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Convergence of randomly weighted sums of Banach space valued random elements and uniform integrability concerning the random weights ☆

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Abstract

Some notions of uniform integrability of an array of random elements in a separable Banach space with respect to an array of random variables are introduced and characterized, in order to obtain weak laws of large numbers for randomly weighted sums. The paper contains results which generalize some previous results for weighted sums with nonrandom weights, and one of them is used to obtain a result of convergence for sums with a random number of addends. Furthermore, a result of almost everywhere convergence of the sequence of certain conditional expectations of the row sums is obtained.

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1. Introduction

There exists an extensive literature about the weak or strong convergence of weighted partial sums $\sum_{j=1}^{n} a_{nj}X_j$, where $\{X_n, n\geqslant 1\}$ is a sequence of random variables, and $\{a_{nj}, 1\leqslant j\leqslant n, n\geqslant 1\}$ is an array of (nonrandom) constants. In this scope, Rosalsky and Sreehari (1998) provide a complete list of references from 1965 to 1995.

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Starting from the 1970s, the random nature of many problems arising in the applied sciences is noted. This leads to mathematical models which deal with the limiting behaviour of weighted sums of random elements in normed linear spaces, where the weights are random variables.

Taylor and Padgett (1972, 1974, 1976) obtain (still in the scope of constant weights) some basic results by considering a sequence $\{A_n, n\geqslant 1\}$ of random weights. From 1978 on, it begins to be studied directly the convergence of randomly weighted partial sums of random elements in separable Banach spaces or in separable normed linear spaces, in general. The reader may refer to Wei and Taylor (1978a, b), Taylor and Calhoun (1983), Taylor et al. (1984), Ordoñez Cabrera (1988), Adler et al. (1992), Wang and Rao (1995) and Hu and Chang (1999). In these papers, the (weak or strong) convergence of sums $\sum_{j=1}^{n} A_{nj}V_j$ is analyzed, where $\{A_{nj}, 1\leqslant j\leqslant n, n\geqslant 1\}$ is an array of random variables, and $\{V_n, n\geqslant 1\}$ is a sequence of random elements taking values in a separable normed linear space (or in a Banach space). This structure can be subsumed in the general structure of randomly weighted partial sums $\sum_{j=1}^{n} A_{nj}V_{nj}$, by putting $V_{nj} = V_j$, $1\leqslant j\leqslant n$, $n\geqslant 1$.

The limiting behaviour of randomly weighted partial sums $\sum_{j=1}^{n} A_{nj}V_{nj}$ plays an important role in various applied and theoretical problems. On the matter, see the Example of Rosalsky and Sreehari (1998), in queueing theory, where the sums $\sum_{j=1}^{n} A_{nj}V_{nj}$ can be used to represent the total output for a customer being served by n machines.

At once, this structure can be subsumed in a more general structure, where the sums are not necessarily partial sums. Let $\{u_n\geqslant -\infty, n\geqslant 1\}$ and $\{v_n\leqslant +\infty, n\geqslant 1\}$ be two sequences of integers, $v_n > u_n$ for all $n\geqslant 1$. Consider an array of random elements $\{V_{nj}, u_n\leqslant j\leqslant v_n, n\geqslant 1\}$ defined on a probability space $(\Omega, \mathscr{A}, P)$ and taking values in a real separable Banach space $\mathscr{X}$ with norm $\|\cdot\|$. Let $\{A_{nj}, u_n\leqslant j\leqslant v_n, n\geqslant 1\}$ be an array of random variables defined on the same probability space $(\Omega, \mathscr{A}, P)$. We consider the randomly weighted sums $\sum_{j=u_n}^{v_n} A_{nj}V_{nj}$. When $u_n = 1$, $v_n = n$, $n\geqslant 1$, we have randomly weighted partial sums.

In the case of a triangular array of constant weights, the notion of uniform integrability of the array of random elements or the notion of uniform integrability of this array concerning the constant weights have been useful in order to obtain weak laws of large numbers. We refer, among others, to Gut (1992), Ordoñez Cabrera (1994) and Sung (1999).

In this note, we introduce some notions of uniform integrability of an array $\{V_{nj}, u_n\leqslant j\leqslant v_n, n\geqslant 1\}$ of random elements with respect to an array $\{A_{nj}, u_n\leqslant j\leqslant v_n, n\geqslant 1\}$ of random variables. Definitions 2 and 3 are close to corresponding ones for an array of constants (non random weights). An interesting feature of the current work is the notion of $\{A_{nj}\}$-conditional uniform integrability relative to a sequence $\{\mathscr{B}_n\}$ of σ-algebras (Definition 5). This notion is of the greatest interest when $\mathscr{B}_n = \sigma(A_{nj}, u_n\leqslant j\leqslant v_n)$, i.e., when $\{\mathscr{B}_n\}$ is the σ-algebra generated by $\{A_{nj}, u_n\leqslant j\leqslant v_n\}$, for each $n\geqslant 1$.

