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APPLICATIONS OF THE WEAK l_p EXPONENTIAL INEQUALITIES TO THE LAWS OF LARGE NUMBERS FOR WEIGHTED SUMS

BY

A. I. VOLODIN (KAZAN)

Abstract. We yield new laws of large numbers for weighted sums of random elements taking values in Banach space with the help of the Heinkel's and Pisier's weak l_p exponential inequalities [6].

Exponential bounds (see the table given in [6]) are an important tool in proving limit theorems. We use the weak l_p exponential bounds in the proof of the laws of large numbers (LLN) for weighted sums of random elements taking values in Banach spaces. With the help of Pisier's inequality ([6], Proposition 1.1, and [13], Lemma 2.7) we examine the Strong LLN for certain random elements with values in an arbitrary Banach space and with the help of Heinkel's inequality ([6], Theorem 3.1) we prove the LLN with respect to complete convergence in Banach spaces of type p.

Some results of this paper were announced in [15].

Let E be a separable real Banach space. In the sequel we shall distinguish the notions of random element (taking values in a Banach space) and random variable (assuming values in $\boldsymbol{R}$). A random variable ε is said to be a *Bernoulli random variable* if $\boldsymbol{P}\{\varepsilon = 1\} = \boldsymbol{P}\{\varepsilon = -1\} = 1/2$. Let $(X_k)_{k \leqslant n}$ be independent random elements and $\boldsymbol{a} = \{a_k(n)\colon 1 \leqslant k \leqslant n,\ n \in N\}$ be a triangular array of real numbers which in the sequel will be called a *weight*. We call $T_n = \sum_{k=1}^{n} a_k(n) X_k$ a *weighted sum of random elements* and $S_n = \sum_{k=1}^{n} X_k$ the *unweighted partial sum of random elements*. If $(X_k)_{k \leqslant n}$ are independent copies of the random element X, then we use the notation $T_n(X)$ and $S_n(X)$, respectively.

In the sequel we shall require the following condition on our weight $\boldsymbol{a}$:

(A) There exist $A > 0$ and $p < 2$ such that

$$\max_{k \leqslant n} |a_k(n)| \leqslant A n^{-1/p}$$

for all sufficiently large $n \in N$.

Let $1 < s < \infty$ be given and let $(b_k)_{k \geqslant 1}$ be a sequence of real numbers. Let us set

$$\|(b_k)_{k \leqslant n}\|_{s,\infty} = \max_{k \leqslant n} k^{1/s} b_k^*,$$

where $(b_k^*)_{k \leqslant n}$ is the non-increasing rearrangement of $(|b_k|)_{k \leqslant n}$. For a random variable ξ let us put

$$\Lambda_s(\xi) = (\sup_{t > 0} t^s \boldsymbol{P}\{|\xi| > t\})^{1/s}.$$

To facilitate the formulation of the first theorem we introduce the random element $X = \sum_{i=1}^{\infty} \eta_i x_i$ (in which the series converges almost surely), where (x_i) is a non-random sequence of elements in E and (η_i) is a sequence of independent random variables. Note that we do not require that the sequence (η_i) be identically distributed. We should mention that Matsak [11] proved the central limit theorem for such random elements.

Denote by β_q the symmetric random variable for which the absolute value has the Weibull distribution with parameter $q > 0$, that is, $\boldsymbol{P}\{|\beta_q| > t\} = \exp\{-t^q\}$ and let $(\beta_{q,i})_{i \geqslant 1}$ be the independent copies of β_q.

THEOREM 1. *Let* $X = \sum_{i=1}^{\infty} \eta_i x_i$ *and assume that the weight* $\boldsymbol{a}$ *satisfies the condition* (A). *If there exists* s, $2 > s > p$, *such that*

(1) $W = \sum_{i=1}^{\infty} \beta_{q,i} x_i$ *converges a.s. for* $1/q + 1/s = 1$,

(2) $\Lambda_s(\sup_{i \geqslant 1} |\eta_i|) < \infty$,

then $T_n(X) \to 0$ *a.s. as* $n \to \infty$.

We need some lemmas in order to prove out first theorem. In the sequel we shall use the same notation as in the formulation of the theorem.

LEMMA 1. *If* $W = \sum_{i=1}^{\infty} \beta_{q,i} x_i$ *converges a.s., then it converges in* $\boldsymbol{L}_p(E)$.

