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MEAN CONVERGENCE THEOREM FOR ARRAYS OF RANDOM ELEMENTS IN MARTINGALE TYPE p BANACH SPACES

BY

S. E. AHMED, S. H. SUNG AND A. I. VOLODIN (沃洛金)

Abstract. For weighted sums of the form $S_n = \sum_{j=u_n}^{v_n} a_{nj}(V_{nj} - c_{nj})$ where $\{u_n, n \geq 1\}$ and $\{v_n, n \geq 1\}$ are sequences of integers, $\{a_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ are constants, $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ are random elements in a real separable martingale type p Banach space, and $\{c_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ are suitable conditional expectations, a mean convergence theorem is established. This result takes the forms $||S_n|| \xrightarrow{\mathcal{L}_r} 0$. No conditions are imposed on the joint distributions of the $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$. The mean convergence theorem is proved assuming that $\{||V_{nj}||^r, u_n \leq j \leq v_n, n \geq 1\}$ is $\{|a_{nj}|^r\}$-uniformly integrable with respect to $\{u_n, v_n\}$ which is weaker than Cesàro uniform integrability. The current work extends that of Gut (1992), Adler, Rosalsky and Volodin (1997) and Sung (1999).

1. Introduction. Let $\{u_n \geq -\infty, n \geq 1\}$ and $\{v_n \leq +\infty, n \geq 1\}$ be two sequences of integers. Consider an array of constants $\{a_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ and an array of random elements $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and taking values in a real separable Banach space $\mathcal{X}$ with norm $||\cdot||$. Let $\{c_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ be a "centering" array consisting of (suitably selected) conditional expectations. In this

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paper, a mean convergence theorem will be established. This convergence result is of the form

$$||\sum_{j=u_n}^{v_n} a_{nj}(V_{nj} - c_{nj})|| \xrightarrow{\mathcal{L}_r} 0.$$

This expression is referred to as *weighted sums* with *weights* $\{a_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ which implies convergence in probability of the weighted sums by the Markov inequality.

The hypotheses to the main result impose conditions on the growth behavior of the weights $\{a_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ and on the marginal distributions of the random variables $\{||V_{nj}||, u_n \leq j \leq v_n, n \geq 1\}$. These two types of conditions are conjoined in Theorem in Sections 2 by a uniform integrability type conditions which generalizes a uniform integrability type condition formulated by Cabrera (1994). In the present inverstigation random elements in the array are not assumed to be rowwise independent. Further, no conditions are imposed on the joint distributions of the random elements comprising the array. However, the Banach space $\mathcal{X}$ is assumed to be of martingale type p.

The main result is an extension to a martingale type p Banach space setting of results of Gut (1992) and Sung (1999), which were proved for arrays of (real-valued) random variables. The main result is analogous to Theorem 1 of Adler, Rosalsky and Volodin (1997). Theorem (which establishes the $\mathcal{L}_r$ convergence result) concerns arrays of random elements satisfying condtion which is more general than the uniform integrability type conditons of Carbrera (1994) and hence more general than the Cesàro uniform integrability condition employed by Gut (1992).

As usual, the symbol C denotes throughout a generic constant ($0 < C < \infty$) which is not necessarily the same one in each appearance. The symbol $I(A)$ denotes the indicator function of an event A.

A real separable Banach space $\mathcal{X}$ is said to be of *martingale type* $p(1 \leq p \leq 2)$ if there exists a finite constant C such that for all martingales $\{S_n, n \geq 1\}$ with values in $\mathcal{X}$,

$$\sup_{n\geq 1} E||S_n||^p \leq C \sum_{n=1}^{\infty} E||S_n - S_{n-1}||^p,$$

where $S_0 \equiv 0$. It can be shown using classical methods from martingale theory that if $\mathcal{X}$ is of martingale type p, then for all $1 \leq r < \infty$ there exists a finite constant C' such that for all $\mathcal{X}$-valued martingales $\{S_n, n \leq 1\}$

$$E\sup_{n\geq 1} ||S_n||^r \geq C' E(\sum_{n=1}^{\infty} ||S_n - S_{n-1}||^p)^{r/p}. \tag{1}$$

For more details, the reader may refer to Pisier (1986).