Under the condition $\sup_{n\geqslant 1} \sum_{j=u_n}^{v_n} |A_{nj}|\leqslant C$ a.e. we obtain a characterization of this notion, which involves respective characterizations of the notions of $\{A_{nj}\}$-uniform integrability in the strong and weak senses. These results extend in a natural way the characterizations of $\{a_{nj}\}$-uniform integrability (for nonrandom weights) in Ordoñez Cabrera (1994).

Theorems 3 and 4 and Corollary 3 give results of convergence for randomly weighted sums of random elements which generalize some previous results for weighted sums with nonrandom weights. Independence between weights and random elements is required. By supposing the hypothesis of $\{A_{nj}\}$-conditional uniform integrability relative to a sequence $\{\mathscr{B}_n\}$ of σ-algebras, we prove Theorems 5 and 6 for randomly weighted sums of random elements. Theorem 5 gives a result of convergence in L_1 and Theorem 6 gives a result of almost everywhere (a.e.) convergence of the sequence of conditional expectations of the row sums.

2. Definitions

Let $\{u_n, n\geqslant 1\}$ and $\{v_n, n\geqslant 1\}$ be two sequences of integers (not necessary positive or finite) such that $v_n > u_n$ for all $n\geqslant 1$ and $v_n - u_n \to \infty$ as $n\to\infty$. Consider two arrays of random variables $\{X_{nj}, u_n\leqslant j\leqslant v_n,$

$n \geqslant 1\}$ and $\{A_{nj},\, u_n \leqslant j \leqslant v_n,\, n \geqslant 1\}$ defined on a probability space $(\Omega,\, \mathscr{A},\, P)$ and an array of constants $\{a_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$.

(1) The following concept was introduced in Ordoñez Cabrera (1994), with $u_n = 1$.

We say that $\{X_{nj},\, u_n \leqslant j \leqslant v_n,\, n \geqslant 1\}$ is $\{a_{nj}\}$-*uniformly integrable* if

$$\lim_{a\to\infty} \sup_{n\geqslant 1} \left(\sum_{j=u_n}^{v_n} |a_{nj}| E(|X_{nj}| I_{[|X_{nj}|>a]}) \right) = 0.$$

(2) We say that $\{X_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ is $\{A_{nj}\}$-*uniformly integrable in the strong sense* if for all $\varepsilon > 0$, there exists $a_0 > 0$ such that

$$\sup_{n\geqslant 1} \left(\sum_{j=u_n}^{v_n} |A_{nj}| E(|X_{nj}| I_{[|X_{nj}|>a_0]}) \right) < \varepsilon \quad \text{a.e.}$$

(3) Let the random variables $\{A_{nj},\, u_n \leqslant j \leqslant v_n,\, n \geqslant 1\}$ be integrable.

We say that $\{X_{nj},\, u_n \leqslant j \leqslant v_n,\, n \geqslant 1\}$ is $\{A_{nj}\}$-*uniformly integrable in the weak sense* if $\{X_{nj},\, u_n \leqslant j \leqslant v_n,\, n \geqslant 1\}$ is $\{E|A_{nj}|\}$-uniformly integrable, i.e., if

$$\lim_{a\to\infty} \sup_{n\geqslant 1} \left(\sum_{j=u_n}^{v_n} E|A_{nj}| E(|X_{nj}| I_{[|X_{nj}|>a]}) \right) = 0,$$

It is easy to check that if $\sup_{n\geqslant 1} \sum_{j=u_n}^{v_n} E|A_{nj}| < \infty$, then (2) $\Rightarrow$ (3).

Let $\mathscr{B}_n$ be a sequence of sub σ-algebras of $\mathscr{A}$. For each $n \geqslant 1$, denote by $E^{\mathscr{B}_n}(Y)$ the conditional expectation of the random variable Y relative to $\mathscr{B}_n$, and by $P^{\mathscr{B}_n}(A)$ the conditional probability of the event $A \in \mathscr{A}$ relative to $\mathscr{B}_n$.

(4) We say that $\{X_{nj},\, u_n \leqslant j \leqslant v_n,\, n \geqslant 1\}$ is *conditionally uniformly integrable relative to* $\mathscr{B}_n$ if

$$\lim_{a\to\infty} \sup_{n\geqslant 1} \sup_{u_n \leqslant j \leqslant v_n} E^{\mathscr{B}_n}(|X_{nj}| I_{[|X_{nj}|>a]}) = 0 \quad \text{a.e.}$$

(5) We say that $\{X_{nj},\, u_n \leqslant j \leqslant v_n,\, n \geqslant 1\}$ is $\{A_{nj}\}$-*conditionally uniformly integrable relative to* $\mathscr{B}_n$ if for all $\varepsilon > 0$, there exists $a_0 > 0$ such that

$$\sup_{n\geqslant 1} \left(\sum_{j=u_n}^{v_n} |A_{nj}| E^{\mathscr{B}_n}(|X_{nj}| I_{[|X_{nj}|>a_0]}) \right) < \varepsilon \quad \text{a.e.}$$

In particular, it is of interest when $\mathscr{B}_n = \sigma(A_{nj}, u_n \leqslant j \leqslant v_n)$ is the σ-algebra generated by $\{A_{nj}, u_n \leqslant j \leqslant v_n\}$ for each $n \geqslant 1$.

