Convergence of randomly weighted sums of Banach space valued random elements and uniform integrability concerning the random weights

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Abstract

Some notions of uniform integrability of an array of random elements in a separable Banach space with respect to an array of random variables are introduced and characterized, in order to obtain weak laws of large numbers for randomly weighted sums. The paper contains results which generalize some previous results for weighted sums with nonrandom weights, and one of them is used to obtain a result of convergence for sums with a random number of addends. Furthermore, a result of almost everywhere convergence of the sequence of certain conditional expectations of the row sums is obtained.

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1. Introduction

There exists an extensive literature about the weak or strong convergence of weighted partial sums \( \sum_{j=1}^{n} a_{nj} X_j \), where \( \{X_n, n \geq 1\} \) is a sequence of random variables, and \( \{a_{nj}, 1 \leq j \leq n, n \geq 1\} \) is an array of (nonrandom) constants. In this scope, Rosalsky and Sreehari (1998) provide a complete list of references from 1965 to 1995.

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Starting from the 1970s, the random nature of many problems arising in the applied sciences is noted. This leads to mathematical models which deal with the limiting behaviour of weighted sums of random elements in normed linear spaces, where the weights are random variables.

Taylor and Padgett (1972, 1974, 1976) obtain (still in the scope of constant weights) some basic results by considering a sequence \( \{A_n, n \geq 1\} \) of random weights. From 1978 on, it begins to be studied directly the convergence of randomly weighted partial sums of random elements in separable Banach spaces or in separable normed linear spaces, in general. The reader may refer to Wei and Taylor (1978a,b), Taylor and Calhoun (1983), Taylor et al. (1984), Ordoñez Cabrera (1988), Adler et al. (1992), Wang and Rao (1995) and Hu and Chang (1999). In these papers, the (weak or strong) convergence of sums \( \sum_{j=1}^{n} A_{nj} V_j \) is analyzed, where \( \{A_{nj}, 1 \leq j \leq n, n \geq 1\} \) is an array of random variables, and \( \{V_n, n \geq 1\} \) is a sequence of random elements taking values in a separable normed linear space (or in a Banach space). This structure can be subsumed in the general structure of randomly weighted partial sums \( \sum_{j=1}^{n} A_{nj} V_{nj} \), by putting \( V_{nj} = V_j, 1 \leq j \leq n, n \geq 1 \).

The limiting behaviour of randomly weighted partial sums \( \sum_{j=1}^{n} A_{nj} V_{nj} \) plays an important role in various applied and theoretical problems. On the matter, see the Example of Rosalsky and Sreehari (1998), in queuing theory, where the sums \( \sum_{j=1}^{n} A_{nj} V_{nj} \) can be used to represent the total output for a customer being served by \( n \) machines.

At once, this structure can be subsumed in a more general structure, where the sums are not necessarily partial sums. Let \( \{u_n \geq -\infty, n \geq 1\} \) and \( \{v_n \leq +\infty, n \geq 1\} \) be two sequences of integers (not necessary positive or finite) such that \( u_n < v_n \) for all \( n \geq 1 \). Consider an array of random elements \( \{V_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) defined on a probability space \((\Omega, \mathcal{A}, P)\) and taking values in a real separable Banach space \( X \) with norm \( \|\cdot\| \). Let \( \{A_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) be an array of random variables defined on the same probability space \((\Omega, \mathcal{A}, P)\). We consider the randomly weighted sums \( \sum_{j=u_n}^{v_n} A_{nj} V_{nj} \).

In the case of a triangular array of constant weights, the notion of uniform integrability of the array of random elements or the notion of uniform integrability of this array concerning the constant weights have been useful in order to obtain weak laws of large numbers. We refer, among others, to Gut (1992), Ordoñez Cabrera (1994) and Sung (1999).

In this note, we introduce some notions of uniform integrability of an array \( \{V_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) of random elements with respect to a sequence \( \{A_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) of random variables. Definitions 2 and 3 are close to corresponding ones for an array of constants (non random weights). An interesting feature of the current work is the notion of \( \{A_{nj}\}\text{-conditional uniform integrability relative to a sequence} \{\mathcal{B}_n\} \) of \( \sigma \)-algebras (Definition 5). This notion is of the greatest interest when \( \mathcal{B}_n = \sigma(A_{nj}, u_n \leq j \leq v_n) \), i.e., when \( \{\mathcal{B}_n\} \) is the \( \sigma \)-algebra generated by \( \{A_{nj}, u_n \leq j \leq v_n\} \), for each \( n \geq 1 \).

