THE COMPLETE CONVERGENCE RATES OF THE BOOTSTRAP MEAN

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SUMMARY

For a sequence of random variables $\{X_n, n \geq 1\}$, the convergence rate (that is Baum-Katz/Hsu-Robbins/Spitzer complete convergence type result) is obtained for bootstrapped means. In this investigation, the sequence of random variables need not necessarily to be independent or identically distributed. Further, no assumptions are made concerning either the marginal or joint distributions of the random variables.

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1. INTRODUCTION

The main focus of the present investigation is to obtain the convergence rates in the form of Baum-Katz/Hsu-Robbins/Spitzer type result for bootstrapped means from the sequence of random variables which are not necessarily independent or identically distributed. It is important to note that the strong laws of large numbers are of practical use in establishing the strong asymptotic validity of the bootstrapped mean for random variables. The work on the consistency of bootstrap estimators has received a lot of attention in recent years due to a growing demand of the procedure both theoretically and practically.

In the present investigation we do not make any assumptions regarding the marginal or joint distributions of the random variables taken from the sample. In this case, the main result of Hu and Taylor (1997) can be seen as a special case of the result given in Theorem 1 of this paper.

For expository purpose, first we wish to give a brief description of results related to independent an identically distributed (i.i.d) random variables. To this end, let $\{X_n;\ n \geq 1\}$ be a sequence of independent and identically distributed random variables defined on some complete probability space $(\Omega, \mathcal{F}, P)$ and write $S_n = \sum_{i=1}^n X_i, n \geq 1$. For $\omega \in \Omega$ and each $n \geq 1$, let $P_n(\omega) = n^{-1} \sum_{i=1}^n \delta_{X_i(\omega)}$ denote the empirical measure and let $\{\hat{X}_{n,j}^{\omega};\ 1 \leq j \leq m(n)\}$ be i.i.d. random variables with law $P_n(\omega)$, where $\{m(n);\ n \geq 1\}$ (resampling size) is a sequence of positive integers. Let $\bar{X}_n(\omega)$ be the sample mean of $\{X_i(\omega);\ 1 \leq i \leq n\},\ n \geq 1$. Moreover, assume that $\{X, X_n;\ n \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables, X is nondegenerate and $E(X^2) < \infty$. Bickel and Freedman (1981) and Singh (1981) showed the following weak convergence of distributions for $m(n) = n,\ n \geq 1$ and almost every $\omega \in \Omega$,

$$\mathcal{L}\left(n^{1/2}(n^{-1}\hat{S}_n^{\omega} - \bar{X}_n(\omega))\right) \rightarrow_w N(0, \sigma^2). \tag{1.1}$$

Here $\hat{S}_n^{\omega} = \sum_{j=1}^n \hat{X}_{n,j}^{\omega},\ n \geq 1$ and $\sigma^2 = E(X - E(X))^2$. Note that by the Glivenko-Cantelli Theorem $P_n(\omega)$ is close to $\mathcal{L}(X)$ and by the Lévy central limit theorem

$$\mathcal{L}\left(n^{1/2}(n^{-1}S_n - E(X))\right) \rightarrow_w N(0, \sigma^2).$$

It follows that if $E(X^2) < \infty$, then the statistic $n^{1/2}(n^{-1}S_n - E(X))$ is close in distribution to the bootstrap statistic $n^{1/2}(n^{-1}\hat{S}_n^{\omega} - \bar{X}_n(\omega))$ for large n ω-almost surely (a.s.). This is, very roughly, the idea of the bootstrap. See Efron (1979), where this nice idea is made explicit and where it is substantiated with several important examples. Giné and Zinn (1989) proved that the existence of the second moment is necessary for there to exist positive scalars $a_n \uparrow \infty$, centering $c_n(\omega)$, and a random probability measure $\nu(\omega)$ nondegenerate with positive probability, such that $\mathcal{L}(a_n^{-1}\hat{S}_n^{\omega} - c_n(\omega)) \rightarrow_w \nu(\omega)$ for almost every $\omega \in \Omega$. The limit law (1) tells us just the right rate at which to magnify the difference $n^{-1}\hat{S}_n^{\omega} - \bar{X}_n(\omega)$, which is tending a.s. to zero, in order to obtain convergence in distribution to a nondegenerate law ω-a.s. We note from (1.1) that, for almost every $\omega \in \Omega$,