Note that if $\sup_{n\geqslant 1} \sum_{j=u_n}^{v_n} |A_{nj}| < \infty$, a.e., then (4)$\Rightarrow$(5).

If $A_{nj} = a_{nj}$ (nonrandom) a.s for all $u_n \leqslant j \leqslant v_n$, $n \geqslant 1$, definitions (2), (3) and (5) (when $\mathscr{B}_n = \{\emptyset, \Omega\}\ \forall_n \in N$) coincide with Definition 1).

(6) Let $\{V_{nj},\, u_n \leqslant j \leqslant v_n,\, n \geqslant 1\}$ be an array of random elements in a separable Banach space $\mathscr{X}$ with norm $\|\cdot\|$. We say that $\{V_{nj},\, u_n \leqslant j \leqslant v_n,\, n \geqslant 1\}$ is uniformly (or conditionally uniformly) integrable in each one of the preceding senses if the array of random variables $\{\|V_{nj}\|,\, u_n \leqslant j \leqslant v_n,\, n \geqslant 1\}$ is so.

3. Characterizations

In this section we will obtain characterizations of the various concepts of uniform integrability which have been introduced in the previous section.

Theorem 1. *Let* $\{X_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ *and* $\{A_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ *be two arrays of random variables with* $\sup_{n\geqslant 1} \sum_{j=u_n}^{v_n} |A_{nj}| \leqslant C$ *a.e., for some constant* $C < \infty$ *and let* $\{\mathscr{B}_n, n \geqslant 1\}$ *be a sequence of sub* σ*-algebras of* $\mathscr{A}$.

Then, $\{X_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ *is* $\{A_{nj}\}$*-conditionally uniformly integrable relative to* $\mathscr{B}_n$ *if, and only if:*
(a) $\sup_{n\geqslant 1} \sum_{j=u_n}^{v_n} |A_{nj}| E^{\mathscr{B}_n}|X_{nj}| = M < \infty$ *a.e.*
(b) *for each* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that whenever* $\{B_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ *is an array of events satisfying* $\sup_{n\geqslant 1} \sum_{j=u_n}^{v_n} |A_{nj}| P^{\mathscr{B}_n}(B_{nj}) < \delta$ *a.e., then* $\sup_{n\geqslant 1} \sum_{j=u_n}^{v_n} |A_{nj}| E^{\mathscr{B}_n}(|X_{nj}| I_{B_{nj}}) < \varepsilon$ *a.e.*

Proof. Let $\{X_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ be an array of random variables which is $\{A_{nj}\}$-conditionally uniformly integrable relative to $\mathscr{B}_n$.

Then, given $\varepsilon > 0$, there exists $a > 0$ such that

$$\sup_{n\geqslant 1} \sum_{j=u_n}^{v_n} |A_{nj}| E^{\mathscr{B}_n}(|X_{nj}| I_{[|X_{nj}|>a]}) < \frac{\varepsilon}{2} \quad \text{a.e.}$$

Then

$$E^{\mathscr{B}_n}|X_{nj}| = E^{\mathscr{B}_n}(|X_{nj}| I_{[|X_{nj}|\leqslant a]} + |X_{nj}| I_{[|X_{nj}|>a]}) \leqslant a + E^{\mathscr{B}_n}(|X_{nj}| I_{[|X_{nj}|>a]}) \quad \text{a.e.}$$

Therefore, for every $n \in N$:

$$\sum_{j=u_n}^{v_n} |A_{nj}| E^{\mathscr{B}_n}|X_{nj}| \leqslant a \sum_{j=u_n}^{v_n} |A_{nj}| + \sum_{j=u_n}^{v_n} |A_{nj}| E^{\mathscr{B}_n}(|X_{nj}| I_{[|X_{nj}|>a]}) \quad \text{a.e.}$$

and so

$$\sup_{n\geqslant 1} \sum_{j=u_n}^{v_n} |A_{nj}| E^{\mathscr{B}_n}|X_{nj}| = M < \infty \quad \text{a.e.}$$

Now let $\varepsilon > 0$, and let $\{B_{nj}, u_n \leqslant j \leqslant v_n, n \geqslant 1\}$ be an array of events with

$$\sup_{n\geqslant 1} \sum_{j=u_n}^{v_n} |A_{nj}| P^{\mathscr{B}_n}(B_{nj}) < \frac{\varepsilon}{2a} = \delta \quad \text{a.e.}$$