Under the condition \( \sup_{n \geq 1} \sum_{j=u_n}^{v_n} |A_{nj}| \leq C \) a.e., we obtain a characterization of this notion, which involves respective characterizations of the notions of \( \{A_{nj}\}\text{-uniform integrability in the strong and weak senses. These results extend in a natural way the characterizations of} \{a_{nj}\}\text{-uniform integrability (for nonrandom weights) in Ordoñez Cabrera (1994).} \)

Theorems 3 and 4 and Corollary 3 give results of convergence for randomly weighted sums of random elements which generalize some previous results for weighted sums with nonrandom weights. Independence between weights and random elements is required. By supposing the hypothesis of \( \{A_{nj}\}\text{-conditional uniform integrability relative to a sequence} \{\mathcal{B}_n\} \) of \( \sigma \)-algebras, we prove Theorems 5 and 6 for randomly weighted sums of random elements. Theorem 5 gives a result of convergence in \( L_1 \) and Theorem 6 gives a result of almost everywhere (a.e.) convergence of the sequence of conditional expectations of the row sums.

2. Definitions

Let \( \{u_n, n \geq 1\} \) and \( \{v_n, n \geq 1\} \) be two sequences of integers (not necessary positive or finite) such that \( v_n > u_n \) for all \( n \geq 1 \) and \( v_n - u_n \to \infty \) as \( n \to \infty \). Consider two arrays of random variables \( \{X_{nj}, u_n \leq j \leq v_n, n \geq 1\} \),
(1) The following concept was introduced in Ordoñez Cabrera (1994), with \( u_n = 1 \).
We say that \( \{X_n, u_n \leq j \leq v_n, n \geq 1\} \) is \( \{a_n\} \)-uniformly integrable if
\[
\lim_{a \to \infty} \sup_{n \geq 1} \left( \sum_{j = u_n}^{v_n} |a_n| E(|X_n| |I_j| |X_n| > a) \right) = 0.
\]

(2) We say that \( \{X_n, u_n \leq j \leq v_n, n \geq 1\} \) is \( \{A_{n}\} \)-uniformly integrable in the strong sense if for all \( \varepsilon > 0 \), there exists \( a_0 > 0 \) such that
\[
\sup_{n \geq 1} \left( \sum_{j = u_n}^{v_n} |A_{n}| E(|X_n| |I_j| |X_n| > a_0) \right) < \varepsilon \quad \text{a.e.}
\]

(3) Let the random variables \( \{A_{n}, u_n \leq j \leq v_n, n \geq 1\} \) be integrable.
We say that \( \{X_n, u_n \leq j \leq v_n, n \geq 1\} \) is \( \{A_{n}\} \)-uniformly integrable in the weak sense if \( \{X_n, u_n \leq j \leq v_n, n \geq 1\} \) is \( \{E|A_{n}|\} \)-uniformly integrable, i.e., if
\[
\lim_{a \to \infty} \sup_{n \geq 1} \left( \sum_{j = u_n}^{v_n} E[A_{n}] E(|X_n| |I_j| |X_n| > a) \right) = 0,
\]

It is easy to check that if \( \sup_{n \geq 1} \sum_{j = u_n}^{v_n} E|A_{n}| < \infty \), then (2) \( \Rightarrow \) (3).
Let \( \mathcal{B}_n \) be a sequence of sub-\( \sigma \)-algebras of \( \mathcal{A} \). For each \( n \geq 1 \), denote by \( E^{\mathcal{B}_n}(Y) \) the conditional expectation of the random variable \( Y \) relative to \( \mathcal{B}_n \), and by \( P^{\mathcal{B}_n}(A) \) the conditional probability of the event \( A \in \mathcal{A} \) relative to \( \mathcal{B}_n \).

