ELSEVIER

Statistics & Probability Letters 55 (2001) 301–309

STATISTICS & PROBABILITY LETTERS

www.elsevier.com/locate/stapro

On conditional compactly uniform pth-order integrability of random elements in Banach spaces ☆

Manuel Ordóñez Cabrera[a,*], Andrei I. Volodin[b]

[a] Department of Mathematical Analysis, University of Sevilla, Apdo. De Correos 1160, Sevilla 41080, Spain
[b] Department of Mathematics and Statistics, University of Regina, Regina, Saskatchewan, Canada S4S 0A2

Received March 2001; received in revised form June 2001

Abstract

The notion of conditional compactly uniform pth-order integrability of an array of random elements in a separable Banach space concerning an array of random variables and relative to a sequence of σ-algebras is introduced and characterized. We state a conditional law for randomly weighted sums of random elements in a Banach space with the bounded approximation property, and we prove that, under the introduced condition, the problem can be reduced to a similar problem for random elements in a finite-dimensional space. © 2001 Elsevier Science B.V. All rights reserved

MSC: 60B11; 60B12

Keywords: Random elements; Randomly weighted sums; Conditional compactly uniform pth-order integrability; Conditional tightness; Conditional uniform integrability; Bounded approximation property; Schauder basis

1. Introduction

In view of the random nature of many problems arising in the applied sciences, the limiting behaviour of weighted partial sums of random elements in normed linear spaces with random weights (i.e., when the weights are random variables) has started being studied since the begining of the 1970s. The interested reader can find a complete list of references about this area in Rosalsky and Sreehari (1998).

We consider now: (a) two sequences of integers $\{u_n \geqslant -\infty, n \geqslant 1\}$ and $\{v_n \leqslant +\infty, n \geqslant 1\}$, with $v_n > u_n$ for all $n \geqslant 1$; (b) an array of random elements $\{V_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ defined on a probability space $(\Omega, \mathscr{A}, P)$ and taking values in a real separable Banach space $(\mathscr{X}, \|.\|)$; (c) an array of random variables $\{A_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ defined on $(\Omega, \mathscr{A}, P)$. We consider the randomly weighted sums $\sum_{j=u_n}^{v_n} A_{nj} V_{nj}$. When

☆ The research of M. Ordóñez Cabrera has been partially supported by DGICYT grant BFM2000-0344-C02-01 and Junta de Andalucia FQM 127.

* Corresponding author.
E-mail address: cabrera@cica.es (M.O. Cabrera).

PII: S0167-7152(01)00159-6

$u_n = -\infty$ or $v_n = +\infty$, we assume that the series converges. When $u_n = 1, v_n = n, n \geqslant 1$, we have the usual randomly weighted partial sums.

When a triangular array of nonrandom weights $A_{nj} \equiv a_{nj}$ is considered, the uniform integrability of $\{V_{nj}\}$- or $\{a_{nj}\}$-uniform integrability of $\{V_{nj}\}$ are powerful notions for obtaining weak laws of large numbers. The interested reader can refer to Gut (1992), Cabrera (1994) and Sung (1999).

These concepts are extended in a natural way, and they give rise to the concepts of compactly uniform pth-order integrability or $\{a_{nj}\}$-compactly uniform pth-order integrability of $\{V_{nj}\}$. The interested reader can refer to Wang and Rao (1987), Cuesta and Matrán (1988) and Cabrera (1997).

Hu et al. (2001) introduced some notions of uniform integrability of an array $\{V_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ of random elements with respect to an array $\{A_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ of random variables; from among those, they emphazise the notion of $\{A_{nj}\}$-conditional uniform integrability relative to a sequence $\{\mathscr{B}_n, n \geqslant 1\}$ of sub σ-algebras of $\mathscr{A}$, which allows one to obtain conditional laws of large numbers for randomly weighted sums of random elements.

In this note, the notion of $\{A_{nj}\}$-conditional compactly uniform pth-order integrability relative to $\{\mathscr{B}_n\}$ is introduced and characterized in terms of $\{A_{nj}\}$-conditional uniform integrability relative to $\{\mathscr{B}_n\}$- and $\{A_{nj}\}$-conditional tightness relative to $\{\mathscr{B}_n\}$. In the particular case of an array of constants $\{a_{nj}\}$, this characterization allows one to obtain the characterization of $\{a_{nj}\}$-compactly uniform pth-order integrability of $\{V_{nj}\}$, which was an open problem.