Then, for every $n \in N$:

$$\begin{aligned}\sum_{j=u_n}^{v_n} |A_{nj}| E^{\mathscr{B}_n}(|X_{nj}| I_{B_{nj}}) &= \sum_{j=u_n}^{v_n} |A_{nj}| E^{\mathscr{B}_n}(|X_{nj}| I_{B_{nj}\cap[|X_{nj}|\leqslant a]} + |X_{nj}| I_{B_{nj}\cap[|X_{nj}|>a]}) \\ &\leqslant a \sum_{j=u_n}^{v_n} |A_{nj}| P^{\mathscr{B}_n}(B_{nj}) + \sum_{j=u_n}^{v_n} |A_{nj}| E^{\mathscr{B}_n}(|X_{nj}| I_{[|X_{nj}|>a]}) < a\frac{\varepsilon}{2a} + \frac{\varepsilon}{2} = \varepsilon \quad \text{a.e.}\end{aligned}$$

Conversely, for each $a>0$ and every $n\in N$:

$$\sum_{j=u_n}^{v_n}|A_{nj}|P^{\mathscr{B}_n}([|X_{nj}|>a])=\sum_{j=u_n}^{v_n}|A_{nj}|E^{\mathscr{B}_n}I_{[|X_{nj}|>a]}\leqslant\frac{1}{a}\sum_{j=u_n}^{v_n}|A_{nj}|E^{\mathscr{B}_n}|X_{nj}|\leqslant\frac{M}{a}\quad\text{a.e.}$$

since $aI_{[|X_{nj}|>a]}\leqslant|X_{nj}|$ a.e.

Given $\varepsilon>0$, we have, for each $a\geqslant a_0=2M/\delta$ and every $n\in N$:

$$\sum_{j=u_n}^{v_n}|A_{nj}|P^{\mathscr{B}_n}([|X_{nj}|>a])\leqslant\frac{M}{a_0}=\frac{\delta}{2}<\delta\quad\text{a.e.}$$

Therefore, the array of events $\{B_{nj}\}=\{[|X_{nj}|>a]\}$, for each $a>a_0$, verifies condition (b). So:

$$\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|E^{\mathscr{B}_n}(|X_{nj}|I_{[|X_{nj}|>a]})<\varepsilon\quad\text{a.e.}$$

for each $a>a_0$, i.e., $\{X_{nj},\ u_n\leqslant j\leqslant v_n,\ n\geqslant 1\}$ is $\{A_{nj}\}$-conditionally uniformly integrable relative to $\mathscr{B}_n$. □

By considering the sequence of σ-algebras $\mathscr{B}_n=\{\emptyset,\Omega\}$ for every $n\in N$, we obtain the characterization of $\{A_{nj}\}$-uniform integrability in the strong sense and in the weak sense:

Corollary 1. *Let $\{X_{nj},\ u_n\leqslant j\leqslant v_n,\ n\geqslant 1\}$ and $\{A_{nj},\ u_n\leqslant j\leqslant v_n,\ n\geqslant 1\}$ be two arrays of random variables with* $\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|\leqslant C$ *a.e.*

Then, $\{X_{nj},\ u_n\leqslant j\leqslant v_n,\ n\geqslant 1\}$ is $\{A_{nj}\}$-uniformly integrable in the strong sense if and only if:

(a) $\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|E|X_{nj}|=M<\infty$ *a.e.*

(b) *for each $\varepsilon>0$, there exists $\delta>0$ such that whenever $\{B_{nj},\ u_n\leqslant j\leqslant v_n,\ n\geqslant 1\}$ is an array of events satisfying* $\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|P(B_{nj})<\delta$ *a.e., then* $\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}|A_{nj}|E(|X_{nj}|I_{B_{nj}})<\varepsilon$ *a.e.*

Corollary 2. *Let $\{X_{nj},\ u_n\leqslant j\leqslant v_n,\ n\geqslant 1\}$ and $\{A_{nj},\ u_n\leqslant j\leqslant v_n,\ n\geqslant 1\}$ be two arrays of random variables with* $\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}E|A_{nj}|<\infty$.

Then, $\{X_{nj},\ u_n\leqslant j\leqslant v_n,\ n\geqslant 1\}$ is $\{A_{nj}\}$-uniformly integrable in the weak sense if and only if:

(a) $\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}E|A_{nj}|E|X_{nj}|=M<\infty$

(b) *for each $\varepsilon>0$, there exists $\delta>0$ such that whenever $\{B_{nj},\ u_n\leqslant j\leqslant v_n,\ n\geqslant 1\}$ is an array of events satisfying* $\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}E|A_{nj}|P(B_{nj})<\delta$, *then* $\sup_{n\geqslant 1}\sum_{j=u_n}^{v_n}E|A_{nj}|E(|X_{nj}|I_{B_{nj}})<\varepsilon$.

