ON COMPLETE CONVERGENCE FOR ARRAYS OF ROWWISE INDEPENDENT RANDOM ELEMENTS IN BANACH SPACES

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ABSTRACT

We extend and generalize some recent results on complete convergence (cf. Hu, Moricz, and Taylor [14], Gut [11], Wang, Bhaskara Rao, and Yang [26], Kuczmaszewska and Szynal [17], and Sung [23]) for arrays of rowwise independent Banach space valued random elements. In the main result, no assumptions are made concerning the existence of expected values or absolute moments of the random elements and no assumptions are made concerning the geometry of the underlying Banach space. Some well-known results from the literature are obtained easily as corollaries. The corresponding convergence rates are also established.

1 Introduction

The concept of complete convergence was introduced by Hsu and Robbins [13] as follows. A sequence of random variables $\{U_n, n \geq 1\}$ is said to *converge completely* to a constant C if $\sum_{n=1}^{\infty} P\{|U_n - C| > \epsilon\} < \infty$ for all $\epsilon > 0$. In view of the Borel-Cantelli lemma, this implies that $U_n \to C$ almost surely (a.s.). The converse is true if the $\{U_n, n \geq 1\}$ are independent. Hsu and Robbins [13] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdös [8] proved the converse. The Hsu-Robbins-Erdös result may be formulated as follows.

Theorem 1.1. (Hsu and Robbins [13], Erdös [8]) If $\{X, X_n, n \geq 1\}$ are i.i.d. random variables, then $\frac{1}{n}\sum_{k=1}^{n} X_k$ converges completely to 0 if and only if $EX = 0$ and $EX^2 < \infty$.

This result has been generalized and extended in several directions (cf. Rohatgi [21], Hu, Moricz, and Taylor [14], Gut [11], Wang, Bhaskara Rao, and Yang [26], Kuczmaszewska and Szynal [17], and Sung [23] among others). Some of these articles concern a Banach space setting. A sequence of Banach space valued random elements is said to *converge completely* to the 0 element of the Banach space if the corresponding sequence of norms converges completely to 0.

In Pruitt [20], weighted sums of i.i.d. random variables were considered permitting a more general normalization than that in Theorem 1.1. Relying heavily on the techniques of Pruitt [20], Rohatgi [21] generalized Pruitt's result to the case of independent stochastically dominated random variables.

Theorem 1.2. (Pruitt [20], Rohatgi [21]) Let $\{X_n, n \geq 1\}$ be a sequence of independent random variables which are stochastically dominated by a random variable X in the sense that there exists a constant $D > 0$ such that $P\{|X_n| > x\} \leq DP\{D|X| > x\}$ for all $x \geq 0$ and $n \geq 1$. Let $\{a_{nk}, k \geq 1, n \geq 1\}$ be a Toeplitz array such that $\sup_{k\geq 1} |a_{nk}| = \mathcal{O}(n^{-\gamma})$ for some $\gamma > 0$. If $E|X|^{1+\gamma^{-1}} < \infty$, then $\sum_{k=1}^{\infty} a_{nk}X_k$ converges completely to 0.

Hu, Moricz, and Taylor [14] generalized Theorem 1.1 for triangular arrays of rowwise independent (but not necessarily identically distributed) random variables and obtained a complete convergence theorem with a Marcinkiewicz-Zygmund type normalization.

Theorem 1.3. (Hu, Moricz, and Taylor [14]) Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be a triangular array of rowwise independent mean 0 random variables which are stochastically dominated in the Cesàro sense by a random variables X, that is, there exists a constant $D > 0$ such that $\frac{1}{n}\sum_{k=1}^{n} P\{|X_{nk}| > x\} \leq DP\{D|X| > x\}$ for all $x \geq 0$ and $n \geq 1$. If $E|X|^{2t} < \infty$ where $1 \leq t < 2$, then $n^{-1/t}\sum_{k=1}^{n} X_{nk}$ converges completely to 0.

Gut [11] simplified the proof of Theorem 1.3 and considered more general arrays. Taylor and Hu [25] considered arrays of random elements taking values in a Banach space of Rademacher type p. Wang, Bhaskara Rao, and Yang [26] obtained the corresponding convergence rate for one of the Taylor and Hu [25] results as a corollary of the following theorem.

Theorem 1.4. (Wang, Bhaskara Rao, and Yang [26]) Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be a triangular array of rowwise independent random elements which are stochastically dominated by a random variable X. If $E|X|^{rt} < \infty$ where $r \geq 1, 1 \leq t < 2$, $rt > 1$ and

$$\max_{1\leq i\leq n} P\left\{\frac{||\sum_{k=1}^{i} X_{nk}||}{n^{1/t}} > \epsilon\right\} \to 0 \text{ as } n \to \infty \text{ for all } \epsilon > 0, \tag{1.1}$$

then

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{\left|\left|\sum_{k=1}^{n} x_{nk}\right|\right| > \epsilon n^{1/t}\right\} < \infty \text{ for all } \epsilon > 0.$$

In Kuczmaszewska and Szynal [17], the same normalization as in Theorem 1.1 was considered, but no assumption concerning identical distributions of the random elements was made and no stochastic domination conditions were imposed. Also the case of Banach space valued random elements was considered.

Sung [23] generalized the work of Stout [22, p. 226] to the case of Banach space valued random elements by establishing the following result.

Theorem 1.5. (Sung [23]) Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be a triangular array of rowwise independent random elements which are stochastically dominated by a random variable X. Let $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ be a triangular array of real numbers satisfying $\sum_{k=1}^{n} a_{nk}^2 = o(1/\log\, n)$ and $\max_{1\leq k\leq n}|a_{nk}| = \mathcal{O}(n^{-1/p})$ for some $p \geq 1$. If $\sum_{k=1}^{n} a_{nk}X_{nk} \xrightarrow{P} 0$ and $E|X|^{2p} < \infty$, then $\sum_{k=1}^{n} a_{nk}X_{nk}$ converges completely to 0.

Sung [23] also improved a result of Bozorgnia, Patterson, and Taylor [6] by establishing the following theorem with a Marcinkiewicz-Zygmund type normalization.

Theorem 1.6. (Sung [23]) Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be a triangular array of rowwise independent random elements such that

$$\max_{1\leq k\leq n} E||X_{nk}||^{\nu} = \mathcal{O}(n^{\alpha}) \tag{1.2}$$

where $\frac{\nu}{p} - \alpha > \max\{\frac{\nu}{2}, 2\}, \nu \geq 1$, and $\alpha \geq 0$. If $\sum_{k=1}^{n} X_{nk}/n^{1/p} \xrightarrow{P} 0$, then $\sum_{k=1}^{n} X_{nk}/n^{1/p}$ converges completely to 0.

Our work unifies and extends the ideas in the previously cited results. Versions of those results can be obtained from our work (except for the necessity half of Theorem 1.1) and some of them will be presented as corollaries. The current work is devoted to an extension of the Hsu-Robbins [13] theorem to general arrays of rowwise independent but not necessarily identically distributed Banach space valued random elements.

In Section 2, we recall some definitions and we present some inequalities and lemmas which will be used in the proofs of our results. In Section 3, we obtain complete convergence for row sums with the corresponding rates of convergence. In the main result (Theorem 3.1), no assumptions are made concerning the existence of expected values or absolute moments of the random elements and no assumptions are made concerning the geometry of the underlying Banach space. Finally, in Section 4, we present some well-known results as corollaries of our results.