The various concepts of compactly uniform integrability (in respect of constants or not) have often been used by their authors to obtain limit laws for sums of random elements in a Banach space $\mathscr{X}$ which has a Schauder basis. The crucial point for the fruitfulness of the obtained results is the fact that the identity operator in $\mathscr{X}$ can be approximated by the partial sums operators corresponding to the Schauder basis, uniformly on compact sets. This is a consequence of the fact that the existence of a Schauder basis in a Banach space implies that the Banach space has the bounded approximation property (BAP in short) and the approximation property.

It was a very famous open problem for many years as to whether the existence of a Schauder basis in a separable Banach space is equivalent to the BAP. Pelczynski (1971) and Johnson et al. (1971), independently, discovered that a separable Banach space $\mathscr{X}$ has the BAP if and only if $\mathscr{X}$ is isomorphic to a complemented subspace of a space with a basis (see also Lindenstrauss and Tzafriri, 1977), but the question remained open until Szarek (1987) proved that there exists a separable Banach space which has the BAP but fails to have a basis, i.e., the BAP is a condition weaker that the existence of a Schauder basis.

In the last section of this note, we extend the habitual field of applications of the concepts of compactly uniform integrability to the scope of separable Banach spaces with BAP, and we show that under the hypothesis of conditional compactly uniform pth-order integrability, the study of a conditional law of large numbers in a separable Banach space with BAP can be reduced to the study of a similar limit law for finite-dimensional random elements.

2. Definitions

Let $\{u_n, n \geqslant 1\}$ and $\{v_n, n \geqslant 1\}$ be two sequences of integers (not necessary positive or finite) such that $v_n > u_n$ for all $n \geqslant 1$ and $v_n - u_n \to \infty$ as $n \to \infty$. Consider an array of random variables $\{A_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ and an array $\{V_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ of random elements in a real separable Banach space $\mathscr{X}$ with norm $\|\cdot\|$, defined on a probability space $(\Omega, \mathscr{A}, P)$. Let $\{\mathscr{B}_n, n \geqslant 1\}$ be a sequence of sub σ-algebras of $\mathscr{A}$. For each $n \geqslant 1$, denote by $E^{\mathscr{B}_n}(Y)$ the conditional expectation of the random variable Y relative to $\mathscr{B}_n$, and by $P^{\mathscr{B}_n}(A)$ the conditional probability of the event $A \in \mathscr{A}$ relative to $\mathscr{B}_n$.

(1) We recall that $\{V_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ is said to be *tight* if for all $\varepsilon > 0$ there exists a compact subset K of $\mathscr{X}$ such that

$$\sup_{n \geqslant 1} \sup_{u_n \leqslant j \leqslant v_n} P[V_{nj} \notin K] < \varepsilon.$$

(2) The following definition was introduced in Cabrera (1997).

Let $\{a_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ be an array of constants. Let $p > 0$. $\{V_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ is said to be $\{a_{nj}\}$-*compactly uniformly pth-order integrable* if for all $\varepsilon > 0$, there exists a compact subset K of $\mathscr{X}$ such that

$$\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|a_{nj}|E(\|V_{nj}\|^p I_{[V_{nj}\notin K]}) < \varepsilon.$$

(3) The following definition was introduced in Hu et al. (2001).

$\{V_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ is said to be $\{A_{nj}\}$-*conditionally uniformly integrable relative to* $\{\mathscr{B}_n\}$ if for all $\varepsilon > 0$, there exists $a_0 > 0$ such that

$$\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|E^{\mathscr{B}_n}(\|V_{nj}\| I_{[\|V_{nj}\|>a_0]}) < \varepsilon \quad \text{a.e.}$$

We introduce the following new definitions now.

(4) Let $\{a_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ be an array of constants. We say that $\{V_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ is $\{a_{nj}\}$-*tight* if for all $\varepsilon > 0$ there exists a compact subset K of $\mathscr{X}$ such that

$$\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|a_{nj}|P[V_{nj}\notin K] < \varepsilon.$$

Remark. If $\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|a_{nj}| \leqslant C$ for some constant $C > 0$, then (1) $\Rightarrow$ (4).