Note that if, in particular, $A_{nj}=a_{nj}$ (nonrandom) for all $u_n\leqslant j\leqslant v_n,\ n\geqslant 1$, Corollary 1 gives a characterization of the $\{a_{nj}\}$-uniform integrability of an array $\{X_{nj},\ u_n\leqslant j\leqslant v_n,\ n\geqslant 1\}$ which extends the characterization of the $\{a_{nj}\}$-uniform integrability of a sequence $\{X_n,\ n\geqslant 1\}$ in Ordoñez Cabrera (1994). From this point of view, Corollary 2 is the characterization of the $\{a_{nj}\}$-uniform integrability of $\{X_{nj},\ u_n\leqslant j\leqslant v_n,\ n\geqslant 1\}$ when we consider $a_{nj}\equiv E|A_{nj}|$ for all $u_n\leqslant j\leqslant v_n,\ n\geqslant 1$.

By using a similar technique to that we used in the proof of Theorem 1, the following characterization of the conditional uniform integrability relative to a sequence of σ-algebras can be obtained:

Theorem 2. *Let $\{X_{nj},\ u_n\leqslant j\leqslant v_n,\ n\geqslant 1\}$ be an array of random variables, and let $\{\mathscr{B}_n,\ n\geqslant 1\}$ be a sequence of sub σ-algebras of $\mathscr{A}$.*

Then, $\{X_{nj},\ u_n\leqslant j\leqslant v_n,\ n\geqslant 1\}$ is conditionally uniformly integrable relative to $\mathscr{B}_n$ if, and only if:

(a) $\sup_{n\geqslant 1}\sup_{u_n\leqslant j\leqslant v_n}E^{\mathscr{B}_n}|X_{nj}|=M<\infty$ *a.e.*

(b) *for each $\varepsilon>0$, there exists $\delta>0$ such that whenever $\{B_{nj},\ u_n\leqslant j\leqslant v_n,\ n\geqslant 1\}$ is an array of events satisfying* $\sup_{n\geqslant 1}\sup_{u_n\leqslant j\leqslant v_n}P^{\mathscr{B}_n}(B_{nj})<\delta$ *a.e., then* $\sup_{n\geqslant 1}\sup_{u_n\leqslant j\leqslant v_n}E^{\mathscr{B}_n}(|X_{nj}|I_{B_{nj}})<\varepsilon$ *a.e.*

4. Convergence of randomly weighted sums

In the following results, we suppose that all the random elements and the random variables are defined on the same probability space $(\Omega, \mathscr{A}, P)$.

Theorem 3. *Let $\{V_{nj},\ u_n \leqslant j \leqslant v_n,\ n \geqslant 1\}$ be an array of random elements taking values in a real separable Banach space and $\{A_{nj},\ u_n \leqslant j \leqslant v_n,\ n \geqslant 1\}$ be an array of random variables such that*:
(1) *A_{nj} and V_{nj} are independent for each j, $u_n \leqslant j \leqslant v_n$ and every $n \geqslant 1$,*
(2) $\lim_{n\to\infty} \sum_{j=u_n}^{v_n} E|A_{nj}| = 0$,
(3) *$\{\|V_{nj}\|^q,\ u_n \leqslant j \leqslant v_n,\ n \geqslant 1\}$ is $\{|A_{nj}|^q\}$-uniformly integrable in weak sense for some $0 < q \leqslant 1$.*
Then $\sum_{j=u_n}^{v_n} |A_{nj}|\|V_{nj}\| \to 0$ in L_q as $n \to \infty$ and, consequently,

$$\left\| \sum_{j=u_n}^{v_n} A_{nj} V_{nj} \right\| \to 0 \quad \textit{in } L_q.$$

Proof. For any $a > 0$ define

$$V'_{nj} = V_{nj} I_{[\|V_{nj}\| \leqslant a]}, \quad V''_{nj} = V_{nj} I_{[\|V_{nj}\| > a]}.$$

Then since $q \leqslant 1$

$$\begin{aligned} E\left[\sum_{j=u_n}^{v_n} |A_{nj}|\|V_{nj}\|\right]^q &\leqslant E\left[\sum_{j=u_n}^{v_n} |A_{nj}|\|V'_{nj}\|\right]^q + E\left[\sum_{j=u_n}^{v_n} |A_{nj}|\|V''_{nj}\|\right]^q \\ &\leqslant E\left[\sum_{j=u_n}^{v_n} |A_{nj}|\|V'_{nj}\|\right]^q + \sum_{j=u_n}^{v_n} E|A_{nj}|^q E\|V''_{nj}\|^q \\ &\leqslant a^q \left[\sum_{j=u_n}^{v_n} E|A_{nj|}\right]^q + \sum_{j=u_n}^{v_n} E|A_{nj}|^q E\|V''_{nj}\|^q \end{aligned}$$

Now, the first sum tends to zero by assumption (2) and the second one tends to zero by assumption (3). □

Remark. If $A_{nj} = a_{nj}$ (nonrandom) a.s for all $u_n \leqslant j \leqslant v_n$, $n \geqslant 1$, then Theorem 3 gives Theorem 6 of Ordoñez Cabrera (1994).