The pertinent devices employed in the proof of Theorem 3.1 are:

(i) an iterated version of the Hoffmann-Jϕrgensen [12] inequality due to Jain [16],

(ii) a Banach space version of the classical Marcinkiewicz-Zygmund inequality due to de Acosta [1],

(iii) a modified version of a result of Kuelbs and Zinn [18] concerning the relationship between convergence in probability and mean convergence for sums of independent bounded random variables.

Jain [16] had applied (i) to obtain a complete convergence theorem for a sequence of i.i.d. Banach space valued random elements. On the other hand, for the random variable case, Gut [10, 11] used (i) and the classical Marcinkiewicz-Zygmund inequality to establish complete convergence results for rowwise independent arrays.

2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let B be a real separable Banach space with norm $||\cdot||$. A *random element* is defined to be an $\mathcal{F}$-measurable mapping of Ω into B with the Borel σ-algebra (that is, the σ-algebra generated by the open sets determined by $||\cdot||$). The concept of independent random elements is a direct extension of the concept of independent random variables. A detailed account of basic properties of random elements in real separable Banach spaces can be found in Taylor [24].

Let $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise independent, but not necessarily identically distributed, random elements taking values in B. In general the case $k_n = \infty$ is not being precluded. Rowwise independence means that the random elements within each row are independent but that no independence is assumed between rows.

Before proceeding further, we will recall some definitions for an array of random elements. An array $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ is said to be *symmetric* if X_{nk} is symmetrically distributed for all $1 \le k \le k_n$ and all $n \ge 1$. An array $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ is said to be *infinitesimal* if for all $\epsilon > 0$, $\lim_{n\to\infty} \sup_{1\le k\le k_n} P\{||X_{nk}|| > \epsilon\} = 0$. For a random element Y, its *symmetrization* will be denoted by $Y^s = Y - \tilde{Y}$ where $\tilde{Y}$ is an independent copy of Y.

The concepts of stochastic domination and stochastic domination in the Cesàro sense by a random variable also have direct extensions for arrays of random elements as follows. An array $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ of random elements is said to be:

(i) *stochastically dominated* by a random variable X if there exists a constant $D > 0$ such that $P\{||X_{nk}|| > x\} \le DP\{D|X| > x\}$ for all $x \ge 0$ and for all $1 \le k \le k_n$ and $n \ge 1$,

(ii) *stochastically dominated in the Cesàro sense* by a random variable X if $k_n < \infty$ for all $n \geq 1$ and there exists a constant $D > 0$ such that $\sum_{k=1}^{k_n} P\{||X_{nk}|| > x\} \leq Dk_n P\{D|X| > x\}$ for all $x \geq 0$ and $n \geq 1$.

A double array $\{a_{nk}, k \geq 1, n \geq 1\}$ of real numbers is said to be a *Toeplitz array* if $\lim_{n\to\infty} a_{nk} = 0$ for each $k \geq 1$ and $\sum_{k=1}^{\infty} |a_{nk}| \leq C$ for all $n \geq 1$ where C is a positive constant.

Throughout, let $S_n = \sum_{k=1}^{k_n} X_{nk}, n \geq 1$. If $k_n = \infty$ for any $n \geq 1$, we will assume that the series S_n converges a.s. if the a.s. convergence is not automatic from the hypotheses.

We are now ready to present some well-known inequalities and some lemmas which will be useful in Section 3. The first proposition is an iterated form of the Hoffmann-Jϕrgensen [12] inequality and is due to Jain [16]. When $j = 1$, C_1 and D_1 can be taken to be 1 and 4, respectively, according to the Hoffmann-Jϕrgensen [12] inequality.

Proposition 2.1. (Jain [16]) If an array $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ of rowwise independent random elements is symmetric, then for all $n \geq 1, j \geq 1$, and $t \geq 0$

$$P\{||S_n|| > 3^j t\} \leq C_j P\left\{\sup_{1\leq k\leq k_n} ||X_{nk}|| > t\right\} + D_j (P\{||S_n|| > t\})^{2^j}$$

where C_j and D_j are positive constants depending only on j.

The next inequality is a Banach space analogue of the classical Marcinkiewicz-Zygmund inequality due to de Acosta [1] (cf. also Berger [5]).

Proposition 2.2. (de Acosta [1]) Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise independent random elements. Then for every $p \geq 1/2$, there is a positive constant A_p depending only on p such that for all $n \geq 1$

$$E\Big|||S_n|| - E||S_n||\Big|^{2p} \leq A_p E\left(\sum_{k=1}^{k_n} ||X_{nk}||^2\right)^p.$$

Proposition 2.3. (Etemadi [9]) If an array $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ of rowwise independent random elements is symmetric, then for every $\epsilon > 0$ and $n \geq 1$

$$\sum_{k=1}^{k_n} P\{||X_{nk}|| > \epsilon\} \leq \frac{P\{||S_n|| > \frac{\epsilon}{8}\}}{1 - 8P\{||S_n|| > \frac{\epsilon}{8}\}}$$

provided $P\{||S_n|| > \frac{\epsilon}{8}\} \leq \frac{1}{8}$.

The first lemma is a modification of a result of Kuelbs and Zinn [18] concerning the relationship between convergence in probability and mean convergence for sums of independent bounded random variables.

Lemma 2.1. Let the array $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ of rowwise independent random elements be symmetric and suppose there exists $\delta > 0$ such that $||X_{nk}|| \leq \delta$ a.s. for all $1 \leq k \leq k_n, n \geq 1$. If $S_n \xrightarrow{P} 0$, then $E||S_n|| \to 0$ as $n \to \infty$.

Proof. Fix $\epsilon > 0$ and $A > 0$. Let the integer N be large enough so that $\sup_{n\geq N} P\{||S_n|| \geq \epsilon\} \leq 1/24$. By the Hoffmann-J$\phi$rgensen [12] inequality (Proposition 2.1 with $j = 1$), we have for $n \geq N$ that

$$\int_0^A P\{||S_n|| \geq t\}dt = 3\int_0^{A/3} P\{||S_n|| \geq 3t\}dt$$

$$\leq 3\left(4\int_0^{A/3} (P\{||S_n|| \geq t\})^2 dt + \int_0^{A/3} P\left\{\sup_{1\leq k\leq k_n} ||X_{nk}|| \geq t\right\} dt\right)$$

$$\leq 12\epsilon + 12\int_{\epsilon}^{A/3} \frac{1}{24} P\{||S_n|| \geq t\}dt + 3\int_0^{\delta} P\left\{\sup_{1\leq k\leq k_n} ||X_{nk}|| \geq t\right\} dt$$
$$\leq 12\epsilon + \frac{1}{2}\int_0^{A} P\{||S_n|| \geq t\}dt + 3\int_0^{\delta} P\left\{\sup_{1\leq k\leq k_n} ||X_{nk}|| \geq t\right\} dt.$$

Hence for $n \geq N$

$$\int_0^{A} P\{||S_n|| \geq t\}dt \leq 24\epsilon + 6\int_0^{\delta} P\left\{\sup_{1\leq k\leq k_n} ||X_{nk}|| \geq t\right\} dt. \quad (2.1)$$

Next, by Lévy's maximal inequality (cf. Araujo and Giné [2, p. 102]), (2.1) yields for $n \geq N$

$$\int_0^{A} P\{||S_n|| \geq t\}dt \leq 24\epsilon + 12\int_0^{\delta} P\{||S_n|| \geq t\}dt.$$

Letting $A \to \infty$, it follows that

$$E||S_n|| = \int_0^{\infty} P\{||S_n|| \geq t\}dt \leq 24\epsilon + 12\int_0^{\delta} P\{||S_n|| \geq t\}dt.$$

Now the last integral is $o(1)$ as $n \to \infty$ by the Lebesgue bounded convergence theorem. Thus $\limsup_{n\to\infty} E||S_n|| \leq 24\epsilon$ and the result follows since $\epsilon > 0$ is arbitrary. □

We also need the following symmetrization inequalities in Section 3.