(5) We say that $\{V_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ is *conditionally tight relative to* $\{\mathscr{B}_n\}$ if for all $\varepsilon > 0$ there exists a compact subset K of $\mathscr{X}$ such that

$$\sup_{n\geqslant 1}\sup_{u_n\leqslant j\leqslant v_n} P^{\mathscr{B}_n}[V_{nj}\notin K] < \varepsilon \quad \text{a.e.}$$

Remark. Note that

$$P[V_{nj}\notin K] = E(I_{[V_{nj}\notin K]}) = E(E^{\mathscr{B}_n}(I_{[V_{nj}\notin K]})) = E(P^{\mathscr{B}_n}[V_{nj}\notin K]).$$

Thus, (5) $\Rightarrow$ (1).

(6) We say that $\{V_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ is $\{A_{nj}\}$-*conditionally tight relative to* $\{\mathscr{B}_n\}$ if for all $\varepsilon > 0$ there exists a compact subset K of $\mathscr{X}$ such that

$$\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|P^{\mathscr{B}_n}[V_{nj}\notin K] < \varepsilon \quad \text{a.e.}$$

Remark. If $\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}| \leqslant C$ a.e. for some constant $C > 0$, then (5) $\Rightarrow$ (6).

(7) Let $p > 0$. We say that $\{V_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ is $\{A_{nj}\}$-*conditionally compactly uniformly pth-order integrable relative to* $\{\mathscr{B}_n\}$ if for all $\varepsilon > 0$ there exists a compact subset K of $\mathscr{X}$ such that

$$\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|E^{\mathscr{B}_n}(\|V_{nj}\|^p I_{[V_{nj}\notin K]}) < \varepsilon \quad \text{a.e.}$$

If $p = 1$, we say that $\{V_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ is $\{A_{nj}\}$-*conditionally compactly uniformly integrable relative to* $\{\mathscr{B}_n\}$.

3. Characterization

In this section we will obtain the characterization of the concept of conditional compactly uniform pth-order integrability which has been introduced in the previous section. We will also obtain, as a particular case, a characterization of the concept of $\{a_{nj}\}$-compactly uniform pth-order integrability (with $\{a_{nj}\}$ being an array of constants).

Theorem 1. *Let $\{A_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ be an array of random variables and let $\{V_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ be an array of random elements in a separable Banach space $\mathscr{X}$ with norm $\|\cdot\|$, defined on a probability space $(\Omega, \mathscr{A}, P)$, with $\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}| \leqslant C$ a.e., for some constant $C < \infty$. Let $\{\mathscr{B}_n, n \geqslant 1\}$ be a sequence of sub σ-algebras of $\mathscr{A}$. Let $p > 0$.*

Then, $\{V_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ is $\{A_{nj}\}$-conditionally compactly uniformly pth-order integrable relative to $\{\mathscr{B}_n\}$ if, and only if, $\{\|V_{nj}\|^p, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ is $\{A_{nj}\}$-conditionally uniformly integrable relative to $\{\mathscr{B}_n\}$ and $\{V_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ is $\{A_{nj}\}$-conditionally tight relative to $\{\mathscr{B}_n\}$.

Proof. Let $\{\|V_{nj}\|^p, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ be $\{A_{nj}\}$-conditionally uniformly integrable relative to $\{\mathscr{B}_n\}$. By Theorem 1 of Hu et al. (2001), given $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $\{B_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ is an array of events satisfying $\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|P^{\mathscr{B}_n}(B_{nj}) < \delta$ a.e., then

$$\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|E^{\mathscr{B}_n}(\|V_{nj}\|^p I_{B_{nj}}) < \varepsilon \quad \text{a.e.}$$

By hypothesis of $\{A_{nj}\}$-conditional tightness, there exists a compact subset K of $\mathscr{X}$ such that $\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|P^{\mathscr{B}_n}[V_{nj} \notin K] < \delta$ a.e., and therefore

$$\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|E^{\mathscr{B}_n}(\|V_{nj}\|^p I_{[V_{nj}\notin K]}) < \varepsilon \quad \text{a.e.}$$

So, $\{V_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ is $\{A_{nj}\}$-conditionally compactly uniformly pth-order integrable relative to $\{\mathscr{B}_n\}$.

