ON THE WEAK LAWS WITH RANDOM INDICES FOR PARTIAL SUMS FOR ARRAYS OF RANDOM ELEMENTS IN MARTINGALE TYPE $p$ BANACH SPACES

Soo Hak Sung, Tien-Chung Hu, and Andrei I. Volodin

Abstract. Sung et al. [13] obtained a WLLN (weak law of large numbers) for the array \( \{X_{ni}, u_n \leq i \leq v_n, n \geq 1\} \) of random variables under a Cesàro type condition, where \( \{u_n \geq -\infty, n \geq 1\} \) and \( \{v_n \leq +\infty, n \geq 1\} \) are two sequences of integers. In this paper, we extend the result of Sung et al. [13] to a martingale type $p$ Banach space.

1. Introduction

The classical weak law of large numbers (WLLN) says that if \( \{X_n, n \geq 1\} \) is a sequence of independent and identically distributed (i.i.d.) random variables satisfying \( nP(|X_1| > n) = o(1) \), then \( \sum_{i=1}^{n}(X_i - EX_1I(\{|X_1| \leq n\}))/n \to 0 \) in probability as \( n \to \infty \). The WLLN has been extended to the arrays of random variables or random elements (for random variables, see Hong and Lee [5], Hong and Oh [6], and Sung [12], and for random elements, see Adler et al. [1], Ahmed et al. [2], and Hong et al. [7]).

Recently, Sung et al. [13] obtained a WLLN for the array \( \{X_{ni}, u_n \leq i \leq v_n, n \geq 1\} \) of a random variables under a Cesàro type condition, where \( \{u_n \geq -\infty, n \geq 1\} \) and \( \{v_n \leq +\infty, n \geq 1\} \) are two sequences of
So in this paper, we extend the result of Sung et al. [13] to a martingale type $p$ Banach space.

2. Preliminary definitions

Technical definitions relevant to the current work will be discussed in this section. Scalora [11] introduced the idea of the conditional expectation of a random element in a Banach space. For a random element $V$ and sub $\sigma$-algebra $G$ of $\mathcal{F}$, the conditional expectation $E(V|G)$ is defined analogously to that in the random variable case and enjoys similar properties. See Scalora [11] for a complete development, as well as for a development of Banach space valued martingales including martingale convergence theorems.

A real separable Banach space $X$ is said to be of martingale type $p$ $(1 \leq p \leq 2)$ if there exists a finite constant $C$ such that for all martingales $\{S_n, n \geq 1\}$ with values in $X$,

$$\sup_{n \geq 1} E||S_n||^p \leq C \sum_{n=1}^{\infty} E||S_n - S_{n-1}||^p,$$

where $S_0 \equiv 0$. It can be shown using classical methods from martingale theory that if $X$ is of martingale type $p$, then for all $1 \leq r < \infty$ there exists a finite constant $C'$ such that for all $X$-valued martingales $\{S_n, n \geq 1\}$

$$E \sup_{n \geq 1} ||S_n||^r \leq C' E(\sum_{n=1}^{\infty} ||S_n - S_{n-1}||^p)^{r/p}.$$

Clearly every real separable Banach space is of martingale type 1 and the real line (the same as any Hilbert space) is of martingale type 2. It follows from the Hoffmann-Jørgensen and Pisier [4] characterization of Rademacher type $p$ Banach spaces that if a Banach space is of martingale type $p$, then it is of Rademacher type $p$. But the notion of martingale type $p$ is only superficially similar to that of Rademacher type $p$ and has a geometric characterization in terms of smoothness. For proofs and more details, the reader may refer to Pisier [9, 10].

We say that a sequence $\{X_n, n \geq 1\}$ of random elements is uniformly bounded by a random variable $X$ if there exists a constant $C > 0$ such that for all $n \geq 1$ and all $t > 0$:

$$P(||X_n|| > t) \leq CP(|X| > Ct).$$

Without loss of generality we assume that $C = 1$. 
3. Main results

Throughout this section, let \( \{X_{ni}, -\infty < i < \infty, n \geq 1\} \) be an array of random elements defined on a probability space \((\Omega, \mathcal{F}, P)\) and taking values in a real separable Banach space. Let \( \{U_n, n \geq 1\} \) and \( \{V_n, n \geq 1\} \), where \( U_n \leq V_n \) almost surely for all \( n \geq 1 \), be sequences of integer valued random variables.

Let \( \{k_n, n \geq 1\} \) and \( \{b_n, n \geq 1\} \) be sequences of positive constants such that \( k_n \rightarrow \infty, b_n \rightarrow \infty \). Next, assume that \( \{u_n, n \geq 1\} \) and \( \{v_n, n \geq 1\} \) are two sequences of integers, \( u_n \geq -\infty, v_n \leq \infty \) such that \( u_n \leq v_n \) for all \( n \geq 1 \). Set \( \mathcal{F}_{nj} = \sigma\{X_{ni}, u_n \leq i \leq j\} \) if \( j \geq u_n \), and \( \mathcal{F}_{nj} = \{\emptyset, \Omega\} \) if \( j < u_n, n \geq 1 \).