Lemma 2.2.

(i) If an array $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ of rowwise independent random elements is infinitesimal, then for all $t > 0$ and all sufficiently large n and all $1 \leq k \leq k_n$, we have

$$P\{||X_{nk}|| > t\} \leq 2P\{||X_{nk}^s|| > t/2\}. \quad (2.2)$$

(ii) If $\{Y_n, n \geq 1\}$ is a sequence of random elements with $Y_n \xrightarrow{P} 0$, then for all $t > 0$ and sufficiently large n

$$P\{||Y_n|| > t\} \leq 2P\{||Y_n^s|| > t/2\}. \quad (2.3)$$

Proof.

(i) Let m_{nk} denote any median of the random variable $||X_{nk}||, 1 \leq k \leq k_n, n \geq 1$. Since the array is infinitesimal, we have $\sup_{1\leq k\leq k_n} m_{nk} \to 0$ as $n \to \infty$. Let the integer N be large enough so that $\sup_{n\geq N}\sup_{1\leq k\leq k_n} m_{nk} \leq t/2$. Then for $n \geq N$

$$P\{||X_{nk}|| > t\} \leq P\{||X_{nk}|| - m_{nk} > t/2\}$$
$$\leq P\left\{\left|||X_{nk}|| - m_{nk}\right| > t/2\right\}$$
$$\leq 2P\left\{\left|||X_{nk}|| - ||\tilde{X}_{nk}||\right| > t/2\right\} \text{ (cf. Loève [19, p. 257])}$$
$$\leq 2P\{||X_{nk}^s|| > t/2\}.$$

(ii) Use the same argument as in part (i). □

The following lemma will also be used in Section 3.

Lemma 2.3. Let $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise independent random elements such that for some $\delta > 0$

$$\sum_{k=1}^{k_n} P\{||X_{nk}|| > \delta\} \to 0 \text{ as } n \to \infty. \tag{2.4}$$

Then $S_n \xrightarrow{P} 0$ if and only if $S'_n = \sum_{k=1}^{k_n} X_{nk} I\{||X_{nk}|| \le \delta\} \xrightarrow{P} 0$.

Proof. Let $S''_n = \sum_{k=1}^{k_n} X_{nk} I\{||X_{nk}|| > \delta\}, n \ge 1$. Observe that for arbitrary $\epsilon > 0$

$$\begin{aligned} P\{||S_n|| \ge \epsilon\} &\le P\{||S'_n|| \ge \epsilon/2\} + P\{||S''_n|| \ge \epsilon/2\} \\ &\le P\{||S'_n|| \ge \epsilon/2\} + \sum_{k=1}^{k_n} P\{||X_{nk}|| > \delta\} \end{aligned}$$

and the sufficiency half follows from (2.4).

To prove the necessity half, observe that for arbitrary $\epsilon > 0$

$$\begin{aligned} P\{||S_n|| \ge \epsilon\} &\ge P\left\{||S_n|| \ge \epsilon, \sup_{1 \le k \le k_n} ||X_{nk}|| \le \delta\right\} \\ &= P\left\{||S'_n|| \ge \epsilon, \sup_{1 \le k \le k_n} ||X_{nk}|| \le \delta\right\} \\ &\ge P\{||S'_n|| \ge \epsilon\} - P\left\{\sup_{1 \le k \le k_n} ||X_{nk}|| > \delta\right\} \end{aligned}$$

implying by (2.4) that

$$P\{||S'_n|| \ge \epsilon\} \le P\{||S_n|| \ge \epsilon\} + \sum_{k=1}^{k_n} P\{||X_{nk}|| > \delta\} \to 0 \text{ as } n \to \infty. \quad \square$$

In order to formulate the next lemma we recall a definition and introduce some notation. Let $1 \le p \le 2$ and let $\{\epsilon_n, n \ge 1\}$ be a sequence of i.i.d. Bernoulli random variables with $P\{\epsilon_1 = \pm 1\} = 1/2$. A real separable Banach space B is said to be of *Rademacher type* p if $\sum_{n=1}^{\infty} \epsilon_n x_n$ converges a.s. whenever $\{x_n, n \ge 1\} \subseteq B$ with $\sum_{n=1}^{\infty} ||x_n||^p < \infty$. Every real separable Banach space is of Rademacher type 1. For a real separable Banach space B, let $p(B) = \sup\{p \in [1, 2] : B \text{ is of Rademacher type } p\}$.

The following lemma is a slight modification of Theorem 2 of Wei and Taylor [27] which holds for sequences of random elements. The modification concerns arrays of random elements; its proof can be obtained from Theorem 2 of Wei and Taylor [27] line by line and so will be omitted. We remark that the condition $E|X|^p < \infty$ can be weakened as was done in Theorem 4.1 of Adler, Rosalsky, and Volodin [3].

Lemma 2.4. Let $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise independent mean 0 random elements in a real separable Banach space B with $p(B) > 1$. Suppose that $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ is stochastically dominated in the Cesàro sense by a random variable X. If $E|X|^p < \infty$ for some $1 < p < p(B)$, then $\sum_{k=1}^{k_n} X_{nk}/k_n^{1/p} \xrightarrow{P} 0$.

3 Mainstream

With the preliminaries accounted for, the main theorem can now be presented. It should be noted that (3.5) is immediate if $\sum_{n=1}^{\infty} c_n < \infty$ and thus Theorem 3.1 only has content if $\sum_{n=1}^{\infty} c_n = \infty$.

Theorem 3.1. Let $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise independent random elements in a real separable Banach space and let $\{c_n, n \ge 1\}$ be a sequence of positive constants such that

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P\{||X_{nk}|| > \epsilon\} < \infty \text{ for all } \epsilon > 0, \tag{3.1}$$

there exist $p \ge 1/2, J \ge 2$, and $\delta > 0$ such that

$$\sum_{n=1}^{\infty} c_n \left(E\left[\sum_{k=1}^{k_n} ||X_{nk} I\{||X_{nk}|| \le \delta\}||^2 \right]^p \right)^J < \infty, \tag{3.2}$$

and

$$S_n \xrightarrow{P} 0. \tag{3.3}$$

Furthermore, suppose that

$$\sum_{k=1}^{k_n} P\{||X_{nk}|| > \delta\} = o(1) \text{ as } n \to \infty \tag{3.4}$$

if $\liminf_{n\to\infty} c_n = 0$. Then

$$\sum_{n=1}^{\infty} c_n P\{||S_n|| > \epsilon\} < \infty \text{ for all } \epsilon > 0. \tag{3.5}$$