Conversely, suppose that $\{V_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ is $\{A_{nj}\}$-conditionally compactly uniformly pth-order integrable relative to $\{\mathscr{B}_n\}$.

Given $\varepsilon > 0$, for each $i \in \mathbb{N}$ there exists a compact $K_i \subset \mathscr{X}$ such that

$$\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|E^{\mathscr{B}_n}(\|V_{nj}\|^p I_{[V_{nj}\notin K_i]}) < \frac{\varepsilon}{i^p 2^i} \quad \text{a.e.}$$

Denote by $B(0, 1/i)$ the open ball in $\mathscr{X}$ with center 0 and radius $1/i$. Then

$$\begin{aligned} E^{\mathscr{B}_n}(\|V_{nj}\|^p I_{[V_{nj}\notin K_i]}) &\geqslant E^{\mathscr{B}_n}(\|V_{nj}\|^p I_{[V_{nj}\in K_i^{\mathrm{c}}\cap B^{\mathrm{c}}(0,1/i)]}) \\ &\geqslant \frac{1}{i^p}E^{\mathscr{B}_n} I_{[V_{nj}\in K_i^{\mathrm{c}}\cap B^{\mathrm{c}}(0,\frac{1}{i})]} = \frac{1}{i^p}P^{\mathscr{B}_n}\left[V_{nj}\in K_i^{\mathrm{c}}\cap B^{\mathrm{c}}\left(0,\frac{1}{i}\right)\right] \quad \text{a.e.,} \end{aligned}$$

which implies that

$$\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|P^{\mathscr{B}_n}\left[V_{nj}\in K_i^{\mathrm{c}}\cap B^{\mathrm{c}}\left(0,\frac{1}{i}\right)\right] \leqslant \sup_{n\geqslant 1} i^p\sum_{j=u_n}^{v_n}|A_{nj}|E^{\mathscr{B}_n}(\|V_{nj}\|^p I_{[V_{nj}\notin K_i]}) < \frac{\varepsilon}{2^i} \quad \text{a.e.}$$

Let now $K=\bigcap_{i\in\mathbb{N}}(K_i\cup B(0,1/i))$. The closure of K in $\mathscr{X}$, $\bar{K}$, is a compact set (see Lemma 2.2 in Wang and Rao, 1987). Then

$$P^{\mathscr{B}_n}[V_{nj}\in\bar{K}^{\mathrm{c}}]\leqslant P^{\mathscr{B}_n}[V_{nj}\in K^{\mathrm{c}}]$$

$$=P^{\mathscr{B}_n}\left[V_{nj}\in\bigcup_{i\in\mathbb{N}}\left(K_i^{\mathrm{c}}\cap B^{\mathrm{c}}\left(0,\frac{1}{i}\right)\right)\right]\leqslant\sum_{i=1}^{\infty}P^{\mathscr{B}_n}\left[V_{nj}\in K_i^{\mathrm{c}}\cap B^{\mathrm{c}}\left(0,\frac{1}{i}\right)\right]\quad\text{a.e.}$$

Therefore, for each $n\in\mathbb{N}$

$$\sum_{j=u_n}^{v_n}|A_{nj}|P^{\mathscr{B}_n}[V_{nj}\notin\bar{K}]\leqslant\sum_{j=u_n}^{v_n}|A_{nj}|\sum_{i=1}^{\infty}|P^{\mathscr{B}_n}\left[V_{nj}\in K_i^{\mathrm{c}}\cap B^{\mathrm{c}}\left(0,\frac{1}{i}\right)\right]$$

$$=\sum_{i=1}^{\infty}\sum_{j=u_n}^{v_n}|A_{nj}|P^{\mathscr{B}_n}\left[V_{nj}\in K_i^{\mathrm{c}}\cap B^{\mathrm{c}}\left(0,\frac{1}{i}\right)\right]<\sum_{i=1}^{\infty}\frac{\varepsilon}{2^i}=\varepsilon\quad\text{a.e.}$$

So, $\{V_{nj},u_n\leqslant j\leqslant v_n,n\geqslant 1\}$ is $\{A_{nj}\}$-conditionally tight relative to $\{\mathscr{B}_n\}$.