Corollary 3. *Let $\{V_{nj}, -\infty < j < +\infty,\ n \geqslant 1\}$ be an array of random elements taking values in a real separable Banach space, $\{A_{nj},\ -\infty < j < +\infty,\ n \geqslant 1\}$ be an array of random variables and $\{N_n,\ n \geqslant 1\}$ and $\{M_n,\ n \geqslant 1\}$ be two sequence of (not necessarily positive) integer-valued random variables with $N_n \leqslant M_n$ a.e., $n \geqslant 1$, and such that for some nonrandom sequences $\{u_n,\ n \geqslant 1\}$ and $\{v_n,\ n \geqslant 1\}$, we have*

$$P[N_n < u_n] = \mathrm{o}(1) \quad \textit{and} \quad P[M_n > v_n] = \mathrm{o}(1) \textit{ as } n \to \infty.$$

Suppose also that assumptions (1)–(3) of Theorem 3 hold. Then $\sum_{j=N_n}^{M_n} |A_{nj}|\|V_{nj}\| \to 0$ in probability as $n \to \infty$ and, consequently, $\| \sum_{j=u_n}^{v_n} A_{nj} V_{nj}\| \to 0$ in probability.

Proof. For arbitrary $\varepsilon > 0$ and $n \geqslant 1$:

$$P\left[\sum_{j=N_n}^{M_n} |A_{nj}|\|V_{nj}\| > \varepsilon\right] = P\left[\sum_{j=N_n}^{M_n} |A_{nj}|\|V_{nj}\| > \varepsilon, N_n \geqslant u_n, M_n \leqslant v_n\right]$$

$$+P\left[\sum_{j=N_n}^{M_n} |A_{nj}|\|V_{nj}\| > \varepsilon, N_n \geqslant u_n, M_n > v_n\right] + P\left[\sum_{j=N_n}^{M_n} |A_{nj}|\|V_{nj}\| > \varepsilon, N_n < u_n\right]$$

$$\leqslant P\left[\sum_{j=u_n}^{v_n} |A_{nj}|\|V_{nj}\| > \varepsilon\right] + P[M_n > v_n] + P[N_n < u_n] = o(1)$$

by Theorem 3 and assumptions of Corollary 3. $\square$

We need the following lemma for proof of Theorem 4.

Lemma. *Suppose that $\{X_{nj},\ u_n \leqslant j \leqslant v_n,\ n \geqslant 1\}$ is an array of $\{a_{nj}\}$-uniformly integrable random variables satisfying $\sup_{n\geqslant 1} \sum_{j=u_n}^{v_n} |a_{nj}|E|X_{nj}| < \infty$.*

Denote $m_n = 1/\sup_{u_n \leqslant j \leqslant v_n} |a_{nj}|$. If $m_n \to \infty$ as $n \to \infty$ and $p > 1$ then

$$\sum_{j=u_n}^{v_n} |a_{nj}|^p E|X_{nj}|^p I[|X_{nj}| \leqslant m_n] = o(1).$$

Proof. For any $a < m_n$, we have

$$\sum_{j=u_n}^{v_n} |a_{nj}|^p E|X_{nj}|^p I[|X_{nj}| \leqslant m_n] = \sum_{j=u_n}^{v_n} |a_{nj}|^p E|X_{nj}|^p (I[|X_{nj}| \leqslant a] + I[a < |X_{nj}| \leqslant m_n])$$

$$\leqslant \sum_{j=u_n}^{v_n} |a_{nj}|^p a^{p-1} E|X_{nj}| I[|X_{nj}| \leqslant a] + \sum_{j=u_n}^{v_n} |a_{nj}|^p m_n^{p-1} E|X_{nj}| I[|X_{nj}| > a]$$

$$\leqslant m_n^{1-p} a^{p-1} \sup_{m\geqslant 1} \sum_{j=u_m}^{v_m} |a_{mj}|E|X_{mj}| + \sup_{m\geqslant 1} \sum_{j=u_m}^{v_m} |a_{mj}|E|X_{mj}| I[|X_{mj}| > a].$$

By the assumption to the Lemma, the first term above is o(1) as $n \to \infty$ since $m_n \to \infty$; and the second term above is o(1) as $a \to \infty$. $\square$

Theorem 4. *Let $\{V_{nj},\ u_n \leqslant j \leqslant v_n,\ n \geqslant 1\}$ be an array of random elements taking values in a real separable Banach space, and $\{A_{nj},\ u_n \leqslant j \leqslant v_n,\ n \geqslant 1\}$ be an array of random variables such that, for some $0 < q < 1$,*
(1) *A_{nj} and V_{nj} are independent for each j, $u_n \leqslant j \leqslant v_n$ and every $n \geqslant 1$,*
(2) *$\lim_{n\to\infty} \sup_{u_n \leqslant j \leqslant v_n} E|A_{nj}| = 0$,*
(3) *$\{\|V_{nj}\|^q\}$ is $\{(E|A_{nj}|)^q\}$-uniformly integrable,*
(4) *$\sup_{n\geqslant 1} \sum_{j=u_n}^{v_n} (E|A_{nj}|)^q E\|V_{nj}\|^q < \infty$.*

Then $\|\sum_{j=u_n}^{v_n} A_{nj}V_{nj}\| \to 0$ in L_q, as $n \to \infty$.