To prove our main results, we will need the following lemma.

**Lemma 1.** Assume that

\[
\frac{k_n}{b_n^p} \rightarrow 0 \text{ for some } p > 0.
\]

Suppose that there exists a positive nondecreasing function \( g \) on \([0, \infty)\) satisfying

\[
\lim_{a \to 0} g(a) = 0, \quad \sum_{j=1}^{\infty} g^p(1/j) < \infty,
\]

and

\[
\frac{k_n \sum_{j=1}^{k_n-1} g^p(j + 1) - g^p(j)}{j} = O(1).
\]

Moreover, let

\[
\sup_{a > 0} \sup_{n \geq 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} aP(||X_{ni}|| > g(a)) < \infty
\]

and

\[
\lim_{a \to \infty} \sup_{n \geq 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} aP(||X_{ni}|| > g(a)) = 0.
\]

Then

\[
\sum_{i=u_n}^{v_n} E||X_{ni}||^p I(||X_{ni}|| \leq g(k_n)) = o(b_n^p).
\]

**Proof.** The proof is same as that of Sung et al. [13] except that \( p \) and \( ||X_{ni}|| \) are used instead of \( \beta \) and \( |X_{ni}| \), respectively.

Now we state and prove one of our main results.
Theorem 1. Let $0 < p \leq 2$. Assume that

$$P(U_n < u_n) = o(1) \text{ and } P(V_n > v_n) = o(1) \text{ as } n \to \infty.$$ 

When $1 \leq p \leq 2$, we assume further that the underlying Banach space is of martingale type $p$. Under the same conditions of Lemma 1,

$$\sum_{i=U_n}^{V_n} (X_{ni} - c_{ni})/b_n \to 0 \text{ in probability},$$

where $c_{ni} = 0$ if $0 < p \leq 1$ and $c_{ni} = E(X_{ni}I(||X_{ni}|| \leq g(k_n))|\mathcal{F}_{n,i-1})$ if $1 < p \leq 2$.

Proof. Let $X'_{ni} = X_{ni}I(||X_{ni}|| \leq g(k_n))$ for $-\infty < i < \infty, n \geq 1$. Then

$$P(||\sum_{i=U_n}^{V_n} X_{ni}/b_n - \sum_{i=U_n}^{V_n} X'_{ni}/b_n|| > \epsilon)$$

$$\leq P(U_n < u_n) + P(V_n > v_n) + P(\bigcup_{i=U_n}^{V_n} (X_{ni} \neq X'_{ni}))$$

$$= o(1) + P(\bigcup_{i=U_n}^{V_n} ||X_{ni}|| > g(k_n))$$

$$\leq o(1) + \sum_{i=u_n}^{v_n} P(||X_{ni}|| > g(k_n))$$

$$= o(1) + k_n^{-1} \sum_{i=u_n}^{v_n} k_n P(||X_{ni}|| > g(k_n)),$$

so that $\sum_{i=U_n}^{V_n} X_{ni}/b_n - \sum_{i=U_n}^{V_n} X'_{ni}/b_n \to 0$ in probability. Thus, to prove the theorem it is enough to show that

$$\sum_{i=U_n}^{V_n} (X'_{ni} - c_{ni})/b_n \to 0 \text{ in probability}.$$ 

For $n \geq 1$ and any integers $j < m$ denote

$$B_{j,m}^n = \{||\sum_{i=j}^{m} (X'_{ni} - c_{ni})|| > b_n \epsilon\}$$

and $D_n = \cup_{u_n \leq j < m \leq v_n} B_{j,m}^n$. Then

$$P(B_{U_n,V_n}^n) \leq P(B_{U_n,V_n}^n, U_n \geq u_n, V_n \leq v_n) + P(U_n < u_n) + P(V_n > v_n)$$

$$\leq P(D_n) + o(1),$$

and hence it is sufficient to show that $P(D_n) = o(1)$. 


First, we consider the case of $0 < p \leq 1$. Since $c_{ni} = 0$, it follows by the Markov’s inequality and Lemma 1 that

$$P(D_n) = P\left( \max_{u_n \leq j < m \leq v_n} \left\| \sum_{i=j}^{m} (X'_{ni} - c_{ni}) \right\| > b_n \epsilon \right)$$

$$\leq \frac{1}{\epsilon p \beta_n^p} E \max_{u_n \leq j < m \leq v_n} \left\| \sum_{i=j}^{m} (X'_{ni} - c_{ni}) \right\|^p$$

$$\leq \sum_{i=u_n}^{v_n} E \left\| X'_{ni} \right\|^p / (\epsilon p \beta_n^p) \to 0.$$