Proof. Set $Y_{nk} = X_{nk} I\{||X_{nk}|| \le \delta\}, 1 \le k \le k_n, n \ge 1$ and

$$S'_n = \sum_{k=1}^{k_n} Y_{nk} \text{ and } S''_n = \sum_{k=1}^{k_n} X_{nk} I\{||X_{nk}|| > \delta\}, n \ge 1$$

where $\delta > 0$ is as in (3.2). Then for arbitrary $\epsilon > 0$

$$\begin{aligned} P\{||S_n|| \ge \epsilon\} &\le P\{||S'_n|| \ge \epsilon/2\} + P\{||S''_n|| \ge \epsilon/2\} \\ &\le P\{||S'_n|| \ge \epsilon/2\} + \sum_{k=1}^{k_n} P\{||X_{nk}|| > \delta\} \end{aligned}$$

and so in view of (3.1) it suffices to estimate $P\{||S'_n|| \ge \epsilon\}$. Before estimating, note that if $\liminf_{n\to\infty} c_n > 0$, then (3.1) ensures that $\sum_{k=1}^{k_n} P\{||X_{nk}|| > \delta\} = o(1)$ as $n \to \infty$ and recall that this condition holds by hypothesis if $\liminf_{n\to\infty} c_n = 0$. Then by (3.3) and Lemma 2.3, $S'_n \xrightarrow{P} 0$. Thus, $S'^s_n \xrightarrow{P} 0$ and by Lemma 2.1 we have

$$E||S'^s_n|| \to 0 \text{ as } n \to \infty. \tag{3.6}$$

Now by Lemma 2.2(ii), Proposition 2.1 with $j = [\text{Log } J]+1$ where Log denotes the logarithm to the base 2 and $[\cdot]$ is the integer part function, and the weak symmetrization inequality (cf. Loève [19, p. 257]), we have for all large n

$$\begin{aligned} P\{||S'_n|| &\ge \epsilon\} \le 2P\{||S'^s_n|| \ge \epsilon/2\} \\ &\le 2C_j \sum_{k=1}^{k_n} P\{||X^s_{nk}|| \ge \frac{\epsilon}{2 \cdot 3^j}\} + 2D_j \left(P\left\{ ||S'^s_n|| \ge \frac{\epsilon}{2 \cdot 3^j} \right\} \right)^J \\ &\le 4C_j \sum_{k=1}^{k_n} P\{||X_{nk}|| \ge \frac{\epsilon}{4 \cdot 3^j}\} + 2D_j \left(P\left\{ ||S'^s_n|| \ge \frac{\epsilon}{2 \cdot 3^j} \right\} \right)^J. \end{aligned}$$

Hence by (3.1) it suffices to estimate $(P\{||S_n'^s|| \geq \epsilon\})^J$. Now for all large n,

$$\begin{aligned} P\{||S_n'^s|| \geq \epsilon\} &\leq P\left\{\left| ||S_n'^s|| - E||S_n'^s|| \right| \geq \epsilon/2\right\} \text{ (by (3.6))} \\ &\leq (2/\epsilon)^{2p} E\left| ||S_n'^s|| - E||S_n'^s|| \right|^{2p} \text{ (by the Markov inequality)} \\ &\leq (2/\epsilon)^{2p} A_p E\left(\sum_{k=1}^{k_n} ||Y_{nk}^s||^2\right)^p \text{ (by Proposition 2.2)} \\ &\leq (2/\epsilon)^{2p} A_p E\left(\sum_{k=1}^{k_n} 2(||Y_{nk}||^2 + ||\tilde{Y}_{nk}||^2)\right)^p \\ &\leq \text{Const. } E\left(\sum_{k=1}^{k_n} ||Y_{nk}||^2\right)^p. \end{aligned}$$

In view of (3.2), the proof of (3.5) is completed. □

Remark. Apropos of the last assumption in Theorem 3.1, it will now be shown that (3.3) and

$$\sup_{1 \leq k \leq k_n} P\{||X_{nk}|| > \delta'\} < \frac{1}{2} \text{ for some } 0 < \delta' < \delta \text{ and all large } n \tag{3.7}$$

imply (3.4).

Proof. By (3.3), $S_n^s \xrightarrow{P} 0$. Then employing a random element version of Lévy's maximal inequality (cf. Araujo and Giné [2, p. 102]) and (if any $k_n = \infty$) Theorem 8.1.3 of Chow and Teicher [7, p. 278], we have for arbitrary $\epsilon > 0$

$$P\left\{\sup_{1 \leq k \leq k_n} ||X_{nk}^s|| > \epsilon\right\} \leq 2P\{||S_n^s|| \geq \epsilon\} = o(1) \text{ as } n \to \infty$$

implying

$$\begin{aligned} 1 &\geq \exp\left\{-\sum_{k=1}^{k_n} P\{||X_{nk}^s|| > \epsilon\}\right\} \geq \prod_{k=1}^{k_n}(1 - P\{||X_{nk}^s|| > \epsilon\}) \\ &= P\left\{\bigcap_{k=1}^{k_n}[||X_{nk}^s|| \leq \epsilon]\right\} = P\left\{\sup_{1 \leq k \leq k_n} ||X_{nk}^s|| \leq \epsilon\right\} \to 1 \text{ as } n \to \infty. \end{aligned}$$

Thus, for all $\epsilon > 0$

$$\sum_{k=1}^{k_n} P\{||X_{nk}^s|| > \epsilon\} = o(1) \text{ as } n \to \infty. \tag{3.8}$$

Let $\{m_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be as in the proof of Lemma 2.2. Now (3.7) implies (cf. Chow and Teicher [7, p. 72]) that $|m_{nk}| \leq \delta'$ for all large n and all $1 \leq k \leq k_n$. Then arguing as in the proof of Lemma 2.2, for all large n and all $1 \leq k \leq k_n$ we have

$$P\{||X_{nk}|| > \delta\} \leq P\{||X_{nk}|| - m_{nk} > \delta - \delta'\} \leq 2P\{||X_{nk}^s|| > \delta - \delta'\}$$

and (3.4) follows from (3.8). □

In the ensuing proposition, it will be shown that (3.1) is necessary for the convergence rate in Theorem 3.1 provided that c_n is bounded from 0 or $S_n \xrightarrow{P} 0$.

Proposition 3.1. Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an infinitesimal array of rowwise independent random elements and let $\{c_n, n \geq 1\}$ be a sequence of positive constants. Suppose that c_n is bounded from 0 or $S_n \xrightarrow{P} 0$. Then (3.5) implies (3.1).

Proof. Let $\epsilon > 0$ be arbitrary. If $S_n \xrightarrow{P} 0$, then $P\{||S_n|| > \epsilon/32\} = o(1)$ as $n \to \infty$. Alternatively, if c_n is bounded from 0, then (3.5) ensures that $P\{||S_n|| > \epsilon/32\} = o(1)$ as $n \to \infty$. Thus, in either case, employing the weak symmetrization inequality (cf. Loève [19, p. 257]) it follows that for all large n

$$P\{||S_n^s|| > \epsilon/16\} \leq 2P\{||S_n|| > \epsilon/32\} \leq 1/16.$$

Then by Lemma 2.2(i) and Proposition 2.3, for all large n

$$\begin{aligned}\sum_{k=1}^{k_n} P\{||X_{nk}|| > \epsilon\} &\leq 2\sum_{k=1}^{k_n} P\{||X_{nk}^s|| > \epsilon/2\} \\ &\leq \frac{2P\{||S_n^s|| > \epsilon/16\}}{1 - 8P\{||S_n^s|| > \epsilon/16\}} \\ &\leq \frac{4P\{||S_n|| > \epsilon/32\}}{1 - 16P\{||S_n|| > \epsilon/32\}} \\ &\leq 8P\{||S_n|| > \epsilon/32\}\end{aligned}$$

and the implication (3.5) $\Rightarrow$ (3.1) follows.

Remark.. Different values of p in the condition (3.2) of Theorem 3.1 yield different results. For example, we consider $p = 1/2$ in Corollary 4.7, $p = 1$ in Theorem 3.2, and $p = 2$ in Corollary 4.3.