Proof. It is sufficient to show that $W \in L_p(E)$ ([14], the Corollary to Theorem V.3.2). It suffices to prove the following inequality for all $t > 0$ (see [14], Theorem V.5.1):

$$\int_t^{\infty} \boldsymbol{P}\{|\beta_q| > u\} u^{p-1} du \leqslant Ct^p \boldsymbol{P}\{|\beta_q| > t\},$$

where the constant C does not depend on t. In the particular case of β_q being a Weibull random variable this inequality can be rewritten as

$$\int_t^{\infty} \exp\{-u^q\} u^{p-1} du \leqslant Ct^p \exp\{-t^q\}.$$

Even a more precise estimation, i.e.,

$$\int_t^{\infty} \exp\{-u^q\} u^{p-1} du \leqslant Ct^{p-q} \exp\{-t^q\}, \tag{1}$$

holds since

$$(2)\qquad t^{p-1}\exp\{-t^q\} \leqslant t^{p-1}[\exp\{-t^q\}]\left(q+\frac{q-p}{p}t^{-q}\right)$$

and $p < 2 < q$. The terms of this inequality are the derivatives of the corresponding terms of (1) with the opposite sign. Hence (1) follows from (2) after the integration of (2) on the half-line (t, ∞). ■

PISIER'S INEQUALITY ([6], Remark after Proposition 1.1; [13], Lemma 2.7). *Let $1 < s < 2$, let q be the conjugate of s, $1/q+1/s = 1$, $(b_k)_{k\leqslant n} \subset \boldsymbol{R}$, and let $(\varepsilon_k)_{k\leqslant n}$ be independent Bernoulli random variables. Then, for $k_s = q(q-2)/2$,*

$$\boldsymbol{P}\{|\sum_{k=1}^{n} b_k\varepsilon_k| > t\} \leqslant 2\exp\{-t^q/(k_s\|(b_k)_{k\leqslant n}\|_{s,\infty})^q\}.$$

The next lemma is only a reformulation of Pisier's inequality.

LEMMA 2. *Let $\tau = \sum_{k=1}^{n} b_k\varepsilon_k$ and $v = \|(b_k)_{k\leqslant n}\|_{s,\infty}\beta_q$, where $(b_k)_{k\leqslant n} \subset \boldsymbol{R}$, and let (ε_k) be a sequence of independent Bernoulli random variables. Then, for any $t > 0$,*

$$\boldsymbol{P}\{|\tau| > t\} \leqslant 2\boldsymbol{P}\{k_s|v| > t\}.$$

Let $(\eta_{k,i})_{k\geqslant 1}$ be independent copies of the sequence (η_i). We introduce the random variables $\mu(i, s, n) = \|(\eta_{k,i})_{k\leqslant n}\|_{s,\infty}$.

LEMMA 3. *Let η_i be a sequence of symmetric random variables, let*

$$X_k = \sum_{i=1}^{\infty} \eta_{k,i}x_i \quad \text{and} \quad Y = \sum_{i=1}^{\infty} \beta_{q,i}\mu(i, s, n)x_i.$$

Then the inequality

$$\boldsymbol{E}\|\sum_{k=1}^{n} X_k\|^p \leqslant C\boldsymbol{E}\|Y\|^p, \quad \text{where } C = 2^{1+p}k_s,$$

holds.

Proof. First consider the case $\eta_{k,i} = \varepsilon_{k,i}b_{k,i}$, where $b_{k,i} \in \boldsymbol{R}$ and $\varepsilon_{k,i}$ are independent Bernoulli random variables. Write

$$\tau_i = \sum_{k=1}^{n} b_{k,i}\varepsilon_{k,i} \quad \text{and} \quad v_i = \beta_{q,i}\|(b_{k,i})_{k\leqslant n}\|_{s,\infty}.$$

It follows from Lemma 2 that $\boldsymbol{P}\{|\tau_i| > t\} \leqslant 2\boldsymbol{P}\{k_s|v_i| > t\}$ for all $i \in \boldsymbol{N}$. Using Theorem V.4.5 of [14] we conclude that

$$\boldsymbol{E}\|\sum_{k=1}^{n} X_k\|^p = \boldsymbol{E}\|\sum_{i=1}^{\infty} \tau_i x_i\|^p \leqslant 2^{1+p}k_s^p\boldsymbol{E}\|\sum_{i=1}^{\infty} v_i x_i\|^p = C\boldsymbol{E}\|Y\|^p.$$