(4) We say that \( \{X_n, u_n \leq j \leq v_n, n \geq 1\} \) is conditionally uniformly integrable relative to \( \mathcal{B}_n \) if
\[
\lim_{a \to \infty} \sup_{n \geq 1} \sup_{u_n \leq j \leq v_n} E^{\mathcal{B}_n}(|X_n| |I_j| |X_n| > a) = 0 \quad \text{a.e.}
\]

(5) We say that \( \{X_n, u_n \leq j \leq v_n, n \geq 1\} \) is \( \{A_{n}\} \)-conditionally uniformly integrable relative to \( \mathcal{B}_n \) if for all \( \varepsilon > 0 \), there exists \( a_0 > 0 \) such that
\[
\sup_{n \geq 1} \left( \sum_{j = u_n}^{v_n} |A_{n}| E^{\mathcal{B}_n}(|X_n| |I_j| |X_n| > a_0) \right) < \varepsilon \quad \text{a.e.}
\]

In particular, it is of interest when \( \mathcal{B}_n = \sigma(A_{n}, u_n \leq j \leq v_n) \) is the \( \sigma \)-algebra generated by \( \{A_{n}, u_n \leq j \leq v_n\} \) for each \( n \geq 1 \).
Note that if \( \sup_{n \geq 1} \sum_{j = u_n}^{v_n} |A_{n}| \leq \infty \), a.e., then (4) \( \Rightarrow \) (5).
If \( A_{n} = a_{n} \) (nonrandom) a.s for all \( u_n \leq j \leq v_n, n \geq 1 \), definitions (2), (3) and (5) (when \( \mathcal{B}_n = \{\emptyset, \Omega\} \forall n \in N \)) coincide with Definition 1.

(6) Let \( \{V_{n}, u_n \leq j \leq v_n, n \geq 1\} \) be an array of random elements in a separable Banach space \( X \) with norm \( \| \cdot \| \). We say that \( \{V_{n}, u_n \leq j \leq v_n, n \geq 1\} \) is uniformly (or conditionally uniformly) integrable in each one of the preceding senses if the array of random variables \( \{\|V_{n}\|, u_n \leq j \leq v_n, n \geq 1\} \) is so.

### 3. Characterizations

In this section we will obtain characterizations of the various concepts of uniform integrability which have been introduced in the previous section.
Theorem 1. Let \( \{X_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) and \( \{A_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) be two arrays of random variables with \( \sup_{n \geq 1} \sum_{j=1}^{v_n} |A_{nj}| \leq C \) a.e., for some constant \( C < \infty \) and let \( \{\mathcal{B}_n, n \geq 1\} \) be a sequence of sub \( \sigma \)-algebras of \( \mathcal{F} \).

Then, \( \{X_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) is \( \{A_{nj}\} \)-conditionally uniformly integrable relative to \( \mathcal{B}_n \) if, and only if:

(a) \( \sup_{n \geq 1} \sum_{j=1}^{v_n} |A_{nj}| E^{\mathcal{B}_n}[X_{nj}] = M < \infty \) a.e.

(b) for each \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that whenever \( \{B_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) is an array of events satisfying \( \sup_{n \geq 1} \sum_{j=1}^{v_n} |A_{nj}| P^{\mathcal{B}_n}(B_{nj}) < \delta \) a.e., then \( \sup_{n \geq 1} \sum_{j=1}^{v_n} |A_{nj}| E^{\mathcal{B}_n}(X_{nj} | I_{B_{nj}}) < \varepsilon \) a.e.

Proof. Let \( \{X_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) be an array of random variables which is \( \{A_{nj}\} \)-conditionally uniformly integrable relative to \( \mathcal{B}_n \).

Then, given \( \varepsilon > 0 \), there exists \( a > 0 \) such that

\[
\sup_{n \geq 1} \sum_{j=1}^{v_n} |A_{nj}| E^{\mathcal{B}_n}(X_{nj} | I_{|X_{nj}| > a}) < \frac{\varepsilon}{2} \quad \text{a.e.}
\]

Then

\[
E^{\mathcal{B}_n}(X_{nj}) = E^{\mathcal{B}_n}(X_{nj} I_{|X_{nj}| \leq a} + |X_{nj} I_{|X_{nj}| > a}) \leq a + E^{\mathcal{B}_n}(X_{nj} I_{|X_{nj}| > a}) \quad \text{a.e.}
\]

Therefore, for every \( n \in \mathbb{N} \):

\[
\sum_{j=1}^{v_n} |A_{nj}| E^{\mathcal{B}_n}(X_{nj}) \leq a \sum_{j=1}^{v_n} |A_{nj}| + \sum_{j=1}^{v_n} |A_{nj}| E^{\mathcal{B}_n}(X_{nj} I_{|X_{nj}| > a}) \quad \text{a.e.}
\]

and so

\[
\sup_{n \geq 1} \sum_{j=1}^{v_n} |A_{nj}| E^{\mathcal{B}_n}(X_{nj}) = M < \infty \quad \text{a.e.}
\]