Let $\{u_n, n \geq 1\}$ and $\{v_n, n \geq 1\}$ be two sequences of integers (not necessarily positive or finite) and $\{a_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ be an array of constants. An array of random variables $\{X_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ is said to be $\{a_{nj}\}$-*uniformly integrable with respect to* (u_n, v_n) if

$$\sup_{n\geq 1} \sum_{j=u_n}^{v_n} |a_{nj}| E|X_{nj}| < \infty \tag{2}$$

and

$$\lim_{a\to\infty} \sup_{n\geq 1} \sum_{j=u_n}^{v_n} |a_{nj}| E|X_{nj}| I(|X_{nj}| > a) = 0. \tag{3}$$

See Sung (1999) for details.

It is easy to prove that condition (3) implies condtions (2) if $\sum_{j=u_n}^{v_n} |a_{nj}| < \infty$. Of course, $\{a_{nj}\}$-uniformly integrability with respect to (u_n, v_n) reduces to $\{a_{nj}\}$-uniform integrability of Cabrera (1994) when $u_n = 1$ and $v_n = k_n$.

2. The main result. To prove the main result we will need the following lemma.

Lemma. *Suppose that* $\{X_{nj},\ u_n \le j \le v_n,\ n \ge 1\}$ *is an array of* $\{\alpha_{nj}\}$*-uniformly integrable with respect to* $(u_n,\ v_n)$ *random variables, where* $\{\alpha_{nj},\ u_n \le j \le v_n,\ n \ge 1\}$ *is an array of constants. Denote* $m_n = 1/\sup_{u_n \le j \le u_n} |\alpha_{nj}|$. *If* $m_n \to \infty$ *as* $n \to \infty$ *and* $q > 1$ *then*

$$\sum_{j=u_n}^{v_n} |\alpha_{nj}|^q E|X_{nj}|^q I\{|X_{nj}| \le m_n\} = o(1).$$

Proof. Take $q = \beta/r, m_n = k_n^{1/r}$ and $m_n \alpha_{nj} X_{nj}$ instead of X_{nj} in Lemma 1(ii) of Sung (1999).

Now we are able to formulate and prove the main result of this paper.

Theorem. *Let* $1 \le r < p \le 2$ *and* $\{V_{nj}, u_n \le j \le v_n, n \ge 1\}$ *be an array of random elements with values in real separable martingale type p Banach space and suppose that array* $\{||V_{nj}||^r, u_n \le j \le v_n, n \ge 1\}$ *is* $\{|a_{nj}|^r\}$*-uniformly integrable with respect to* (u_n, v_n) *where* $\{a_{nj}, u_n \le j \le v_n, n \ge 1\}$ *is an array of constants satisfying* $m_n = 1/\sup_{u_n \le j \le v_n} |a_{nj}| \to \infty$ *as* $n \to \infty$. *Then*

$$|| \sum_{j=u_n}^{v_n} a_{nj}(V_{nj} - E(V_{nj}|\mathcal{F}_{n,j-1}))|| \xrightarrow{\mathcal{L}_r} 0.$$

otbains where $\mathcal{F}_{nj} = \sigma(V_{nj}, u_n \le j \le j), u_n \le j \le v_n$ *and* $\mathcal{F}_{n,u_n-1} = \{\emptyset, \Omega\}, n \ge 1$.