$$\frac{n^{1/2}}{x_n}(n^{-1}\hat{S}_n^{\omega} - \bar{X}_n(\omega)) \rightarrow 0 \text{ in probability as } n \rightarrow \infty$$

for any sequence of constants $\{x_n\}$ with $x_n \uparrow \infty$. On the other hand, strong laws of large numbers were proved by Athreya (1983) and Csörgő (1992) for the bootstrap mean. Arenal-Gutiérrez et al. (1996) analyzed the results of Athreya (1983) and Csörgő (1992). Then, by taking into account the different growth rates for the resampling size $m(n)$, they gave new and simple proofs of those results. They also provided examples that show that the sizes of resampling required by their results to ensure a.s. convergence are not far from optimal.

An interesting and novel feature of the current investigation is that no assumptions are made concerning either marginal or joint distributions of the random variables $\{X_n;\ n \geq 1\}$. Further we do not assume that either random variables $\{X_n;\ n \geq 1\}$ are independent or they are identically distributed.

2. MAIN RESULTS

Hsu and Robbins (1947) introduced the concept of complete convergence as follows. A sequence $\{U_n,\ n \geq 1\}$ of random variables *converges completely* to the constant C if $\sum\limits_{n=1}^{\infty} P\{|U_n - C| > \varepsilon\} < \infty$ for all $\varepsilon > 0$. Now, we recall the following theorem of complete convergence, which forms the basis for the new results in this article.

Theorem. (Hu *et al.*1998: Remark 2 after the Theorem): *Let $\{m(n), n \geq 1\}$, be a sequence of positive integers, $m(n) \to \infty$ as $n \to \infty$ and $\{Y_{nk}, 1 \leq k \leq k_n, n \geq 1\}$ be an array of rowwise independent mean zero random variables. Suppose, $E|Y_{nk}|^2 < \infty, 1 \leq k \leq m(n), n \geq 1$ and $r \geq 0$. Moreover, assume that*
A1. $\sum_{n=1}^{\infty} n^r \sum_{k=1}^{m(n)} P\{|Y_{nk}| > \epsilon\} < \infty$ *for all* $\epsilon > 0$. *A2.* *There exists $J \geq 1$ such that*

$$\sum_{n=1}^{\infty} n^r \left(\sum_{k=1}^{m(n)} E|Y_{nk}|^2 \right)^J < \infty.$$

Then

$$\sum_{n=1}^{\infty} n^r P\{|\sum_{k=1}^{m(n)} Y_{nk}| > \epsilon\} < \infty \text{ for all } \epsilon > 0.$$

The main result of this investigation is given in the following Theorem 1. The main thrust and unusual feature of Theorem 1 is that no assumptions are required concerning marginal and joint distributions of the random variables $\{X_n\}$. Not only that, it is not assumed that these random variables are either independent or identically distributed. In general, no moment conditions are imposed on the random variables $\{X_n\}$.

Theorem 1: *Let $\{X_n, n \geq 1\}$ be a sequence of random variables and let $\{a_n, n \geq 1\}$ be a sequence of positive constants. Suppose there exists $J \geq 1$ such that*

$$\frac{\max_{1 \leq i \leq n} |X_i|}{m(n) a_n} \to 0 \text{ a.s. as } n \to \infty \text{ and} \tag{2.2}$$

$$\sum_{n=1}^{\infty} n^r \left(\frac{\sum_{i=1}^{n} X_i^2}{a_n^2 m(n) n} \right)^J < \infty \text{ a.s.} \tag{2.3}$$