Now let \( \varepsilon > 0 \), and let \( \{B_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) be an array of events with

\[
\sup_{n \geq 1} \sum_{j=1}^{v_n} |A_{nj}| P^{\mathcal{B}_n}(B_{nj}) < \frac{\varepsilon}{2a} = \delta \quad \text{a.e.}
\]

Then, for every \( n \in \mathbb{N} \):

\[
\sum_{j=1}^{v_n} |A_{nj}| E^{\mathcal{B}_n}(X_{nj} I_{B_{nj}}) = \sum_{j=1}^{v_n} |A_{nj}| E^{\mathcal{B}_n}(X_{nj} I_{B_{nj} \cap |X_{nj}| \leq a} + |X_{nj} I_{B_{nj} \cap |X_{nj}| > a})
\]

\[
\leq a \sum_{j=1}^{v_n} |A_{nj}| P^{\mathcal{B}_n}(B_{nj}) + \sum_{j=1}^{v_n} |A_{nj}| E^{\mathcal{B}_n}(X_{nj} I_{|X_{nj}| > a}) < a \frac{\varepsilon}{2a} + \frac{\varepsilon}{2} = \varepsilon \quad \text{a.e.}
\]
Conversely, for each \( a > 0 \) and every \( n \in N \):
\[
\sum_{j=0}^{v_n} |A_{nj}|P^{\delta_k}(\{|X_{nj}| > a\}) = \sum_{j=0}^{v_n} |A_{nj}|E^{\delta_k}I_{\{|X_{nj}| > a\}} \leq \frac{1}{a} \sum_{j=0}^{v_n} |A_{nj}|E^{\delta_k}|X_{nj}| \leq \frac{M}{a} \text{ a.e.}
\]

since \( aI_{\{|X_{nj}| > a\}} \leq |X_{nj}| \text{ a.e.} \).

Given \( \varepsilon > 0 \), we have, for each \( a \geq a_0 = 2M/\delta \) and every \( n \in N \):
\[
\sum_{j=0}^{v_n} |A_{nj}|P^{\delta_k}(\{|X_{nj}| > a\}) \leq \frac{M}{a_0} = \frac{\delta}{2} < \delta \text{ a.e.}
\]

Therefore, the array of events \( \{B_{nj}\} = \{|X_{nj}| > a\} \), for each \( a > a_0 \), verifies condition (b). So:
\[
\sup_{n \geq 1} \sum_{j=0}^{v_n} |A_{nj}|E^{\delta_k}(\{|X_{nj}|I_{\{|X_{nj}| > a\}}\}) < \varepsilon \text{ a.e.}
\]

for each \( a > a_0 \), i.e., \( \{X_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) is \( \{A_{nj}\} \)-conditionally uniformly integrable relative to \( \mathcal{B}_n \). \( \square \)

By considering the sequence of \( \sigma \)-algebras \( \mathcal{B}_n = \{\emptyset, \Omega\} \) for every \( n \in N \), we obtain the characterization of \( \{A_{nj}\} \)-uniform integrability in the strong sense and in the weak sense:

**Corollary 1.** Let \( \{X_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) and \( \{A_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) be two arrays of random variables with \( \sup_{n \geq 1} \sum_{j=0}^{v_n} |A_{nj}| \leq C a.e. \).
Then, \( \{X_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) is \( \{A_{nj}\} \)-uniformly integrable in the strong sense if and only if:
(a) \( \sup_{n \geq 1} \sum_{j=0}^{v_n} |A_{nj}|E|X_{nj}| = M < \infty \text{ a.e.} \).
(b) for each \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that whenever \( \{B_{nj}\}, u_n \leq j \leq v_n, n \geq 1\) is an array of events satisfying \( \sup_{n \geq 1} \sum_{j=0}^{v_n} |A_{nj}|P(B_{nj}) < \delta \text{ a.e.} \), then \( \sup_{n \geq 1} \sum_{j=0}^{v_n} |A_{nj}|E(|X_{nj}|I_{B_{nj}}) < \varepsilon \text{ a.e.} \).