Proof. Let $V'_{nj} = V_{nj} I\{||V_{nj}|| \le m_n\}$ and $V''_{nj} = V_{nj} I\{||V_{nj}|| > m_n\}$. In the addition to these notations denote

$$c_{nj} = E(V_{nj}|\mathcal{F}_{n,j-1}), c'_{nj} = E(V'_{nj}|\mathcal{F}_{n,j-1}) \text{ and } c''_{nj} = E(V''_{nj}|\mathcal{F}_{n,j-1}).$$

Observe that

$$V_{nj}-c_{nj}=(V'_{nj}-c'_{nj})+(V''_{nj}-c''_{nj}).$$

and for every $n\geq 1$ sequences $\{V'_{nj}-c'_{nj},u_n\leq j\leq v_n\}$ and $\{V''_{nj}-c''_{nj},u_n\leq j\leq v_n\}$ are martingale difference sequences. Hence

$$\begin{aligned}
&E||\sum_{j=u_n}^{v_n}a_{nj}(V_{nj}-c_{nj})||^r\\
&\leq C\Big(E||\sum_{j=u_n}^{v_n}a_{nj}(V'_{nj}-c'_{nj})||^r+E||\sum_{j=u_n}^{v_n}a_{nj}V''_{nj}-c''_{nj})||^r\Big)\text{ by }c_r\text{-inequality}\\
&\leq CE(\sum_{j=u_n}^{v_n}|a_{nj}|^p||V'_{nj}-c'_{nj}||^p)^{r/p}+C''\sum_{j=u_n}^{v_n}|a_{nj}|^rE||V''_{nj}-c''_{nj}||^r\text{ by(1)}\\
&\leq C(\sum_{j=u_n}^{v_n}|a_{nj}|^pE||V'_{nj}-c'_{nj}||^p)^{r/p}+C'\sum_{j=u_n}^{v_n}|a_{nj}|^rE||V''_{nj}||^r\\
&\qquad\qquad\text{by Jensen's inequality}\\
&\leq C(\sum_{j=u_n}^{v_n}|a_{nj}|^pE||V_{nj}||^pI\{||V_{nj}||\leq m_n\})^{r/p}\\
&+C\sum_{j=u_n}^{v_n}|a_{nj}|^rE||V_{nj}||^rI\{||V_{nj}||>m_n\}\\
&=o\ (1)\text{ by Lemma 1 with }q=p/r,X_{nj}=||V_{nj}||^r\text{ and }\alpha_{nj}=|a_{nj}|^r.
\end{aligned}$$

Thus the proof is complete.

Remarks 1. In the case $0<r<1$ we don't need to subtract the conditional expectations. That is, the following result is true.

Let $0<r<1$ and $\{V_{nj},u_n\leq j\leq v_n,n\geq 1\}$ be an array of random elements with values in a real separable Banach space and suppose that array $\{||V_{nj}||^r,u_n\leq j\leq v_n,n\geq 1\}$ is $\{|a_{nj}|^r\}$-uniformly integrable with respect to (u_n,v_n) where $\{a_{nj},u_n\leq j\leq u_n,n\geq 1\}$ is an array of constants satisfying

$m_n = 1/\sup_{u_n \le j \le v_n} |a_{nj}| \to \infty$ as $n \to \infty$. Then

$$\Big\| \sum_{j=u_n}^{v_n} a_{nj} V_{nj} \Big\| \xrightarrow{\mathcal{L}_r} 0.$$

obtains.

The proof is basically the same as for the Theorem.

Remark 2. In the case $u_n = 1$ and $v_n = k_n$ Remark 1 and Theorem should be compared with Theorem 6 of Cabrera (1994) and Theorem 1 of Adler, Rosalsky and Volodin (1997), respectively. In result of Cabrera (1994) and of Adler, Rosalsky and Volodin (1997) conditions on the sum $\sum_{j=1}^{k_n} |a_{nj}|^q$ ($q = r$ in Theorem 6 of Cabrera (1994) and $q = p$ in Theorem 1 of Adler, Rosalsky and Volodin (1997)) are assumed. In the paper we don't use such type of conditions.

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Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, S4S 0A2, Canada.

Department of Applied Mathematics, Pai Chai University, Taejon, 302-735, South Korea.

Department of Mathematics and Statistics, University of Regina, Regina, Saskatchwan, S4S 0A2, Canada.