Proof. Denote $m_n = 1/\sup_{u_n \leqslant j \leqslant v_n} E|A_{nj}|$. We define:

$$V'_{nj} = V_{nj} I_{[\|V_{nj}\| \leqslant m_n]}, \quad V''_{nj} = V_{nj} I_{[\|V_{nj}\| > m_n]}.$$

Given $\varepsilon > 0$, there exists $a > 0$ such that

$$\sup_{n\geqslant 1}\left(\sum_{j=u_n}^{v_n}(E|A_{nj}|)^q E(\|V_{nj}\|^q I_{[\|V_{nj}\|>a]})\right) < \frac{\varepsilon}{2}.$$

As $\lim_{n\to\infty} m_n = \infty$, there exists $n_0 \in N$ such that $m_n > a$ for all $n \geqslant n_0$. Therefore, for all $n \geqslant n_0$:

$$\sum_{j=u_n}^{v_n}(E|A_{nj}|)^q E\|V''_{nj}\|^q < \frac{\varepsilon}{2}.$$

By applying Lemma with $p = 1/q$, $a_{nj} = (E|A_{nj}|)^q$, we can choose $n_0 \in N$ such that for all $n \geqslant n_0$

$$\sum_{j=u_n}^{v_n}((E|A_{nj}|)^q)^p E(\|V_{nj}\|^q)^p I_{[\|V_{nj}\|^q \leqslant 1/\sup_{u_n\leqslant j\leqslant v_n}(E|A_{nj}|)^q]} = \sum_{j=u_n}^{v_n} E|A_{nj}|E\|V'_{nj}\| < \left(\frac{\varepsilon}{2}\right)^{1/q}$$

Then since $q < 1$, we have for all $n \geqslant n_0$

$$\begin{aligned} E\left\|\sum_{j=u_n}^{v_n} A_{nj}V_{nj}\right\|^q &\leqslant E\left\|\sum_{j=u_n}^{v_n} A_{nj}V'_{nj}\right\|^q + E\left\|\sum_{j=u_n}^{v_n} A_{nj}V''_{nj}\right\|^q \\ &\leqslant \left(E\left\|\sum_{j=u_n}^{v_n} A_{nj}V'_{nj}\right\|\right)^q + \sum_{j=u_n}^{v_n} E|A_{nj}|^q E\|V''_{nj}\|^q \\ &\leqslant \left(\sum_{j=u_n}^{v_n} E|A_{nj}|E\|V'_{nj}\|\right)^q + \sum_{j=u_n}^{v_n}(E|A_{nj}|)^q E\|V''_{nj}\|^q < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square \end{aligned}$$

Theorem 5. *Let* $\{V_{nj},\ u_n \leqslant j \leqslant v_n,\ n \geqslant 1\}$ *be an array of random elements taking values in a real separable Banach space and* $\{A_{nj},\ u_n \leqslant j \leqslant v_n,\ n \geqslant 1\}$ *be an array of random variables such that assumption* (2) *of Theorem* 3, *that is,* $\lim_{n\to\infty}\sum_{j=u_n}^{v_n} E|A_{nj}| = 0$, *holds. Let* $\mathscr{B}_n = \sigma(A_{nj},\ u_n \leqslant j \leqslant v_n)$, *for each* $n \geqslant 1$ *and suppose that* $\{V_{nj},\ u_n \leqslant j \leqslant v_n,\ n \geqslant 1\}$ *is* $\{A_{nj}\}$-*conditionally uniformly integrable relative to* $\mathscr{B}_n$.

Then $\sum_{j=u_n}^{v_n}|A_{nj}|\|V_{nj}\| \to 0$ *in* L_1 *as* $n \to \infty$ (*and, consequently,* $\|\sum_{j=u_n}^{v_n} A_{nj}V_{nj}\| \to 0$ *in* L_1).

Proof. Given $\varepsilon > 0$, there exists $a > 0$ such that

$$\sup_{n\geqslant 1}\left(\sum_{j=u_n}^{v_n}|A_{nj}|E^{\mathscr{B}_n}(\|V_{nj}\|I_{[\|V_{nj}\|>a]})\right) < \frac{\varepsilon}{2} \quad \text{a.e.}$$

We define V'_{nj} and V''_{nj}, as in Theorem 3. Then, since the $\{A_{nj},\ u_n \leqslant j \leqslant v_n\}$ are $\mathscr{B}_n$-measurable:

$$\begin{aligned} E^{\mathscr{B}_n}\left(\sum_{j=u_n}^{v_n}|A_{nj}|\|V_{nj}\|\right) &\leqslant E^{\mathscr{B}_n}\left(\sum_{j=u_n}^{v_n}|A_{nj}|\|V'_{nj}\|\right) + E^{\mathscr{B}_n}\left(\sum_{j=u_n}^{v_n}|A_{nj}|\|V''_{nj}\|\right) \\ &\leqslant aE^{\mathscr{B}_n}\left(\sum_{j=u_n}^{v_n}|A_{nj}|\right) + E^{\mathscr{B}_n}\left(\sum_{j=u_n}^{v_n}|A_{nj}|\|V''_{nj}\|\right) \\ &\leqslant a\left(\sum_{j=u_n}^{v_n}|A_{nj}|\right) + \left(\sum_{j=u_n}^{v_n}|A_{nj}|E^{\mathscr{B}_n}\|V''_{nj}\|\right). \end{aligned}$$