**Corollary 2.** Let \( \{X_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) and \( \{A_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) be two arrays of random variables with \( \sup_{n \geq 1} \sum_{j=0}^{v_n} E|A_{nj}| = M < \infty \).
Then, \( \{X_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) is \( \{A_{nj}\} \)-uniformly integrable in the weak sense if and only if:
(a) \( \sup_{n \geq 1} \sum_{j=0}^{v_n} E|A_{nj}|E|X_{nj}| = M < \infty \).
(b) for each \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that whenever \( \{B_{nj}\}, u_n \leq j \leq v_n, n \geq 1\) is an array of events satisfying \( \sup_{n \geq 1} \sum_{j=0}^{v_n} E|A_{nj}|P(B_{nj}) < \delta \text{ a.e.} \), then \( \sup_{n \geq 1} \sum_{j=0}^{v_n} E|A_{nj}|E(|X_{nj}|I_{B_{nj}}) < \varepsilon \).

Note that if, in particular, \( A_{nj} = a_{nj} \) (nonrandom) for all \( u_n \leq j \leq v_n, n \geq 1 \), Corollary 1 gives a characterization of the \( \{a_{nj}\} \)-uniform integrability of an array \( \{X_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) which extends the characterization of the \( \{a_{nj}\} \)-uniform integrability of a sequence \( \{X_n, n \geq 1\} \) in Ordoñez Cabrera (1994). From this point of view, Corollary 2 is the characterization of the \( \{a_{nj}\} \)-uniform integrability of \( \{X_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) when we consider \( a_{nj} = E|A_{nj}| \) for all \( u_n \leq j \leq v_n, n \geq 1 \).

By using a similar technique to that we used in the proof of Theorem 1, the following characterization of the conditional uniform integrability relative to a sequence of \( \sigma \)-algebras can be obtained:

**Theorem 2.** Let \( \{X_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) be an array of random variables, and let \( \{\mathcal{B}_n, n \geq 1\} \) be a sequence of sub-\( \sigma \)-algebras of \( \mathcal{F} \).
Then, \( \{X_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) is conditionally uniformly integrable relative to \( \mathcal{B}_n \) if, and only if:
(a) \( \sup_{n \geq 1} \sup_{u_n \leq j \leq v_n} E^{\delta_k}|X_{nj}| = M < \infty \text{ a.e.} \).
(b) for each \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that whenever \( \{B_{nj}\}, u_n \leq j \leq v_n, n \geq 1\) is an array of events satisfying \( \sup_{n \geq 1} \sup_{u_n \leq j \leq v_n} P^{\delta_k}(B_{nj}) < \delta \text{ a.e.} \), then \( \sup_{n \geq 1} \sup_{u_n \leq j \leq v_n} E^{\delta_k}(|X_{nj}|I_{B_{nj}}) < \varepsilon \text{ a.e.} \).
4. Convergence of randomly weighted sums

In the following results, we suppose that all the random elements and the random variables are defined on the same probability space \((\Omega, \mathcal{F}, P)\).

**Theorem 3.** Let \(\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}\) be an array of random elements taking values in a real separable Banach space and \(\{A_{nj}, u_n \leq j \leq v_n, n \geq 1\}\) be an array of random variables such that:

1. \(A_{nj}\) and \(V_{nj}\) are independent for each \(j, u_n \leq j \leq v_n\) and every \(n \geq 1\),
2. \(\lim_{n \to \infty} \sum_{j = u_n}^{v_n} E[A_{nj}] = 0\),
3. \(\{\|V_{nj}\|^q, u_n \leq j \leq v_n, n \geq 1\}\) is \(\{|A_{nj}|^q\}\)-uniformly integrable in weak sense for some \(0 < q \leq 1\).

Then \(\sum_{j = u_n}^{v_n} A_{nj} V_{nj} \to 0\) in \(L^q\) as \(n \to \infty\) and, consequently,

\[
\left\| \sum_{j = u_n}^{v_n} A_{nj} V_{nj} \right\| \to 0 \quad \text{in} \ L^q.
\]

**Proof.** For any \(a > 0\) define

\[
V_{nj}' = V_{nj} I[\|V_{nj}\| \leq a], \quad V_{nj}'' = V_{nj} I[\|V_{nj}\| > a].
\]

Then since \(q \leq 1\)

\[
E \left[ \sum_{j = u_n}^{v_n} |A_{nj}||V_{nj}'| \right]^q \leq E \left[ \sum_{j = u_n}^{v_n} |A_{nj}||V_{nj}''| \right]^q + E \left[ \sum_{j = u_n}^{v_n} |A_{nj}||V_{nj}''| \right]^q
\]

\[
\leq E \left[ \sum_{j = u_n}^{v_n} |A_{nj}||V_{nj}''| \right]^q + \sum_{j = u_n}^{v_n} E|A_{nj}|^q E\|V_{nj}''\|^q
\]

\[
\leq a^q \left[ \sum_{j = u_n}^{v_n} E|A_{nj}| \right]^q + \sum_{j = u_n}^{v_n} E|A_{nj}|^q E\|V_{nj}''\|^q
\]