The general case follows from this by a standard procedure. Replace the symmetric random variables η_i by the random variables $\eta_i\varepsilon_i$ which have the same distributions. Here ε_i are independent Bernoulli random variables which are independent of η_i. The left-hand side of the inequality from Lemma 3 is averaged by ε_i for fixed η_i. Finally, by applying Fubini's theorem we obtain the conclusion (see, e.g., the proof of Lemma V.2.1 in [14]). ∎

Note, moreover,

KWAPIEŃ'S INEQUALITY ([14], Lemma V.4.1 (a)). *Let $(X_k)_{k\leqslant n}$ be independent symmetric random elements and $(b_k)_{k\leqslant n}\subset \boldsymbol{R}$. Then for any $t>0$*

$$\boldsymbol{P}\{\|\sum_{k=1}^{n} b_k X_k\| > t\} \leqslant 2\boldsymbol{P}\{\max_{k\leqslant n}|b_k|\,\|\sum_{k=1}^{n} X_k\| > t\}.$$

Proof of Theorem 1. Consider at first the case of symmetric η_i. Let $X_k=\sum_{i=1}^{\infty}\eta_{k,i}x_i$ be independent copies of X. Applying Kwapień's inequality we see that

$$\boldsymbol{P}\{\|T_n(X)\|>\varepsilon\}\leqslant 2\boldsymbol{P}\{\|S_n(X)\|/\max_{k\leqslant n}|a_k(n)|>\varepsilon\}\leqslant 2\boldsymbol{P}\{\|S_n(X)\|/n^{1/p}>\varepsilon/A\}$$

by the condition (A). Hence it is sufficient to prove that $S_n(X)/n^{1/p}\to 0$ a.s. as $n\to\infty$. Let $Z_m=\sum_{i=1}^{m}\eta_i x_i$ and $Z_{k,m}=\sum_{i=1}^{m}\eta_{k,i}x_i$ be independent copies of Z_m. By Lemma 3 and Hoffmann-Jørgensen's inequality ([14], Section V.4, exercise 1 (a)) we obtain

$$\boldsymbol{E}\|S_n(X-Z_m)/n^{1/p}\|^p\leqslant C\boldsymbol{E}\|\sum_{i>m}\beta_{q,i}\mu(i,s,n)\|^p/n$$

$$\leqslant C2^p\boldsymbol{E}(\sup_{i\geqslant 1}\mu(i,s,n))^p\boldsymbol{E}\|\sum_{i>m}\beta_{q,i}x_i\|^p/n.$$

Note that from Lemma 1 we obtain

$$\boldsymbol{E}\|\sum_{i>m}\beta_{q,i}x_i\|^p\to 0\quad \text{as } m\to\infty.$$

Furthermore

$$\boldsymbol{E}(\sup_{i\geqslant 1}\mu(i,s,n))^p\leqslant (1+\int_1^{\infty}\boldsymbol{P}\{\sup_{i\geqslant 1}\|(\eta_{k,i})_{k\leqslant n}\|_{s,\infty}>t\}\,dt^p)$$

$$\leqslant 1+2en\int_1^{\infty}(\Lambda_s(\sup_{i\geqslant 1}|\eta_i|)/t^s)\,dt^p$$

by the Marcus–Pisier result ([6], Proposition 2.2; [13], Lemma 4.11). Since $s>p$, we have

$$\boldsymbol{E}(\sup_{i\geqslant 1}\mu(i,s,n))^p/n\leqslant C\Lambda_s(\sup_{i\geqslant 1}|\eta_i|)+1,$$

whence $\sup_n\boldsymbol{E}\|S_n(X-Z_m)/n^{1/p}\|^p\to 0$ as $m\to\infty$.

Note that the random vector Z_m takes values in the finite-dimensional space $\mathrm{Span}(x_1, \ldots, x_m)$, which has the type p (see the definition below) and $\Lambda_p(\|Z_m\|) < \infty$ since $\Lambda_s(\sup_{i \geq 1}|\eta_i|) < \infty$ and $s > p$. Then $S_n(Z_m)/n^{1/p} \to 0$ in probability as $n \to \infty$ (see [10], Theorem 3.1). From Lemma 3.6 of [12] we may conclude that $S_n(X)/n^{1/p} \to 0$ in probability as $n \to \infty$.