Now we consider the case of $1 < p \leq 2$. In this case, $X'_{ni} - c_{ni}, u_n \leq i \leq v_n$, form a martingale difference sequence. Since the underlying Banach space is of martingale type $p$,

$$P(D_n) = P\left( \max_{u_n \leq j < m \leq v_n} \left\| \sum_{i=j}^{m} (X'_{ni} - c_{ni}) \right\| > b_n \epsilon \right)$$

$$\leq \frac{1}{\epsilon p \beta_n^p} E \max_{u_n \leq j < m \leq v_n} \left\| \sum_{i=j}^{m} (X'_{ni} - c_{ni}) \right\|^p \quad \text{(by Markov’s inequality)}$$

$$= \frac{1}{\epsilon p \beta_n^p} E \max_{u_n \leq j < m \leq v_n} \left\| \sum_{i=j}^{m} (X'_{ni} - c_{ni}) - \sum_{i=u_n}^{j-1} (X'_{ni} - c_{ni}) \right\|^p$$

$$\leq \frac{2^{p-1}}{\epsilon p \beta_n^p} E \max_{u_n \leq j < m \leq v_n} \left\| \sum_{i=j}^{m} (X'_{ni} - c_{ni}) \right\|^p + \left\| \sum_{i=u_n}^{j-1} (X'_{ni} - c_{ni}) \right\|^p$$

(by $c_r$-inequality)

$$\leq \frac{2^p}{\epsilon p \beta_n^p} E \max_{u_n \leq m \leq v_n} \left\| \sum_{i=u_n}^{m} (X'_{ni} - c_{ni}) \right\|^p$$

$$\leq \frac{C_p 2^p}{\epsilon p \beta_n^p} \sum_{i=u_n}^{v_n} E \|X'_{ni} - c_{ni}\|^p$$

$$\leq \frac{C_p 2^{p-1}}{\epsilon p \beta_n^p} \sum_{i=u_n}^{v_n} E \|X'_{ni}\|^p + E |c_{ni}|^p \quad \text{(by $c_r$-inequality)}$$

$$\leq \frac{C_p 2^p}{\epsilon p \beta_n^p} \sum_{i=u_n}^{v_n} E \|X'_{ni}\|^p \to 0 \quad \text{(by Jensen’s inequality and Lemma 1),}$$

where $C_p$ is a constant depending only on $p$. \qed
Corollary 1. Assume that the underlying Banach space is of martingale type \( p, 1 \leq p \leq 2 \) and \( 0 < r < p \). Suppose that
\[
\sup_{a > 0} \sup_{n \geq 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} a P(\|X_{ni}\|^r > a) < \infty
\]
and
\[
\lim_{a \to \infty} \sup_{n \geq 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} a P(\|X_{ni}\|^r > a) = 0.
\]
Moreover, assume that
\[
P(U_n < u_n) = o(1) \text{ and } P(V_n > v_n) = o(1) \text{ as } n \to \infty.
\]
Then
\[
\frac{1}{k_n} \sum_{i=U_n}^{V_n} (X_{ni} - c_{ni})/k_n^{1/r} \to 0 \text{ in probability,}
\]
where \( c_{ni} = 0 \) if \( 0 < r < 1 \) and \( c_{ni} = E(X_{ni}I(\|X_{ni}\|^r \leq k_n)|F_{n,i-1}) \) if \( 1 \leq r < 2 \).

Proof. The proof is similar to that of Corollary 1 of Sung et al. [13] and is omitted.

Theorem 2. Let \( \{X_n, n \geq 1\} \) be a sequence of random elements taking values in a real separable Banach space of martingale type \( p (1 \leq p \leq 2) \), which is uniformly bounded by a random variable \( X \) such that \( a P(\|X\|^r > a) \to 0 \) as \( a \to \infty \) for some \( 0 < r < p \). Let \( \{|a_{ni}|^r, 1 \leq i < \infty, n \geq 1\} \) be a Toeplitz array of constants, i.e.,
\[
\lim_{n \to \infty} a_{ni} = 0 \text{ for every } i
\]
and
\[
\sup_{n \geq 1} \sum_{i=1}^{\infty} |a_{ni}|^r < C \text{ for some constant } C > 0.
\]
If \( \sup_{i \geq 1} |a_{ni}| \to 0 \) as \( n \to \infty \), then
\[
\sum_{i=1}^{\infty} a_{ni}(X_i - c_{ni}) \to 0 \text{ in probability,}
\]
where \( c_{ni} = 0 \) if \( 0 < r < 1 \) and \( c_{ni} = E(X_{ni}I(|a_{ni}X_i|^r \leq 1)|F_{i-1}) \) if \( 1 \leq r < 2 \) \( (F_n = \sigma\{X_i, 1 \leq i \leq n\} \text{ and } F_0 = \{\emptyset, \Omega\}) \).

Proof. The proof is similar to that of Theorem 3 of Sung et al. [13] and is omitted.
On the weak laws with random indices

References


Soo Hak Sung, Department of Applied Mathematics, Pai Chi University, Taejon 302-735, Korea
E-mail: sungsh@pcu.ac.kr

Tien-Chung Hu, Department of Mathematics, National Tsing Hua University, Hsinchu 300, Taiwan
E-mail: tchu@math.nthu.edu.tw

Andrei I. Volodin, Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, S4S 0A2, Canada
E-mail: andrei@math.uregina.ca