We can simplify the condition (3.2) of Theorem 3.1 when absolute moments of some order $q \leq 2$ exist for the random elements comprising the array.

Theorem 3.2. Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise independent random elements and let $\{c_n, n \geq 1\}$ be a sequence of positive constants. Suppose that $E||X_{nk}||^q < \infty$, $1 \leq k \leq k_n$, $n \geq 1$ for some $0 < q \leq 2$, (3.1) holds, (3.3) holds, and there exists $J \geq 2$ such that

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E||X_{nk}||^q\right)^J < \infty. \tag{3.9}$$

Furthermore, suppose that (3.4) holds for some $\delta > 0$ if $\liminf_{n\to\infty} c_n = 0$. Then (3.5) obtains.

Proof. In view of Theorem 3.1, it suffices to verify that (3.2) holds with $p = 1$. Let $\delta_1 = \delta$ if $\liminf_{n\to\infty} c_n = 0$ and let $\delta_1 > 0$ be arbitrary otherwise. Then for $n \geq 1$

$$\begin{aligned}E\left(\sum_{k=1}^{k_n} ||X_{nk}I\{||X_{nk}|| \leq \delta_1\}||^2\right) &= \sum_{k=1}^{k_n} E||X_{nk}I\{||X_{nk}|| \leq \delta_1\}||^2 \\ &\leq \delta_1^{2-q} \sum_{k=1}^{k_n} E||X_{nk}||^q\end{aligned}$$

and (3.2) with $p = 1$ follows from (3.9). □

4 Corollaries

As applications of Theorem 3.2, we can obtain generalizations to a Banach space setting of the main result of Hu, Moricz, and Taylor [14] and Theorem 4.1 of Gut [11]. Furthermore, we can obtain the rate of convergence as in Wang, Bhaskara Rao, and Yang [26] (cf. Theorems 1.3 and 1.4 above).

Following Gut [11], set $\psi(0) = 0, \psi(x) = \text{Card } \{n : k_n \leq x\}, x > 0$ where $k_0 = 0$, and $M_r(x) = \sum_{n=1}^{[x]} k_n^{r-1}, x \geq 0$. We now obtain the following corollary.

Corollary 4.1. Let $\{X_{nk}, 1 \leq k \leq k_n < \infty, n \geq 1\}$ be an array of rowwise independent random elements which are stochastically dominated in the Cesàro sense by a random variable X and let $r \geq 1$ and $1 \leq t < 2$. If $E|X|^q < \infty$ for some $t < q \leq 2$, $EM_r(\psi(|X|^t)) < \infty, \sum_{n=1}^{\infty} k_n^{-\alpha} < \infty$ for some $\alpha > 0$, and

$$k_n^{-1/t} \sum_{k=1}^{k_n} X_{nk} \xrightarrow{P} 0, \tag{4.1}$$

then

$$\sum_{n=1}^{\infty} k_n^{r-2} P\left\{\left|\left|\sum_{k=1}^{k_n} X_{nk}\right|\right| > \epsilon k_n^{1/t}\right\} < \infty \text{ for all } \epsilon > 0. \tag{4.2}$$

Proof. We will apply Theorem 3.2 with $c_n = k_n^{r-2}, n \geq 1$ and X_{nk} replaced by $X_{nk}/k_n^{1/t}, 1 \leq k \leq k_n, n \geq 1$. The conditions (3.1) and (3.9) of Theorem 3.2 will be rewritten, respectively, as follows using the stochastic domination in the Cesàro sense assumption:

$$\sum_{n=1}^{\infty} k_n^{r-1} P\{|X| > \epsilon k_n^{1/t}\} < \infty \text{ for all } \epsilon > 0 \tag{4.3}$$

and there exists $J \geq 2$ such that

$$\sum_{n=1}^{\infty} k_n^{r-2} (k_n^{1-\frac{q}{t}} E|X|^q)^J < \infty. \tag{4.4}$$

(We will not rewrite condition (3.3) of Theorem 3.2 since it is one of the assumptions.)

As in Lemma 2.1 of Gut [10], $EM_r(\psi(|X|^t)) < \infty$ implies (4.3). The condition (4.4) is satisfied by taking $J \geq \max\{2, \frac{t(r-2+\alpha)}{q-t}\}$ recalling that $\sum_{n=1}^{\infty} k_n^{-\alpha} < \infty$. Finally, if $r < 2$, then $c_n \to 0$ and the condition $\sum_{k=1}^{k_n} P\{||X_{nk}||/k_n^{1/t} > \delta\}$ $= o(1)$ as $n \to \infty$ holds for any $\delta > 0$ using stochastic domination in the Cesàro sense, $E|X|^q < \infty$, the Markov inequality, and $q > t$. □

Remark. The assumptions in Corollary 4.1 are weaker than those of Wang, Bhaskara Rao, and Yang [26] in view of the following:

(i) We only assume (4.1) whereas Wang, Bhaskara Rao, and Yang [26] assume (1.1).

(ii) We consider general arrays of random elements $\{X_{nk}, 1 \leq k \leq k_n < \infty, n \geq 1\}$ instead of the standard triangular array wherein $k_n = n, n \geq 1$.

(iii) The random elements are assumed to be stochastically dominated in the Cesàro sense which is weaker than the random elements being stochastically dominated.

(iv) We allow for $r = t = 1$ whereas Wang, Bhaskara Rao, and Yang [26] require that $rt > 1$.

Moreover, our proof is much simpler than that of Wang, Bhaskara Rao, and Yang [26].

When we place a geometric condition on the Banach space B ($p(B) > 1$), we can obtain the following corollary (cf. Adler and Volodin [4]).

Corollary 4.2. Let $\{X_{nk}, 1 \leq k \leq k_n < \infty, n \geq 1\}$ be an array of rowwise independent mean 0 random elements taking values in a Banach space B with $p(B) > 1$. Suppose that

$\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ is stochastically dominated in the Cesàro sense by a random variable X. Let $r \ge 1$ and $1 < t < q < p(B)$. If $E|X|^q < \infty$, $EM_r(\psi(|X|^t)) < \infty$, and $\sum_{n=1}^{\infty} k_n^{-\alpha} < \infty$ for some $\alpha > 0$, then (4.2) obtains.

Proof. In view of Corollary 4.1, we only have to show that (4.1) holds. But this follows immediately from Lemma 2.4. □

Taking B to be the real line, $r = 2$, and $k_n = n, n \ge 1$ in Corollary 4.2 yields the main result of Hu, Moricz, and Taylor [14] (cf. Theorem 1.3 above). Moreover, taking B to be the real line and $r = 2$ in Corollary 4.2 yields (the main result) Theorem 4.1 of Gut [11].

We now present a modification for arrays of Theorem 2 of Kuczmaszewska and Szynal [17].

Corollary 4.3. Let $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise independent random elements such that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{k_n} P\{||X_{nk}|| > \epsilon\} < \infty \text{ for all } \epsilon > 0,$$

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{k_n} E||X_{nk}I\{||X_{nk}|| \le \delta\}||^4\right)^J < \infty \text{ for some } \delta > 0 \text{ and } J \ge 2,$$

and

$$\sum_{n=1}^{\infty}\left(\sum_{m=2}^{k_n} E||X_{nm}I\{||X_{nm}|| \le \delta\}||^2 \sum_{k=1}^{m-1} E||X_{nk}I\{||X_{nk}|| \le \delta\}||^2\right)^J < \infty.$$

Then S_n converges completely to 0 if and only if $S_n \xrightarrow{P} 0$.