On the other hand, given $\varepsilon>0$, there exists a compact subset K of $\mathscr{X}$ such that

$$\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|E^{\mathscr{B}_n}(\|V_{nj}\|^pI_{[V_{nj}\notin K]})<\varepsilon\text{ a.e.}$$

The compactness of K implies that there exists $r>0$ such that $K\subset\bar{B}(0,r)$, and so $[\|V_{nj}\|>r]\subset[V_{nj}\notin K]$ for every $n\geqslant 1,u_n\leqslant j\leqslant v_n$.

Therefore,

$$\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|E^{\mathscr{B}_n}(\|V_{nj}\|^pI_{[\|V_{nj}\|>r]})\leqslant\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|E^{\mathscr{B}_n}(\|V_{nj}\|^pI_{[V_{nj}\notin K]})<\varepsilon\quad\text{a.e.}$$

Thus, $\{\|V_{nj}\|^p,u_n\leqslant j\leqslant v_n,n\geqslant 1\}$ is $\{A_{nj}\}$-conditionally uniformly integrable relative to $\{\mathscr{B}_n\}$. □

Cabrera (1997) obtained a sufficient condition for $\{a_{nj}\}$-compactly uniform pth-order integrability, but not a characterization (i.e., a necessary and sufficient condition) of this concept. Actually, the desired characterization follows from Theorem 1 by taking $\{A_{nj},\ u_n\leqslant j\leqslant v_n,n\geqslant 1\}$ to be an array of real constants $\{a_{nj},u_n\leqslant j\leqslant v_n,n\geqslant 1\}$ and $\mathscr{B}_n=\{\emptyset,\Omega\}$ for every $n\in\mathbb{N}$:

Corollary 1. *Let $\{V_{nj},u_n\leqslant j\leqslant v_n,n\geqslant 1\}$ be an array of random elements in a separable Banach space $\mathscr{X}$, and let $\{a_{nj},u_n\leqslant j\leqslant v_n,n\geqslant 1\}$ be an array of real constants such that $\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|a_{nj}|\leqslant C$, for some constant $C<\infty$. Let $p>0$.*

Then, $\{V_{nj},u_n\leqslant j\leqslant v_n,n\geqslant 1\}$ is $\{a_{nj}\}$-compactly uniformly pth-order integrable if, and only if, $\{\|V_{nj}\|^p,u_n\leqslant j\leqslant v_n,n\geqslant 1\}$ is $\{a_{nj}\}$-uniformly integrable and $\{V_{nj},u_n\leqslant j\leqslant v_n,\ n\geqslant 1\}$ is $\{a_{nj}\}$-tight.

4. A conditional limit law for randomly weighted sums

In the main result of this section we will prove that the condition of $\{A_{nj}\}$-conditional compactly uniform pth-order integrability relative to a certain sequence of σ-algebras $\{\mathscr{B}_n\}$ allows us to reduce the study of conditional convergence of a randomly weighted sum of random elements in some infinite-dimensional Banach spaces to a similar problem posed for random elements in finite-dimensional spaces.

In this section, the term "operator" means a continuous linear operator.

Recall that a separable Banach space $\mathscr{X}$ has the bounded approximation property (BAP) if the identity operator I on $\mathscr{X}$ is a pointwise limit of a sequence of finite rank operators in the strong operator topology; that is, if there exists a sequence $\{T_n, n \geqslant 1\}$ of finite rank operators on $\mathscr{X}$ such that

$$\lim_{n\to\infty} \left\| x - \sum_{k=1}^{n} T_k(x) \right\| = 0 \quad \text{for } x \in \mathscr{X}.$$

An application of Banach–Steinhaus principle implies that there exists a constant $M < \infty$ such that $\| \sum_{k=1}^{n} T_k \| \leqslant M$ for all $n \geqslant 1$.

Throughout this section, we denote $U_n = \sum_{k=1}^{n} T_k$.