If X is an arbitrary, not necessarily symmetric centered random element which satisfies the hypotheses of Theorem 1, then its symmetrization X^s also satisfies the hypotheses of Theorem 1. Consequently, by Lemma V.3.4 (a) of [14] we obtain

$$\sup_n E\|T_n(X - Z_m)/n^{1/p}\| \leqslant 2 \sup_n E\|T_n(X^s - Z_m)/n^{1/p}\| \to 0 \quad \text{as } m \to \infty.$$

It remains to refer to the above-mentioned result of Norvaisa ([12], Lemma 3.6). So, $S_n(X)/n^{1/p} \to 0$ in probability as $n \to \infty$.

Note that, by [2], Theorem III.2.14, and our Lemma 1,

$$E\|X\|^p \leqslant CE\|W\|^p < \infty,$$

where

$$C = 8^p E(\sup_i |\eta_i|)^p (E|\beta|)^{-p} \quad \text{and} \quad E(\sup_i |\eta_i|)^p < \infty.$$

Consequently, by [3] we have $S_n(X)/n^{1/p} \to 0$ a.s. ■

Now we shall return to the study of LLN with respect to complete convergence.

The sequence of random elements (Y_n) *converges completely* to zero if for all $\varepsilon > 0$ the series $\sum_{n=1}^{\infty} P\{\|Y_n\| > \varepsilon\}$ converges.

This definition was introduced by Hsu and Robbins [7], where it was shown that the sequence of arithmetic means of independent and identically distributed random variables converges completely to the expected value of the sums whenever their variance is finite. The converse was proved by Erdös [4]. This result was generalized in various ways and we can refer to papers of Adler [1], Gut [5] and Klesov [9] for further information. The Banach space situation was examined by T.-C. Hu et al. [8].

Recall (see [13]) that a Banach space E is said to be of *Rademacher type* p $(1 \leqslant p \leqslant 2)$ if for any sequence $(x_k) \subset E$ the convergence of the series $\sum_{k=1}^{\infty} \|x_k\|^p$ implies the a.s. convergence of series $\sum_{k=1}^{\infty} \varepsilon_k x_k$, where ε_k are i.i.d. Bernoulli random variables. Analogously, a Banach space E is said to be of *stable type* p $(1 \leqslant p \leqslant 2)$ if for any sequence $(x_k) \subset E$ the convergence of the series $\sum_{k=1}^{\infty} \|x_k\|^p$ implies the a.s. convergence of series $\sum_{k=1}^{\infty} \gamma_k x_k$, where γ_k are i.i.d. p-stable random variables with characteristic function $\exp\{-|t|^p\}$.

For a Banach space E let us set $p(E) = \sup\{p\colon E \text{ is of stable type } p\}$. Two facts are well known (see [13]):

(1) the interval of stable types is opened, that is, if E is of stable type $p < 2$, then $p(E) > p$;

(2) the interval of Rademacher types is closed, that is, E is of Rademacher type $p(E)$.

We shall say that the sequence of random variables (X_k) is *stochastically dominated* by a positive random variable ξ, and we write $(X_k) \prec \xi$, if for any $t > 0$

$$\sup_k \boldsymbol{P}\{\|X_k\| > t\} \leqslant \boldsymbol{P}\{\xi > t\}.$$

For the proof of the next theorem we need

HEINKEL'S INEQUALITY (see [6], Theorem 3.1). *Let (X_k) be independent symmetric random elements with values in the Banach space E with $p(E) > 1$, $1 < s < p(E)$. Let q be the conjugate of s, i.e., $1/q + 1/s = 1$. Then there exist positive constants $L = L(s)$ and $M = M(s, p(E))$ such that, for all $t > 0$ and $\varepsilon > 0$,*

$$\boldsymbol{P}\{\|S_n\| > \varepsilon\} \leqslant \boldsymbol{P}\{\|(\|X_k\|)_{k \leqslant n}\|_{s,\infty} > t\} + M \exp\{-L(\varepsilon/t)^q\}.$$

At this time we introduce the class of random variables $A_r = \{\xi: \boldsymbol{E}|\xi^-|^r \leqslant 1\}$.

Now we can formulate the law of large numbers with respect to uniform variant of complete convergence.