Proof. The result follows immediately from Theorem 3.1 by taking $c_n = 1, n \ge 1$ and $p = 2$. □

Frequently limit theorems are formulated for *weighted sums* $\sum_{k=1}^{k_n} a_{nk}X_{nk}$ [where $\{a_{nk}, 1 \le k \le k_n, n \ge 1\}$ and $\{X_{nk}, 1 \le k \le k_n, n \ge 1\}$ are arrays of constants (called *weights*) and random elements, respectively] instead of for sums $\sum_{k=1}^{k_n} X_{nk}$ [where the weights can be built into the array of random elements]. We now present two general results for weighted sums. Corollary 4.4 generalizes Theorem 1.2 in two directions, namely:

(i) We consider Banach space valued random elements instead of random variables.

(ii) We consider an array rather than a sequence.

Corollary 4.4. Let $\{X_{nk}, k \ge 1, n \ge 1\}$ be an array of rowwise independent random elements which is stochastically dominated by a random variable X and let $\{a_{nk}, k \ge 1, n \ge 1\}$ be a Toeplitz array. If for some $\gamma > 0$

$$\sup_{k\ge1} |a_{nk}| = \mathcal{O}(n^{-\gamma}), E|X|^{1+\frac{1}{\gamma}} < \infty, \text{ and } \sum_{k=1}^{\infty} a_{nk}X_{nk} \xrightarrow{P} 0,$$

then $\sum_{k=1}^{\infty} a_{nk}X_{nk}$ converges completely to 0.

Proof. Note at the outset that the stochastic domination hypothesis ensures that $E||X_{nk}|| \le D^2E|X|, k \ge 1, n \ge 1$ and hence for all $n \ge 1$

$$E\sum_{k=1}^{\infty} ||a_{nk}X_{nk}|| = \sum_{k=1}^{\infty} E||a_{nk}X_{nk}|| \le D^2E|X| \sum_{k=1}^{\infty} |a_{nk}| \le CD^2E|X| < \infty.$$

Thus for all $n \geq 1, \sum_{k=1}^{\infty} ||a_{nk}X_{nk}||$ converges a.s. Then for all $n \geq 1$ and all $K \geq 1$,

$$\begin{aligned}
&\sup_{L>K}\left|\left|\sum_{k=1}^{L} a_{nk}X_{nk} - \sum_{k=1}^{K} a_{nk}X_{nk}\right|\right| = \sup_{L>k}\left|\left|\sum_{k=K+1}^{L} a_{nk}X_{nk}\right|\right| \\
\leq \;& \sup_{L>K}\sum_{k=K+1}^{L} ||a_{nk}X_{nk}|| = \sum_{k=K+1}^{\infty} ||a_{nk}X_{nk}|| \overset{K\to\infty}{\to} 0 \text{ a.s.}
\end{aligned}$$

Thus for all $n \geq 1$, with probability 1, $\{\sum_{k=1}^{K} a_{nk}X_{nk}, K \geq 1\}$ is a Cauchy sequence whence $\sum_{k=1}^{\infty} a_{nk}X_{nk}$ converges a.s.

Let $c_n = 1, n \geq 1$. Then we only need to verify that the conditions (3.1) and (3.9) of Theorem 3.2 hold with $a_{nk}X_{nk}$ playing the role of $X_{nk}, 1 \leq k \leq k_n$, $n \geq 1$ in the formulation of that theorem. To establish (3.1), for arbitrary $\epsilon > 0$, the stochastic domination hypothesis ensures that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} P\{||a_{nk}X_{nk}|| > \epsilon\} \leq D\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} P\{|a_{nk}X| > \epsilon/D\} < \infty$$

by Lemma 1 of Rohatgi [21] (see also Lemma 1 of Pruitt [20]).

To establish (3.9), let $q = \min\{1+\gamma^{-1}, 2\}$ and let $J > \max\{\gamma^{-1}, 2\}$. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} |a_{nk}|^q E||X_{nk}||^q\right)^J \\
\leq \;& D^{(q+1)J}\sum_{n=1}^{\infty}\left(\sup_{k\geq 1}|a_{nk}|^{q-1}\sum_{k=1}^{\infty}|a_{nk}|E|X|^q\right)^J \quad \text{(by stochastic domination)} \\
\leq \;& \text{Const.} \sum_{n=1}^{\infty}\frac{1}{n^{\gamma(q-1)J}} < \infty. \quad \square
\end{aligned}$$

In the next result for weighted sums we obtain the rate of convergence.

Corollary 4.5. Let $\{X_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise independent random elements which is stochastically dominated by a random variable X and let $\{a_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of constants such that for some $0 < q \leq 2$ and $\gamma > 1$

$$\sum_{k=1}^{k_n} |a_{nk}|^q = \mathcal{O}(n^{-\gamma}). \tag{4.5}$$

If $E|X|^q < \infty$ and $\sum_{k=1}^{k_n} a_{nk}X_{nk} \overset{P}{\to} 0$, then for all $\beta < \gamma - 1$

$$\sum_{n=1}^{\infty} n^\beta P\left\{\left|\left|\sum_{k=1}^{k_n} a_{nk}X_{nk}\right|\right| > \epsilon\right\} < \infty \text{ for all } \epsilon > 0.$$

Proof. Without loss of generality assume that $\beta \geq 0$. Let $c_n = n^\beta, n \geq 1$. Then we only need to verify that the conditions (3.1) and (3.9) of Theorem 3.2 hold with $a_{nk}X_{nk}$ playing the role of $X_{nk}, 1 \leq k \leq k_n, n \geq 1$ in the formulation of that theorem. To establish (3.1), for arbitrary $\epsilon > 0$,

$$\sum_{n=1}^{\infty} n^\beta \sum_{k=1}^{k_n} P\{||a_{nk}X_{nk}|| > \epsilon\}$$

$$\begin{aligned}
&\leq D\sum_{n=1}^{\infty} n^{\beta}\sum_{k=1}^{k_n} P\left\{|X| > \frac{\epsilon}{D|a_{nk}|}\right\} \text{ (by stochastic domination)}\\
&\leq D\sum_{n=1}^{\infty} n^{\beta}\sum_{k=1}^{k_n} D^q\epsilon^{-q}|a_{nk}|^q E|X|^q \text{ (by the Markov inequality)}\\
&= \text{Const.} \sum_{n=1}^{\infty} n^{\beta}\sum_{k=1}^{k_n}|a_{nk}|^q\\
&\leq \text{Const.} \sum_{n=1}^{\infty} n^{\beta-\gamma} \text{ (by (4.5))}\\
&< \infty \text{ (since } \beta < \gamma - 1).
\end{aligned}$$

To establish (3.9), note that for $J \geq 2$

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\beta}\left(\sum_{k=1}^{k_n} E||a_{nk}X_{nk}||^q\right)^J\\
&\leq D^{(q+1)J}\sum_{n=1}^{\infty} n^{\beta}\left(\sum_{k=1}^{k_n}|a_{nk}|^q E|X|^q\right)^J \text{ (by stochastic domination)}\\
&\leq \text{Const.} \sum_{n=1}^{\infty} n^{\beta-J\gamma} \text{ (by (4.5))}\\
&< \infty
\end{aligned}$$

since $0 \leq \beta < \gamma - 1$ ensures for $J \geq 2$ that $\beta - J\gamma < -1$. □

We now present simple proofs of the two results of Sung [23] which were stated in Section 1 (Theorems 1.5 and 1.6 above). However, Theorem 1.5 will be proved under somewhat modified conditions which are formulated in the ensuing Corollary 4.6. After Corollary 4.6 is proved, it will be compared with Theorem 1.5. (The proviso $\sum_{k=1}^{n} a_{nk}^2 = o(1/\log\, n)$ automatically follows from $\max_{1\leq k\leq n}|a_{nk}| = \mathcal{O}(n^{-1/p})$ when $0 < p < 2$.)