Now, the first sum tends to zero by assumption (2) and the second one tends to zero by assumption (3). \(\square\)

**Remark.** If \(A_{nj} = a_{nj}\) (nonrandom) a.s for all \(u_n \leq j \leq v_n, n \geq 1\), then Theorem 3 gives Theorem 6 of Ordoñez Cabrera (1994).

**Corollary 3.** Let \(\{V_{nj}, -\infty < j < +\infty, n \geq 1\}\) be an array of random elements taking values in a real separable Banach space, \(\{A_{nj}, -\infty < j < +\infty, n \geq 1\}\) be an array of random variables and \(\{N_n, n \geq 1\}\) and \(\{M_n, n \geq 1\}\) be two sequence of (not necessarily positive) integer-valued random variables with \(N_n \leq M_n\) a.e., \(n \geq 1\), and such that for some nonrandom sequences \(\{u_n, n \geq 1\}\) and \(\{v_n, n \geq 1\}\), we have

\[
P[N_n < u_n] = o(1) \quad \text{and} \quad P[M_n > v_n] = o(1) \quad \text{as} \quad n \to \infty.
\]

Suppose also that assumptions (1)–(3) of Theorem 3 hold. Then \(\sum_{j = N_n}^{M_n} |A_{nj}||V_{nj}| \to 0\) in probability as \(n \to \infty\) and, consequently, \(\| \sum_{j = u_n}^{v_n} A_{nj} V_{nj} \| \to 0\) in probability.
Proof. For arbitrary \( \varepsilon > 0 \) and \( n \geq 1 \):

\[
P \left[ \sum_{j=N_n}^{M_n} |A_{nj}| \| V_{nj} \| > \varepsilon \right] = P \left[ \sum_{j=N_n}^{M_n} |A_{nj}| \| V_{nj} \| > \varepsilon, N_n \geq u_n, M_n \leq v_n \right] + P \left[ \sum_{j=N_n}^{M_n} |A_{nj}| \| V_{nj} \| > \varepsilon, N_n < u_n \right]
\]

\[
\leq P \left[ \sum_{j=N_n}^{M_n} |A_{nj}| \| V_{nj} \| > \varepsilon \right] + P[M_n > v_n] + P[N_n < u_n] = o(1)
\]

by Theorem 3 and assumptions of Corollary 3.

We need the following lemma for proof of Theorem 4.

**Lemma.** Suppose that \( \{X_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) is an array of \( \{a_{nj}\}\)-uniformly integrable random variables satisfying \( \sup_{n \geq 1} \sum_{j=u_n}^{v_n} |a_{nj}| E|X_{nj}| < \infty \).

Denote \( m_n = 1/\sup_{u_n \leq j \leq v_n} |a_{nj}| \). If \( m_n \to \infty \) as \( n \to \infty \) and \( p > 1 \) then

\[
\sum_{j=u_n}^{v_n} |a_{nj}|^p E|X_{nj}|^p I(|X_{nj}| \leq m_n) = o(1).
\]

**Proof.** For any \( a < m_n \), we have

\[
\sum_{j=u_n}^{v_n} |a_{nj}|^p E|X_{nj}|^p I(|X_{nj}| \leq m_n) = \sum_{j=u_n}^{v_n} |a_{nj}|^p E|X_{nj}|^p (I(|X_{nj}| \leq a) + I[a < |X_{nj}| \leq m_n])
\]

\[
\leq \sum_{j=u_n}^{v_n} |a_{nj}|^p a^{p-1} E|X_{nj}|^p I(|X_{nj}| \leq a) + \sum_{j=u_n}^{v_n} |a_{nj}|^p a_n^{p-1} E|X_{nj}|^p I(|X_{nj}| > a)
\]

\[
\leq m_n^{1-p} a^{p-1} \sup_{m \geq 1} \sum_{j=u_n}^{v_n} |a_{nj}| E|X_{nj}|^p + \sup_{m \geq 1} \sum_{j=u_n}^{v_n} |a_{nj}| E|X_{nj}|^p I(|X_{nj}| > a).
\]

By the assumption to the Lemma, the first term above is \( o(1) \) as \( n \to \infty \) since \( m_n \to \infty \); and the second term above is \( o(1) \) as \( a \to \infty \).