We will need the following lemma:

Lemma 1. *Let $\mathscr{X}$ be a separable Banach space with the BAP. Let K be a compact subset of $\mathscr{X}$. Then, for each $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $\|x - U_n(x)\| < \varepsilon$ for all $x \in K$ and $n \geqslant n_0$.*

Proof. Given $\varepsilon > 0$, consider the covering of K by the family of open balls

$$\left\{ B\left(y, \frac{\varepsilon}{2(M+1)} \right), y \in K \right\}.$$

The compactness of K implies that there exist $y_1, y_2, \ldots, y_p \in K$ such that

$$K \subset \bigcup_{i=1}^{p} B\left(y_i, \frac{\varepsilon}{2(M+1)} \right).$$

Then, for every $x \in K$ there exists $j \in \{1, 2, \ldots, p\}$ such that $\|x - y_j\| < \varepsilon/2(M+1)$.

As $x = \lim_{n\to\infty} U_n(x)$ pointwise, there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that $\|y_i - U_n(y_i)\| < \varepsilon/2$ for all $i \in \{1, 2, \ldots, p\}$ and for every $n \geqslant n_0$.

Therefore, for all $x \in K$ and $n \geqslant n_0$:

$$\begin{aligned} \|x - U_n(x)\| &\leqslant \|x - y_j\| + \|y_j - U_n(y_j)\| + \|U_n(y_j) - U_n(x)\| \\ &\leqslant (M+1)\|x - y_j\| + \|y_j - U_n(y_j)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square \end{aligned}$$

The main result in this section is the following:

Theorem 2. *Let $\mathscr{X}$ be a separable Banach space with BAP. Let $\{A_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ be an array of random variables such that $\sup_{n\geqslant 1} \sum_{j=u_n}^{v_n} |A_{nj}| \leqslant C$ a.e., for some constant $C < \infty$. Let $\mathscr{B}_n = \sigma(A_{nj}, u_n \leqslant j \leqslant v_n)$, for each $n \geqslant 1$, and let $\{V_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ be an array of random elements in $\mathscr{X}$ which is $\{A_{nj}\}$-conditionally compactly uniformly pth-order integrable relative to $\{\mathscr{B}_n\}$, for some $p > 0$.*

Then $E^{\mathscr{B}_n} \sum_{j=u_n}^{v_n} |A_{nj}| \|V_{nj}\|^p \to 0$ a.e. as $n \to \infty$ if, and only if, there exists $t_1 \in \mathbb{N}$ such that $E^{\mathscr{B}_n} \sum_{j=u_n}^{v_n} |A_{nj}| \|U_t(V_{nj})\|^p \to 0$ a.e. for each $t \geqslant t_1$, as $n \to \infty$.

Proof. Suppose that $E^{\mathscr{B}_n} \sum_{j=u_n}^{v_n} |A_{nj}| \|U_t(V_{nj})\|^p \to 0$ a.e. for each $t \geqslant t_1$.

Given $\varepsilon > 0$, there exists a compact $K \subset \mathscr{X}$ such that

$$\sup_{n\geqslant 1} \sum_{j=u_n}^{v_n} |A_{nj}| E^{\mathscr{B}_n}(\|V_{nj}\|^p I_{[V_{nj}\notin K]}) < \frac{\varepsilon}{4C_p^2(M+1)^p} \quad \text{a.e.,}$$

where $C_p = 2^{p-1}$ if $p \geqslant 1$ or $C_p = 1$ if $p \in (0, 1)$.

Define, for each $n \in \mathbb{N}$, $u_n \leqslant j \leqslant v_n$:

$$W_{nj} = V_{nj} I_{[V_{nj} \in K]}, \qquad Y_{nj} = V_{nj} I_{[V_{nj} \notin K]}.$$

The compactness of K implies (see Lemma 1) that there exists $t_0 \in \mathbb{N}$ such that for every $t \geqslant t_0$ and every $n \in \mathbb{N}$, $u_n \leqslant j \leqslant v_n$

$$\|W_{nj} - U_t(W_{nj})\| < \left(\frac{\varepsilon}{4C_p^2 C}\right)^{1/p},$$

which implies

$$E^{\mathscr{B}_n}\|W_{nj} - U_t(W_{nj})\|^p < \frac{\varepsilon}{4C_p^2 C} \quad \text{a.e.}$$