Corollary 4.6. Let $\{X_{nk}, 1 \leq k \leq n, n \geq 1\}$ be a triangular array of rowwise independent random elements which are stochastically dominated by a random variable X. Let $\{a_{nk}, 1 \leq k \leq n, n \geq 1\}$ be a triangular array of real numbers satisfying $\max_{1\leq k\leq n}|a_{nk}| = \mathcal{O}(n^{-1/p})$ for some $0 < p < 2$. If $\sum_{k=1}^{n} a_{nk}X_{nk} \xrightarrow{P} 0$ and $E|X|^{p+1} < \infty$, then $\sum_{k=1}^{n} a_{nk}X_{nk}$ converges completely to 0.

Proof. Let $c_n = 1, n \geq 1$. As in the proof of Corollaries 4.4 and 4.5, we only need to verify that the conditions (3.1) and (3.9) of Theorem 3.2 hold with $a_{nk}X_{nk}$ playing the role of $X_{nk}, 1 \leq k \leq n, n \geq 1$ in the formulation of that theorem. Since $p > 0$, (3.1) follows exactly as it did in the proof of Corollary 4.4.

To establish (3.9), let $q = 2$ and let $J > \max\{p/(2-p), 2\}$. Then recalling $0 < p < 2$

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} E||a_{nk}X_{nk}||^2\right)^J\\
&\leq \text{Const.} \sum_{n=1}^{\infty}(n\, \max_{1\leq k\leq n} a_{nk}^2 EX^2)^J \text{ (by stochastic domination)}\\
&\leq \text{Const.} \sum_{n=1}^{\infty}\frac{1}{n^{(\frac{2}{p}-1)J}} < \infty. \quad \square
\end{aligned}$$

Remark. Corollary 4.6 will now be compared with Theorem 1.5. Suppose that $\max_{1\le k\le n}|a_{nk}| = \mathcal{O}(n^{-1/p})$ for some $p>0$. If $0<p<1$, then Corollary 4.6 requires $E|X|^{1+p}<\infty$ whereas Theorem 1.5 requires the stronger condition $EX^2<\infty$. If $p=1$, Corollary 4.6 and Theorem 1.5 coincide. If $1<p<2$, then Corollary 4.6 requires $E|X|^{p+1}<\infty$ whereas Theorem 1.5 requires the stronger condition $E|X|^{2p}<\infty$. However, if $p\ge 2$, then Corollary 4.6 might not apply whereas Theorem 1.5 can apply provided $\sum_{k=1}^n a_{nk}^2 = o(1/\log n)$.

Proof of Theorem 1.6. We will apply Theorem 3.2 with $k_n = n, n\ge 1, c_n = 1, n\ge 1, q = \min\{\nu, 2\}$, and with X_{nk} replaced by $X_{nk}/n^{1/p}, 1\le k\le n, n\ge 1$. To verify (3.1), note that

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n} P\{||X_{nk}||>\epsilon n^{1/p}\} \le \sum_{n=1}^{\infty}\sum_{k=1}^{n}\frac{E||X_{nk}||^{\nu}}{\epsilon^{\nu}n^{\nu/p}} \text{ (by the Markov inequality)}$$

$$\le \quad \text{Const.} \sum_{n=1}^{\infty}\frac{n^{\alpha+1}}{n^{\nu/p}} \text{ (by (1.2))}$$

$$= \quad \text{Const.} \sum_{n=1}^{\infty}\frac{1}{n^{\frac{\nu}{p}-\alpha-1}} < \infty \text{ (since } \frac{\nu}{p}-\alpha>2).$$

To verify (3.9), we consider the two cases $\nu\le 2$ and $\nu>2$. If $\nu\le 2$, then $q=\nu$ and taking $J=2$ gives

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} E\left|\left|\frac{X_{nk}}{n^{1/p}}\right|\right|^{q}\right)^{2} \le \text{ Const. } \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{n^{\alpha}}{n^{\nu/p}}\right)^{2} \text{ (by (1.2))}$$

$$= \text{ Const. } \sum_{n=1}^{\infty}\frac{1}{n^{2(\frac{\nu}{p}-\alpha-1)}} < \infty \text{ (since } \frac{\nu}{p}-\alpha>2).$$

On the other hand if $\nu>2$, then $q=2$. Note that $\frac{\nu}{p}-\alpha>\frac{\nu}{2}$ ensures that $\frac{2}{p}-\frac{2\alpha}{\nu}>1$ and so $J\ge 2$ can be chosen so that $J(\frac{2}{p}-\frac{2\alpha}{\nu}-1)>1$. Then

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} E\left|\left|\frac{X_{nk}}{n^{1/p}}\right|\right|^{q}\right)^{J} \le \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\left(E\left|\left|\frac{X_{nk}}{n^{1/p}}\right|\right|^{\nu}\right)^{\frac{2}{\nu}}\right)^{J} \text{ (by Liapounov's inequality)}$$

$$\le \quad \text{Const.} \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{n^{\frac{2\alpha}{\nu}}}{n^{\frac{2}{p}}}\right)^{J} \text{ (by (1.2))}$$

$$= \quad \text{Const.} \sum_{n=1}^{\infty}\frac{1}{n^{J(\frac{2}{p}-\frac{2\alpha}{\nu}-1)}} < \infty \text{ (by the choice of } J).$$

Thus (3.9) holds in each case. □

By modifying the condition (3.2) of Theorem 3.1 we can omit the assumption (3.3) of that theorem. This will be accomplished by the following corollary.

Corollary 4.7. Let $\{X_{nk}, 1\le k\le k_n, n\ge 1\}$ be an array of rowwise independent random elements and let $\{c_n, n\ge 1\}$ be a sequence of positive constants such that (3.1) holds and such that there exist $J\ge 2$ and $\delta>0$ with

$$\sum_{n=1}^{\infty} c_n\left(\sum_{k=1}^{k_n} E||X_{nk}I\{||X_{nk}||\le\delta\}||\right)^{J} < \infty. \tag{4.6}$$

Furthermore, suppose that (3.4) holds and

$$\sum_{k=1}^{k_n} E||X_{nk}I\{||X_{nk}|| \le \delta\}|| = o(1) \text{ as } n \to \infty. \tag{4.7}$$

Then (3.5) obtains.

Proof. We will verify that the conditions (3.2) and (3.3) of Theorem 3.1 hold. Condition (3.2) with $p = 1/2$ follows from

$$E\left[\sum_{k=1}^{k_n} ||X_{nk}I\{||X_{nk}|| \le \delta\}||^2\right]^{1/2} \le \sum_{k=1}^{k_n} E||X_{nk}I\{||X_{nk}|| \le \delta\}||$$

and (4.6).

Regarding condition (3.3), we note by (3.4) and Lemma 2.3 that it can be replaced by

$$\sum_{k=1}^{k_n} X_{nk}I\{||X_{nk}|| \le \delta\} \xrightarrow{P} 0. \tag{4.8}$$

But from (4.7) we obtain the convergence in $\mathcal{L}_1$ of the expression in (4.8) to 0 which implies its convergence in probability to 0. □

We will now reformulate Theorem 3.1 for arrays of random variables. For a different proof and for some additional corollaries we refer the reader to Hu, Szynal, and Volodin [15].