Then the bootstrapped mean is strongly consistent, that is, for almost all $\omega \in \Omega$ and all $\epsilon > 0$

$$\sum_{n=1}^{\infty} n^r P \left\{ |\sum_{k=1}^{m(n)} \left(\hat{X}_{n,k}^{\omega} - \bar{X}_n(\omega) \right)| > \epsilon a_n m(n) \right\} < \infty.$$

Proof: We need only check conditions of Theorem (Hu *et al.* 1998) for an array

$$\left\{ Y_{nk} = \frac{\hat{X}_{n,k}^{\omega} - \bar{X}_n(\omega)}{m(n) a_n}, 1 \leq k \leq m(n), n \geq 1 \right\}.$$

These conditions may be rewritten as follows. *Condition 1*: $\sum_{n=1}^{\infty} n^r \sum_{k=1}^{m(n)} P\{|\hat{X}_{n,k}^{\omega} - \bar{X}_n(\omega)| >$

$\epsilon m(n)a_n\} < \infty$ for all $\epsilon > 0$.

Condition 2: There exists $J \geq 1$ such that

$$\sum_{n=1}^{\infty} n^r \left(\sum_{k=1}^{m(n)} \frac{E|\hat{X}^{\omega}_{n,k} - \bar{X}_n(\omega)|^2}{(a_n m(n) n)^2} \right)^J < \infty.$$

Further, for the first condition,

$$|\hat{X}^{\omega}_{n,k} - \bar{X}_n(\omega)| \leq 2 \max_{1 \leq i \leq n} |X_i(\omega)|,$$

and for the second one, since

$$\begin{aligned} E|\hat{X}^{\omega}_{n,k} - \bar{X}_n(\omega)|^2 &= \frac{1}{n} \sum_{i=1}^{n} |X_i(\omega) - \bar{X}_n(\omega)|^2 \\ &\leq \frac{4}{n} \sum_{i=1}^{n} X_i(\omega)^2, \end{aligned}$$

we have

$$\begin{aligned} &\sum_{n=1}^{\infty} n^r \left(\sum_{k=1}^{m(n)} \frac{E|\hat{X}^{\omega}_{n,k} - \bar{X}_n(\omega)|^2}{(a_n m(n) n)^2} \right)^J \\ &\leq C \sum_{n=1}^{\infty} n^r \left(\frac{\sum_{i=1}^{n} X_i^2}{a_n^2 m(n) n} \right)^J < \infty \text{ a.s.} \end{aligned}$$

where C is a positive constant. $\square$

Now we wish to make the following two remarks.

Remark 1: If $m(n)a_n \to \infty$ monotonically, then (2.2) is equivalent to the structurally simpler condition

$$\lim_{n \to \infty} \frac{X_n}{m(n)a_n} = 0 \text{ a.s.} \tag{2.4}$$

Indeed, let $m(n)a_n \to \infty$ monotonically and (2.4) holds. Then, for arbitrary $n \geq k \geq 2$,

$$\begin{aligned} \frac{\max_{1 \leq i \leq n} |X_i|}{m(n)a_n} &\leq \frac{\max_{1 \leq i \leq k-1} |X_i|}{m(n)a_n} + \frac{\max_{k \leq i \leq n} |X_i|}{m(n)a_n} \\ &= \frac{\max_{1 \leq i \leq k-1} |X_i|}{m(n)a_n} + \max_{k \leq i \leq n} \frac{|X_i|}{m(n)a_n} \\ &\leq \frac{\max_{1 \leq i \leq k-1} |X_i|}{m(n)a_n} + \sup_{i \geq k} \frac{|X_i|}{m(n)a_n} \to 0, \end{aligned}$$

as first $n \to \infty$ and then $k \to \infty$. It is noted that the reverse implication is evident.

Remark 2: It appears that the comparison of Theorem 1 of this paper to Theorem 2.1 of Li, Rosalsky and Ahmed (1999) may not be possible. One simple argument is that the difference in the assumptions of both theorems does to provide for comparative analysis. Li, Rosalsky and Ahmed (1999) assumed the convergence of partial sums. On the other hand we use only boundness of partial sums.