Moreover, for every $n \in \mathbb{N}$

$$\sum_{j=u_n}^{v_n} |A_{nj}| E^{\mathscr{B}_n}\|Y_{nj} - U_t(Y_{nj})\|^p \leqslant (M+1)^p \sum_{j=u_n}^{v_n} |A_{nj}| E^{\mathscr{B}_n}\|Y_{nj}\|^p < \frac{\varepsilon}{4C_p^2} \quad \text{a.e.}$$

Therefore, for every $t \geqslant t_0$

$$\begin{aligned}
&\sup_{n\geqslant 1} E^{\mathscr{B}_n} \sum_{j=u_n}^{v_n} |A_{nj}| \|V_{nj} - U_t(V_{nj})\|^p \\
&\quad \leqslant \sup_{n\geqslant 1} \sum_{j=u_n}^{v_n} |A_{nj}| E^{\mathscr{B}_n}(\|W_{nj} - U_t(W_{nj})\| + \|Y_{nj} - U_t(Y_{nj})\|)^p \\
&\quad \leqslant \sup_{n\geqslant 1} C_p \left[\sum_{j=u_n}^{v_n} |A_{nj}| E^{\mathscr{B}_n}\|W_{nj} - U_t(W_{nj})\|^p + \sum_{j=u_n}^{v_n} |A_{nj}| E^{\mathscr{B}_n}\|Y_{nj} - U_t(Y_{nj})\|^p\right] \leqslant \frac{\varepsilon}{2C_p} \quad \text{a.e.}
\end{aligned}$$

We take now $t \geqslant \max\{t_0, t_1\}$. By hypothesis, there exists $n_0 \in \mathbb{N}$ such that

$$\text{for all } n \geqslant n_0,\ E^{\mathscr{B}_n} \sum_{j=u_n}^{v_n} |A_{nj}| \|U_t(V_{nj})\|^p < \varepsilon/2C_p \text{ a.e.}$$

Therefore, for all $n \geqslant n_0$

$$\begin{aligned}
E^{\mathscr{B}_n} \sum_{j=u_n}^{v_n} |A_{nj}| \|V_{nj}\|^p &= E^{\mathscr{B}_n} \sum_{j=u_n}^{v_n} |A_{nj}| \|V_{nj} - U_t(V_{nj}) + U_t(V_{nj})\|^p \\
&\leqslant C_p \left[E^{\mathscr{B}_n} \sum_{j=u_n}^{v_n} |A_{nj}| \|V_{nj} - U_t(V_{nj})\|^p + E^{\mathscr{B}_n} \sum_{j=u_n}^{v_n} |A_{nj}| \|U_t(V_{nj})\|^p\right] < \varepsilon \quad \text{a.e.}
\end{aligned}$$

On the other hand, suppose that $E^{\mathscr{B}_n} \sum_{j=u_n}^{v_n} |A_{nj}| \|V_{nj}\|^p \to 0$ a.e.
Then, it is immediate that for every t, $n \in \mathbb{N}$

$$E^{\mathscr{B}_n} \sum_{j=u_n}^{v_n} |A_{nj}| \|U_t(V_{nj})\|^p \leqslant M^p E^{\mathscr{B}_n} \sum_{j=u_n}^{v_n} |A_{nj}| \|V_{nj}\|^p \quad \text{a.e.}$$

and so, $E^{\mathscr{B}_n} \sum_{j=u_n}^{v_n} |A_{nj}| \|U_t(V_{nj})\|^p \to 0$ a.e. $\square$

In the case $p = 1$ we can improve slightly the result above:

Corollary 2. *Let hypothesis be as in Theorem* 2, *with* $p=1$.

Then $E^{\mathscr{B}_n}\|\sum_{j=u_n}^{v_n} A_{nj}V_{nj}\| \to 0$ *a.e. as* $n\to\infty$ *if, and only if, there exists* $t_1\in\mathbb{N}$ *such that* $E^{\mathscr{B}_n}\|\sum_{j=u_n}^{v_n} A_{nj}U_t(V_{nj})\| \to 0$ *a.e. for each* $t\geqslant t_1$, *as* $n\to\infty$.

Proof. Suppose that $E^{\mathscr{B}_n}\|\sum_{j=u_n}^{v_n} A_{nj}U_t(V_{nj})\| \to 0$ a.e. for each $t\geqslant t_1$.

