty < i < \infty, n \geq 1\}$ be an array of random elements defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a real separable Banach space. Let $\{U_n, n \geq 1\}$ and $\{V_n, n \geq 1\}$, where $U_n \leq V_n$ almost surely for all $n \geq 1$, be sequences of integer valued random variables.

Let $\{k_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of positive constants such that $k_n \to \infty, b_n \to \infty$. Next, assume that $\{u_n, n \geq 1\}$ and $\{v_n, n \geq 1\}$ are two sequences of integers, $u_n \geq -\infty, v_n \leq \infty$ such that $u_n \leq v_n$ for all $n \geq 1$. Set $\mathcal{F}_{nj} = \sigma\{X_{ni}, u_n \leq i \leq j\}$ if $j \geq u_n$, and $\mathcal{F}_{nj} = \{\emptyset, \Omega\}$ if $j < u_n, n \geq 1$.

To prove our main results, we will need the following lemma.

Lemma 1. *Assume that*

$$\frac{k_n}{b_n^p} \to 0 \text{ for some } p > 0.$$

Suppose that there exists a positive nondecreasing function g on $[0, \infty)$ satisfying

$$\lim_{a \to 0} g(a) = 0, \quad \sum_{j=1}^{\infty} g^p(1/j) < \infty,$$

and

$$\frac{k_n}{b_n^p} \sum_{j=1}^{k_n - 1} \frac{g^p(j+1) - g^p(j)}{j} = O(1).$$

Moreover, let

$$\sup_{a>0} \sup_{n \geq 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} aP(||X_{ni}|| > g(a)) < \infty$$

and

$$\lim_{a \to \infty} \sup_{n \geq 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} aP(||X_{ni}|| > g(a)) = 0.$$

Then

$$\sum_{i=u_n}^{v_n} E||X_{ni}||^p I(||X_{ni}|| \leq g(k_n)) = o(b_n^p).$$

Proof. The proof is same as that of Sung et al. [13] except that p and $||X_{ni}||$ are used instead of β and $|X_{ni}|$, respectively. □

Now we state and prove one of our main results.

THEOREM 1. *Let $0 < p \le 2$. Assume that*

$$P(U_n < u_n) = o(1) \text{ and } P(V_n > v_n) = o(1) \text{ as } n \to \infty.$$

When $1 \le p \le 2$, we assume further that the underlying Banach space is of martingale type p. Under the same conditions of Lemma 1,

$$\sum_{i=U_n}^{V_n} (X_{ni} - c_{ni})/b_n \to 0 \text{ in probability,}$$

where $c_{ni} = 0$ if $0 < p \le 1$ and $c_{ni} = E(X_{ni} I(||X_{ni}|| \le g(k_n))|\mathcal{F}_{n,i-1})$ if $1 < p \le 2$.

Proof. Let $X'_{ni} = X_{ni} I(||X_{ni}|| \le g(k_n))$ for $-\infty < i < \infty, n \ge 1$. Then

$$\begin{aligned}
&P(||\sum_{i=U_n}^{V_n} X_{ni}/b_n - \sum_{i=U_n}^{V_n} X'_{ni}/b_n|| > \epsilon) \\
&\le P(U_n < u_n) + P(V_n > v_n) + P(\cup_{i=u_n}^{v_n} (X_{ni} \ne X'_{ni})) \\
&= o(1) + P(\cup_{i=u_n}^{v_n} ||X_{ni}|| > g(k_n)) \\
&\le o(1) + \sum_{i=u_n}^{v_n} P(||X_{ni}|| > g(k_n)) \\
&= o(1) + k_n^{-1} \sum_{i=u_n}^{v_n} k_n P(||X_{ni}|| > g(k_n)),
\end{aligned}$$

so that $\sum_{i=U_n}^{V_n} X_{ni}/b_n - \sum_{i=U_n}^{V_n} X'_{ni}/b_n \to 0$ in probability. Thus, to prove the theorem it is enough to show that

$$\sum_{i=U_n}^{V_n} (X'_{ni} - c_{ni})/b_n \to 0 \text{ in probability.}$$

For $n \ge 1$ and any integers $j < m$ denote

$$B^n_{j,m} = \{||\sum_{i=j}^{m} (X'_{ni} - c_{ni})|| > b_n \epsilon\}$$

and $D_n = \cup_{u_n \le j < m \le v_n} B^n_{j,m}$. Then

$$\begin{aligned}
P(B^n_{U_n,V_n}) &\le P(B^n_{U_n,V_n}, U_n \ge u_n, V_n \le v_n) + P(U_n < u_n) + P(V_n > v_n) \\
&\le P(D_n) + o(1),
\end{aligned}$$

and hence it is sufficient to show that $P(D_n) = o(1)$.