Finally, we wish to provide a generalization of the main result of Hu and Taylor (1997). We give the proof for pairwise i.i.d. random variables. Recall that Hu and Taylor (1997) only considered i.i.d. case in their publication. In addition, the result in this investigation is more sharp in the sense that it establishes convergence rates which was not given in Hu and Taylor (1997). In their paper only a.s. convergence result is being stated. Furthermore, our proof may be viewed simpler than that of Hu and Taylor (1997).

Corollary: *Let $\{X_n, n \geq 1\}$ be pairwise independent identically distributed random variables with $E|X|^{1+\delta} < \infty$ for some $\delta > 0$ and $EX = \mu$. Then for all $\epsilon > 0$, for almost all $\omega \in \Omega$ and any real number r:*

$$\sum_{n=1}^{\infty} n^r P\left\{ |\sum_{k=1}^{n} \left(\hat{X}^{\omega}_{n,k} - \mu \right) | > \epsilon n \right\} < \infty.$$

Proof: Noting that it is sufficient to show that present result for $r \geq 0$, since for $r < -1$ the result is obvious. Let $a_n = 1$ and $m(n) = n$.

In order to prove (2.4), we can write that for arbitrary $\epsilon > 0$

$$\sum_{n=1}^{\infty} P\{|X_n| > \epsilon n\} \leq CE|X_1| < \infty.$$

Thus, by the Borell-Cantelly lemma

$$\lim_{n \to \infty} X_n/n = 0, \text{ a.s.}$$

To prove (2.3), note that $\{X_n^2, n \geq 1\}$ is also pairwise independent identically distributed random variables. Recall that $E|X|^{1+\delta} < \infty$ and by the application of Petrov (1996) results it can be shown that

$$\frac{1}{n^{2/(1+\alpha)}} \sum_{i=1}^{n} X_i^2 \to 0, \text{ a.s., where } 0 < \alpha < \delta.$$

Now let $J > \frac{(1+\alpha)(1+r)}{2\alpha}$, then

$$\sum_{n=1}^{\infty} n^r \left(\frac{1}{n^2} \sum_{i=1}^{n} X_i^2 \right)^J \leq \left(\sup_{n \geq 1} \frac{1}{n^{\frac{2}{1+\alpha}}} \sum_{i=1}^{n} X_i^2 \right)^J \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2\alpha J}{1+\alpha} - r}} < \infty, \text{ a.s. }.$$

Hence, by Theorem 1, for almost all $\omega \in \Omega$ and all $\epsilon > 0$

$$\sum_{n=1}^{\infty} n^r P\left\{ |\sum_{k=1}^{n} \left(\hat{X}^{\omega}_{n,k} - \bar{X}_n(\omega) \right) | > \epsilon n \right\} < \infty.$$

Further, by application of Etemadi (1981) strong law of large numbers we have that $\bar{X}_n \to \mu$ a.s.. Hence, for almost all $\omega \in \Omega$ and all $\epsilon > 0$ there exists $N = N(\epsilon, \omega)$ such that for all $n \geq N$ we have $\bar{X}_n - \mu < \epsilon/2$. Then

$$\sum_{n=1}^{\infty} n^r P\left\{ |\sum_{k=1}^{n} \left(\hat{X}^{\omega}_{n,k} - \mu \right) | > \epsilon n \right\}$$

$$\leq \sum_{n=1}^{\infty} n^r P\left\{|\sum_{k=1}^{n}\left(|\hat{X}_{n,k}^{\omega} - \bar{X}_n(\omega)| + |\bar{X}_n(\omega) - \mu|\right) > \epsilon n\right\}$$

$$\leq \sum_{n=1}^{\infty} n^r P\left\{|\sum_{k=1}^{n}\left(|\hat{X}_{n,k}^{\omega} - \bar{X}_n(\omega)|\right) > \epsilon n/2\right\} < \infty. \quad \square$$

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