**Theorem 4.** Let \( \{V_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) be an array of random elements taking values in a real separable Banach space, and \( \{A_{nj}, u_n \leq j \leq v_n, n \geq 1\} \) be an array of random variables such that, for some \( 0 < q < 1 \),

1. \( A_{nj} \) and \( V_{nj} \) are independent for each \( j, u_n \leq j \leq v_n \) and every \( n \geq 1 \),
2. \( \lim_{n \to \infty} \sup_{u_n \leq j \leq v_n} E|A_{nj}| = 0 \),
3. \( \{\|V_{nj}\|^q\} \) is \( \{E|A_{nj}|^q\}-\)uniformly integrable,
4. \( \sup_{n \geq 1} \sum_{j=u_n}^{v_n} E|V_{nj}|^q < \infty \).

Then \( \|\sum_{j=u_n}^{v_n} A_{nj} V_{nj}\| \to 0 \) in \( L_q \), as \( n \to \infty \).

**Proof.** Denote \( m_n = 1/\sup_{u_n \leq j \leq v_n} E|A_{nj}| \). We define:

\[
V'_{nj} = V_{nj} I(\|V_{nj}\| \leq m_n), \quad V''_{nj} = V_{nj} I(\|V_{nj}\| > m_n).
\]
Given $\varepsilon > 0$, there exists $a > 0$ such that
\[
\sup_{n \geq 1} \left( \sum_{j=1}^{a_n} (E|A_{nj}|)^p E(||V_{nj}||^q I_{||V_{nj}|| > a}) \right) \leq \frac{\varepsilon}{2}.
\]

As $\lim_{n \to \infty} m_n = \infty$, there exists $n_0 \in N$ such that $m_n > a$ for all $n \geq n_0$. Therefore, for all $n \geq n_0$:
\[
\sum_{j=1}^{a_n} (E|A_{nj}|)^p E||V_{nj}||^q \leq \frac{\varepsilon}{2}.
\]

By applying Lemma with $p = 1/q$, $a_{nj} = (E|A_{nj}|)^q$, we can choose $n_0 \in N$ such that for all $n \geq n_0$
\[
\sum_{j=1}^{a_n} ((E|A_{nj}|)^q)^p E(||V_{nj}||^q I_{||V_{nj}|| > 1/sup_{j \leq n}(E|A_{nj}|)^q}) = \sum_{j=1}^{a_n} E|A_{nj}|^q E||V_{nj}||^q \leq \left( \frac{\varepsilon}{2} \right)
\]

Then since $q < 1$, we have for all $n \geq n_0$
\[
E \left( \sum_{j=1}^{a_n} A_{nj} V_{nj}^q \right) \leq E \left( \sum_{j=1}^{a_n} A_{nj} V_{nj}^q \right)^q + E \left( \sum_{j=1}^{a_n} A_{nj} V_{nj}^q \right)^q
\]
\[
\leq \left( E \left( \sum_{j=1}^{a_n} A_{nj} V_{nj}^q \right)^q \right) + \sum_{j=1}^{a_n} E|A_{nj}|^q E||V_{nj}||^q
\]
\[
\leq \left( \sum_{j=1}^{a_n} E|A_{nj}|^q E||V_{nj}||^q \right) + \sum_{j=1}^{a_n} (E|A_{nj}|)^q E||V_{nj}||^q \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

**Theorem 5.** Let $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ be an array of random elements taking values in a real separable Banach space and $\{A_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ be an array of random variables such that assumption (2) of Theorem 3, that is, $\lim_{n \to \infty} \sum_{j=u_n}^{v_n} E|A_{nj}| = 0$, holds. Let $\mathcal{B}_n = \sigma(A_{nj}, u_n \leq j \leq v_n)$, for each $n \geq 1$ and suppose that $\{V_{nj}, u_n \leq j \leq v_n, n \geq 1\}$ is $\{A_{nj}\}$-conditionally uniformly integrable relative to $\mathcal{B}_n$.

Then $\sum_{j=u_n}^{v_n} |A_{nj}| ||V_{nj}|| \to 0$ in $L_1$ as $n \to \infty$ (and, consequently, $\sum_{j=u_n}^{v_n} A_{nj} V_{nj} \to 0$ in $L_1$).