THEOREM 2. *Let E be a Banach space of stable type p $(1 < p < 2)$, of independent symmetric random elements $(X_k) \prec \xi$ and weight a satisfying the condition* (A). *Then, for all $r > 2pp(E)/[p(E)-1]$ and $\varepsilon > 0$,*

$$\sup_{\xi \in A_r} \sum_{n=1}^{\infty} \boldsymbol{P}\{\|T_n\| > \varepsilon\} < \infty.$$

Proof. Fix u such that $r > u > 2pp(E)/[p(E)-1]$. Let $v = 2/u$. Note that $r > 2/v$ and $v < 1/p - 1/p(E)$. Fix some s so that $p < s < p(E)$ satisfying $v < 1/p - 1/s < 1/p - 1/p(E)$, that is, $1/p - 1/s - v > 0$.

Note that, by Kwapień's inequality,

$$\boldsymbol{P}\{\|T_n\| > \varepsilon\} \leqslant 2\boldsymbol{P}\{\max_{k \leqslant n} |a_k(n)| \, \|S_n\| > \varepsilon\} \leqslant 2\boldsymbol{P}\{\|S_n\|/n^{1/p} > \varepsilon/A\}.$$

Hence it is sufficient to prove that

$$\sup_{\xi \in A_r} \sum_{n=N}^{\infty} \boldsymbol{P}\{\|S_n\| > \varepsilon n^{1/p}\} \to 0 \quad \text{as } N \to \infty.$$

Set $X'_{k,n} = X_k I\{\|X_k\| \leqslant n^v\}$ and $U_n = (\sum_{k=1}^{n} X'_{k,n})/n^{1/p}$. Since

$$\{\|S_n\|/n^{1/p} > \varepsilon\} \subset \{\|U_n\| > \varepsilon\} \cup \{S_n/n^{1/p} \neq U_n\},$$

it follows that

$$\sum_{n=N}^{\infty} \boldsymbol{P}\{\|S_n/n^{1/p}\| > \varepsilon\} \leqslant \sum_{n=N}^{\infty} \boldsymbol{P}\{\|U_n\| > \varepsilon\} + \sum_{n=N}^{\infty} \boldsymbol{P}\{S_n/n^{1/p} \neq U_n\} = \sum\nolimits_N^1 + \sum\nolimits_N^2.$$

We estimate each of the sums $\sum_N^1$ and $\sum_N^2$ separately. First note that by Heinkel's inequality with $C = n^{v+1/s-1/p}$ we have

$$P\{\|U_n\| > \varepsilon\} \leqslant P\{\|(X'_{k,n}/n^{1/p})_{k \leqslant n}\|_{s,\infty} > n^{v+1/s-1/p}\} \\ + M \exp\{-L(\varepsilon n^{-v-1/s+1/p})^q\}.$$

Note that the first term on the right-hand side of the last inequality is equal to zero if

$$\|(X'_{k,n}/n^{1/p})_{k \leqslant n}\|_{s,\infty} \leqslant n^v \max_{k \leqslant n} k^{1/s}/n^{1/p} \leqslant n^{v+1/s-1/p}.$$

So $P\{\|U_n\| > \varepsilon\} \leqslant M \exp\{-L\varepsilon^q n^w\}$, where $w = q(1/p-1/s-v) > 0$. Then

$$\sum_N^1 \leqslant M \sum_{n=N}^{\infty} \exp\{-L\varepsilon^q n^w\} \to 0 \quad \text{for } N \to \infty.$$

Next observe that

$$\begin{aligned} \sum_N^2 &= \sum_{n=N}^{\infty} P\{S_n/n^{1/p} \neq U_n\} \leqslant \sum_{n=N}^{\infty} \sum_{k=1}^{n} P\{\|X_k\| > n^v\} \\ &\leqslant \sum_{n=N}^{\infty} nP\{\xi > n^v\} \leqslant \sum_{n=N}^{\infty} n \sum_{k=n}^{\infty} P\{k < \xi^{1/v} < k+1\} \\ &\leqslant \sum_{k=N}^{\infty} P\{k < \xi^{1/v} \leqslant k+1\} \sum_{n=1}^{k} n \leqslant C \sum_{k=N}^{\infty} k^2 P\{k < \xi^{1/v} \leqslant k+1\} \\ &\leqslant CE\xi^{2/v} I\{\xi > N\} \leqslant CN^{2/v-r} E\xi^r \leqslant CN^{2/v-r} \to 0 \quad \text{as } N \to \infty \end{aligned}$$

whenever $r > 2/v$. ■

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Research Institute of Mathematics
and Mechanics of Kazan University
Universitetskaya st. 17
Kazan, 420008, Tatarstan, Russia

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