We prove, as in Theorem 2, with $p=1$, that, given $\varepsilon>0$, there exists $t_0\in\mathbb{N}$ such that for every $t\geqslant t_0$: $\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|E^{\mathscr{B}_n}\|V_{nj}-U_t(V_{nj})\|<\varepsilon/2$ a.e.

Then, taking $t\geqslant \max\{t_0,t_1\}$:

$$E^{\mathscr{B}_n}\left\|\sum_{j=u_n}^{v_n} A_{nj}V_{nj}\right\| \leqslant E^{\mathscr{B}_n}\left\|\sum_{j=u_n}^{v_n} A_{nj}U_t(V_{nj})\right\| + \sum_{j=u_n}^{v_n}|A_{nj}|E^{\mathscr{B}_n}\|V_{nj}-U_t(V_{nj})\| \to 0 \quad \text{a.e.}$$

as $n\to\infty$.

Now, suppose that $E^{\mathscr{B}_n}\|\sum_{j=u_n}^{v_n} A_{nj}V_{nj}\| \to 0$ a.e. as $n\to\infty$.

Then, for every $t\in\mathbb{N}$:

$$\left\|\sum_{j=u_n}^{v_n} A_{nj}U_t(V_{nj})\right\| = \left\|U_t\left(\sum_{j=u_n}^{v_n} A_{nj}V_{nj}\right)\right\| \leqslant M\left\|\sum_{j=u_n}^{v_n} A_{nj}V_{nj}\right\|,$$

which implies that

$$E^{\mathscr{B}_n}\left\|\sum_{j=u_n}^{v_n} A_{nj}U_t(V_{nj})\right\| \leqslant ME^{\mathscr{B}_n}\left\|\sum_{j=u_n}^{v_n} A_{nj}V_{nj}\right\| \quad \text{a.e.}$$

and so $E^{\mathscr{B}_n}\|\sum_{j=u_n}^{v_n} A_{nj}U_t(V_{nj})\| \to 0$ a.e. as $n\to\infty$. □

Remark. Theorem 2 and Corollary 2 remain true if $\mathscr{X}$ is a Banach space with a Schauder basis and, for each $t\in\mathbb{N}$, U_t denotes the tth partial sum operator corresponding to the basis.

Acknowledgements

The authors are grateful to the referee for his careful reading of the manuscript and his valuable comments and suggestions.

References

Cabrera, M.O., 1994. Convergence of weighted sums of random variables and uniform integrability concerning the weights. Collect. Math. 45, 121–132.

Cabrera, M.O., 1997. Convergence in mean of weighted sums of $\{a_{nk}\}$-compactly uniformly integrable random elements in Banach spaces. Internat. J. Math. Math. Sci. 20, 443–450.

Cuesta, J.A., Matrán, C., 1988. Strong convergence of weighted sums of random elements through the equivalence of sequences of distributions. J. Multivariate Anal. 25, 311–322.

Gut, A., 1992. The weak law of large numbers for arrays. Statist. Probab. Lett. 14, 49–52.

Hu, T.C., Cabrera, M.O., Volodin, A.I., 2001. Convergence of randomly weighted sums of Banach space valued random elements and uniform integrability concerning the random weights. Statist. Probab. Lett. 51, 155–164.

Johnson, W.B., Rosenthal, H.P., Zippin, M., 1971. On bases, finite-dimensional decompositions and weaker structures in Banach spaces. Israel J. Math. 9, 488–506.

Lindenstrauss, J., Tzafriri, L., 1977. Classical Banach Spaces I. Springer, Berlin.
Pelczynski, A., 1971. Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with a basis. Stud. Math. 40, 239–243.
Rosalsky, A., Sreehari, M., 1998. On the limiting behavior of randomly weighted partial sums. Statist. Probab. Lett. 40, 403–410.
Sung, S.H., 1999. Weak law of large numbers for arrays of random variables. Statist. Probab. Lett. 42, 293–298.
Szarek, S.J., 1987. A Banach space without a basis which has the bounded approximation property. Acta Math. 159 (1-2), 81–98.
Wang, X.C., Rao, M.B., 1987. Some results on the convergence of weighted sums of random elements in separable Banach spaces. Stud. Math. 86, 131–153.