**Proof.** Given $\varepsilon > 0$, there exists $a > 0$ such that
\[
\sup_{n \geq 1} \left( \sum_{j=1}^{a_n} |A_{nj}| E^{\mathcal{B}_n}(||V_{nj}|| I_{||V_{nj}|| > a}) \right) \leq \frac{\varepsilon}{2} \ a.e.
\]

We define $V'_{nj}$ and $V''_{nj}$, as in Theorem 3. Then, since the $\{A_{nj}, u_n \leq j \leq v_n\}$ are $\mathcal{B}_n$-measurable:
\[
E^{\mathcal{B}_n}\left( \sum_{j=1}^{a_n} |A_{nj}| ||V_{nj}|| \right) \leq E^{\mathcal{B}_n}\left( \sum_{j=1}^{a_n} |A_{nj}| ||V'_{nj}|| \right) + E^{\mathcal{B}_n}\left( \sum_{j=1}^{a_n} |A_{nj}| ||V''_{nj}|| \right)
\]
\[
\leq aE^{\mathcal{B}_n}\left( \sum_{j=1}^{a_n} |A_{nj}| \right) + E^{\mathcal{B}_n}\left( \sum_{j=1}^{a_n} |A_{nj}| ||V'_{nj}|| \right) + E^{\mathcal{B}_n}\left( \sum_{j=1}^{a_n} |A_{nj}| ||V''_{nj}|| \right)
\]
\[
\leq a \left( \sum_{j=1}^{a_n} |A_{nj}| \right) + \left( \sum_{j=1}^{a_n} |A_{nj}| E^{\mathcal{B}_n}||V''_{nj}|| \right) .
\]
There exists $n_0 \in \mathbb{N}$ such that the expectation of the first sum is less than $\varepsilon/2$ for all $n \geq n_0$, and the expectation of the second sum is less than $\varepsilon/2$ by the choice of $a$. Then, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$:

$$E \left( E^{|A_{n_j}||V_{n_j}|} \right) = E \left( \sum_{j = u_n}^{v_n} |A_{n_j}||V_{n_j}| \right) < \varepsilon. \quad \square$$

A light modification of conditions in Theorem 5 allow us to obtain a result of strong convergence of the sequence of conditional expectations.

**Theorem 6.** Let $\{V_{n_j}, u_n \leq j \leq v_n, n \geq 1\}$ be an array of random elements taking values in a real separable Banach space and $\{A_{n_j}, u_n \leq j \leq v_n, n \geq 1\}$ be an array of random variables such that $\sup_{u_n \leq j \leq v_n} |A_{n_j}| = o((v_n - u_n)^{-1})$ a.e.

Let $\mathcal{B}_n = \sigma(A_{n_j}, u_n \leq j \leq v_n)$, for each $n \geq 1$ and suppose that $\{V_{n_j}, u_n \leq j \leq v_n, n \geq 1\}$ is $\{A_{n_j}\}$-conditionally uniformly integrable relative to $\mathcal{B}_n$.

Then $E^{|A_{n_j}||V_{n_j}|} \to 0$ a.e. as $n \to \infty$.

**Proof.** The beginning is the same as in Theorem 5. Next, estimate the first sum in the following way:

$$a \sum_{j = u_n}^{v_n} |A_{n_j}| \leq a(v_n - u_n) \sup_{u_n \leq j \leq v_n} |A_{n_j}|. \quad \square$$

We often use assumption (2) from the Theorem 3. In order to check it, the following statement may be useful.

**Proposition.** Let $\{A_{n_j}, u_n \leq j \leq v_n, n \geq 1\}$ be an array of random variables such that $\sup_{u_n \leq j \leq v_n} (E|A_{n_j}|)^r < \infty$, for some $r \in (0, 1)$ and $\lim_{n \to \infty} \sup_{u_n \leq j \leq v_n} E|A_{n_j}| = 0$. Then assumption (2) of Theorem 3 holds, that is, $\lim_{n \to \infty} \sum_{j = u_n}^{v_n} E|A_{n_j}| = 0$.

**Proof.** It is easy to see that

$$\sum_{j = u_n}^{v_n} E|A_{n_j}| < \left( \sup_{u_n \leq j \leq v_n} E|A_{n_j}| \right)^{1-r} \sum_{j = u_n}^{v_n} (E|A_{n_j}|)^r. \quad \square$$

**References**