Corollary 4.8. Let $\{\xi_{nk}, 1 \le k \le k_n, n \ge 1\}$ be an array of rowwise independent random variables and let $\{c_n, n \ge 1\}$ be a sequence of positive constants which are bounded from 0. Suppose that

$$\sum_{n=1}^{\infty} c_n \sum_{k=1}^{k_n} P\{|\xi_{nk}| > \epsilon\} < \infty \text{ for all } \epsilon > 0, \tag{4.9}$$

there exist $p \ge 1, J \ge 2$, and $\delta > 0$ such that

$$\sum_{n=1}^{\infty} c_n \left(E\left[\sum_{k=1}^{k_n} |\xi_{nk}I\{|\xi_{nk}| \ge \delta\}|^2\right]^p\right)^J < \infty, \tag{4.10}$$

and

$$\sum_{k=1}^{k_n} E\xi_{nk}I\{|\xi_{nk}| \le \delta\} \to 0 \text{ as } n \to \infty. \tag{4.11}$$

Then

$$\sum_{n=1}^{\infty} c_n P\left\{\left|\sum_{k=1}^{k_n} \xi_{nk}\right| > \epsilon\right\} < \infty \text{ for all } \epsilon > 0.$$

Proof. We note at the outset that in the case $k_n = \infty$ for any $n \ge 1$, the series $\sum_{k=1}^{k_n} \xi_{nk}$ converges a.s. by the Kolmogorov three-series criterion (cf. Loève [19, p. 249]).

In view of Theorem 3.1, it only needs to be shown that $\sum_{k=1}^{k_n} \xi_{nk} \xrightarrow{P} 0$. By the degenerate convergence criterion (cf. Loève [19, p. 329]), we must verify that for some $\delta > 0$ and all $\epsilon > 0$

$$\sum_{k=1}^{k_n} P\{|\xi_{nk}| > \epsilon\} \to 0 \text{ as } n \to \infty, \sum_{k=1}^{k_n} E\xi_{nk}I\{|\xi_{nk}| \le \delta\} \to 0 \text{ as } n \to \infty,$$

$$\sum_{k=1}^{k_n} \text{Var } \xi_{nk} I\{|\xi_{nk}| \le \delta\} \to 0 \text{ as } n \to \infty. \tag{4.12}$$

Now (4.13) is exactly (4.11), and (4.12) and (4.14) follow immediately from (4.9) and (4.10), respectively. □

Example 4.1. Let $\{\xi_n, n \ge 1\}$ be a sequence of i.i.d. random variables with $E|\xi_1|^{\frac{4(\alpha+2)}{\alpha+3}} < \infty$ for some $\alpha \ge 0$. Set $\xi_{nk} = \xi_k / n^{\frac{\alpha+3+\lambda}{4}}$, $1 \le k \le n, n \ge 1$ where $\lambda > \max\{1-\alpha, 0\}$. We will verify that the conditions (4.9), (4.10), and (4.11) of Corollary 4.8 hold with $k_n = n, n \ge 1, c_n = n^\alpha, n \ge 1, p = 1, J = 2$, and $\delta = 1$. To verify (4.9), note that for $\epsilon > 0$

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^\alpha \sum_{k=1}^{n} P\{|\xi_{nk}| > \epsilon\} = \sum_{n=1}^{\infty} n^{\alpha+1} P\left\{|\xi_1| > \epsilon n^{\frac{\alpha+3+\lambda}{4}}\right\} \\
&\le \text{ Const. } \sum_{n=1}^{\infty} n^{\alpha+1} \cdot \frac{1}{n^{\left(\frac{\alpha+3+\lambda}{4}\right)\left(\frac{4(\alpha+2)}{\alpha+3}\right)}} \text{ (by the Markov inequality)} \\
&= \text{ Const. } \sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{\lambda(\alpha+2)}{\alpha+3}}} < \infty.
\end{aligned}$$

To verify (4.10), note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^\alpha \left(E\left[\sum_{k=1}^{n} \frac{\xi_k^2 I\{|\xi_k| \le n^{\frac{\alpha+3+\lambda}{4}}\}}{n^{\frac{\alpha+3+\lambda}{2}}} \right] \right)^2 \\
&\le \text{ Const. } \sum_{n=1}^{\infty} \frac{n^\alpha n^2}{n^{\alpha+3+\lambda}} = \text{Const. } \sum_{n=1}^{\infty} \frac{1}{n^{1+\lambda}} < \infty.
\end{aligned}$$

Finally, to verify (4.11), note that

$$\begin{aligned}
&\left| \sum_{k=1}^{n} \frac{E\xi_k I\{|\xi_k| \le n^{\frac{\alpha+3+\lambda}{4}}\}}{n^{\frac{\alpha+3+\lambda}{4}}} \right| \le \sum_{k=1}^{n} \frac{E|\xi_k| I\{|\xi_k| \le n^{\frac{\alpha+3+\lambda}{4}}\}}{n^{\frac{\alpha+3+\lambda}{4}}} \\
\le\ &\frac{\text{Const.} n}{n^{\frac{\alpha+3+\lambda}{4}}} = \frac{\text{Const.}}{n^{\frac{\alpha-1+\lambda}{4}}} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Thus by Corollary 4.8,

$$\sum_{n=1}^{\infty} n^\alpha P\left\{ \frac{|\sum_{k=1}^{n} \xi_k|}{n^{\frac{\alpha+3+\lambda}{4}}} > \epsilon \right\} < \infty \text{ for all } \epsilon > 0.$$

The following modification of an example presented in Kuczmaszewska and Szynal [17] shows that Corollary 4.8 can fail for arrays of independent Banach space valued random elements.

Example 4.2. Let ℓ_1 denote the real separable Banach space of absolutely summable real sequences $x = (x_1, x_2, \cdots)$ with norm $||x|| = \sum_{i=1}^{\infty} |x_i|$. Let e_k denote the k-th element of the standard basis in ℓ_1, that is, the element having 1 for its k-th coordinate and 0 for the other coordinates. Let $\{\epsilon_{nk}, 1 \le k \le n, n \ge 1\}$ be an array of rowwise independent random variables each with a symmetric Bernoulli distribution, that is, $P\{\epsilon_{nk} = \pm 1\} = 1/2$ for $1 \le k \le n, n \ge 1$. Define $X_{nk} = \epsilon_{nk} e_k / n, 1 \le k \le n, n \ge 1$. Thus $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ is a symmetric array of rowwise independent ℓ_1-valued random elements. Let $c_n = 1, n \ge 1$.

Now the array $\{X_{nk}, 1 \le k \le n, n \ge 1\}$ satisfies the conditions (4.9), (4.10), and (4.11) of Corollary 4.8 but

$$||S_n|| = \left|\left|\frac{1}{n}\sum_{k=1}^{n}\epsilon_{nk}e_k\right|\right| = \frac{1}{n}\sum_{k=1}^{n}1 = 1 \text{ a.s. } , n \ge 1.$$

Thus the conclusion $\sum_{n=1}^{\infty} P\{||S_n|| > \epsilon\} < \infty$ fails for all ϵ in $(0, 1)$. We note that $S_n \xrightarrow{P} 0$ also fails.

ACKNOWLEDGMENTS

The authors are grateful to Professor Alan Gut (Uppsala University, Sweden) for sending us a copy of his article Gut [11]. Part of the research of T.-C. Hu was conducted during his visit to the Department of Statistics at the University of Florida for the 1997-1998 academic year and he is grateful to the Department for their kind hospitality and for the use of their facilities.

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