First, we consider the case of $0 < p \le 1$. Since $c_{ni} = 0$, it follows by the Markov's inequality and Lemma 1 that

$$
\begin{aligned}
P(D_n) &= P(\max_{u_n \le j < m \le v_n} ||\sum_{i=j}^{m}(X'_{ni} - c_{ni})|| > b_n\epsilon) \\
&\le \frac{1}{\epsilon^p b_n^p} E \max_{u_n \le j < m \le v_n} ||\sum_{i=j}^{m}(X'_{ni} - c_{ni})||^p \\
&\le \sum_{i=u_n}^{v_n} E||X'_{ni}||^p/(\epsilon^p b_n^p) \to 0.
\end{aligned}
$$

Now we consider the case of $1 < p \le 2$. In this case, $X'_{ni} - c_{ni}, u_n \le i \le v_n$, form a martingale difference sequence. Since the underlying Banach space is of martingale type p,

$$
\begin{aligned}
& P(D_n) = P(\max_{u_n \le j < m \le v_n} ||\sum_{i=j}^{m}(X'_{ni} - c_{ni})|| > b_n\epsilon) \\
\le \;& \frac{1}{\epsilon^p b_n^p} E \max_{u_n \le j < m \le v_n} ||\sum_{i=j}^{m}(X'_{ni} - c_{ni})||^p \quad \text{(by Markov's inequality)} \\
= \;& \frac{1}{\epsilon^p b_n^p} E \max_{u_n \le j < m \le v_n} ||\sum_{i=u_n}^{m}(X'_{ni} - c_{ni}) - \sum_{i=u_n}^{j-1}(X'_{ni} - c_{ni})||^p \\
\le \;& \frac{2^{p-1}}{\epsilon^p b_n^p} E \max_{u_n \le j < m \le v_n} ||\sum_{i=u_n}^{m}(X'_{ni} - c_{ni})||^p + ||\sum_{i=u_n}^{j-1}(X'_{ni} - c_{ni})||^p \\
& \text{(by } c_r\text{-inequality)} \\
\le \;& \frac{2^p}{\epsilon^p b_n^p} E \max_{u_n \le m \le v_n} ||\sum_{i=u_n}^{m}(X'_{ni} - c_{ni})||^p \\
\le \;& \frac{C_p 2^p}{\epsilon^p b_n^p} \sum_{i=u_n}^{v_n} E||X'_{ni} - c_{ni}||^p \\
\le \;& \frac{C_p 2^{2p-1}}{\epsilon^p b_n^p} \sum_{i=u_n}^{v_n} E||X'_{ni}||^p + E||c_{ni}||^p \quad \text{(by } c_r\text{-inequality)} \\
\le \;& \frac{C_p 2^{2p}}{\epsilon^p b_n^p} \sum_{i=u_n}^{v_n} E||X'_{ni}||^p \to 0 \quad \text{(by Jensen's inequality and Lemma 1)},
\end{aligned}
$$

where C_p is a constant depending only on p. □

Corollary 1. *Assume that the underlying Banach space is of martingale type $p, 1 \leq p \leq 2$ and $0 < r < p$. Suppose that*

$$\sup_{a>0} \sup_{n\geq 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} aP(||X_{ni}||^r > a) < \infty$$

and

$$\lim_{a\to\infty} \sup_{n\geq 1} \frac{1}{k_n} \sum_{i=u_n}^{u_n} aP(||X_{ni}||^r > a) = 0.$$

Moreover, assume that

$$P(U_n < u_n) = o(1) \text{ and } P(V_n > v_n) = o(1) \text{ as } n \to \infty.$$

Then

$$\sum_{i=U_n}^{V_n} (X_{ni} - c_{ni})/k_n^{1/r} \to 0 \text{ in probability,}$$

where $c_{ni} = 0$ if $0 < r < 1$ and $c_{ni} = E(X_{ni} I(||X_{ni}||^r \leq k_n)|\mathcal{F}_{n,i-1})$ if $1 \leq r < 2$.

Proof. The proof is similar to that of Corollary 1 of Sung et al. [13] and is omitted. □

Theorem 2. *Let $\{X_n, n \geq 1\}$ be a sequence of random elements taking values in a real separable Banach space of martingale type $p\,(1 \leq p \leq 2)$, which is uniformly bounded by a random variable X such that $aP(|X|^r > a) \to 0$ as $a \to \infty$ for some $0 < r < p$. Let $\{|a_{ni}|^r, 1 \leq i < \infty, n \geq 1\}$ be a Toeplitz array of constants, i.e.,*

$$\lim_{n\to\infty} a_{ni} = 0 \text{ for every } i$$

and

$$\sup_{n\geq 1} \sum_{i=1}^{\infty} |a_{ni}|^r < C \quad \text{for some constant } C > 0.$$

If $\sup_{i\geq 1} |a_{ni}| \to 0$ as $n \to \infty$, then

$$\sum_{i=1}^{\infty} a_{ni}(X_i - c_{ni}) \to 0 \text{ in probability,}$$

where $c_{ni} = 0$ if $0 < r < 1$ and $c_{ni} = E(X_i I(||a_{ni}X_i||^r \leq 1)|\mathcal{F}_{i-1})$ if $1 \leq r < 2$ ($\mathcal{F}_n = \sigma\{X_i, 1 \leq i \leq n\}$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$).

Proof. The proof is similar to that of Theorem 3 of Sung et al. [13] and is omitted. □

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