There exists $n_0 \in N$ such that the expectation of the first sum is less than $\varepsilon/2$ for all $n \geqslant n_0$, and the expectation of the second sum is less than $\varepsilon/2$ by the choice of a. Then, given $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n \geqslant n_0$:

$$E\left(E^{\mathscr{B}_n}\left(\sum_{j=u_n}^{v_n} |A_{nj}|\|V_{nj}\|\right)\right) = E\left(\sum_{j=u_n}^{v_n} |A_{nj}|\|V_{nj}\|\right) < \varepsilon. \qquad \square$$

A light modification of conditions in Theorem 5 allow us to obtain a result of strong convergence of the sequence of conditional expectations.

Theorem 6. *Let* $\{V_{nj},\ u_n \leqslant j \leqslant v_n,\ n \geqslant 1\}$ *be an array of random elements taking values in a real separable Banach space and* $\{A_{nj},\ u_n \leqslant j \leqslant v_n,\ n \geqslant 1\}$ be *an array of random variables such that* $\sup_{u_n \leqslant j \leqslant v_n} |A_{nj}| = \mathrm{o}((v_n - u_n)^{-1})$ *a.e.*

Let $\mathscr{B}_n = \sigma(A_{nj},\ u_n \leqslant j \leqslant v_n)$, *for each* $n \geqslant 1$ *and suppose that* $\{V_{nj},\ u_n \leqslant j \leqslant v_n,\ n \geqslant 1\}$ *is* $\{A_{nj}\}$*-conditionally uniformly integrable relative to* $\mathscr{B}_n$.

Then $E^{B_n} \sum_{j=u_n}^{v_n} |A_{nj}|\|V_{nj}\| \to 0$ *a.e. as* $n \to \infty$.

Proof. The begining is the same as in Theorem 5. Next, estimate the first sum in the following way:

$$a \sum_{j=u_n}^{v_n} |A_{nj}| \leqslant a(v_n - u_n) \sup_{u_n \leqslant j \leqslant v_n} |A_{nj}|. \qquad \square$$

We often use assumption (2) from the Theorem 3. In order to check it, the following statement may be useful.

Proposition. *Let* $\{A_{nj},\ u_n \leqslant j \leqslant v_n,\ n \geqslant 1\}$ *be an array of random variables such that* $\sup_n \sum_{j=u_n}^{v_n} (E|A_{nj}|)^r < \infty$, *for some* $r \in (0,1)$ *and* $\lim_{n\to\infty} \sup_{u_n \leqslant j \leqslant v_n} E|A_{nj}| = 0$. *Then assumption* (2) *of Theorem* 3 *holds, that is,* $\lim_{n\to\infty} \sum_{j=u_n}^{v_n} E|A_{nj}| = 0$.

Proof. It is easy to see that

$$\sum_{j=u_n}^{v_n} E|A_{nj}| < \left(\sup_{u_n \leqslant j \leqslant v_n} E|A_{nj}|\right)^{1-r} \sum_{j=u_n}^{v_n} (E|A_{nj}|)^r. \qquad \square$$

References

Adler, A., Rosalsky, A., Taylor, R.L., 1992. Some strong laws of large numbers for sums of random elements. Bull. Inst. Math. Acad. Sinica 20, 335–357.

Gut, A., 1992. The weak law of large numbers for arrays. Statist Probab. Lett. 14, 49–52.

Hu, T.C., Chang, H.C., 1999. Stability for randomly weighted sums of random elements. Internat. J. Math. Math. Sci. 22, 571–580.

Ordoñez Cabrera, M., 1988. Limit theorems for randomly weighted sums of random elements in normed linear spaces. J. Multivariate Anal. 25, 139–145.

Ordoñez Cabrera, M., 1994. Convergence of weighted sums of random variables and uniform integrability concerning the weights. Collect. Math. 45, 121–132.

Rosalsky, A., Sreehari, M., 1998. On the limiting behavior of randomly weighted partial sums. Statist. Probab. Lett. 40, 403–410.

Sung, S.H., 1999. Weak law of large numbers for arrays of random variables. Statist. Probab. Lett. 42, 293–298.

Taylor, R.L., Calhoun, C.A., 1983. On the almost sure convergence of randomly weighted sums of random elements. Ann. Probab. 11, 795–797.

Taylor, R.L., Padgett, W.J., 1972. Weak laws of large numbers in normed linear spaces. Ann. Math. Statist. 43, 1267–1274.

Taylor, R.L., Padgett, W.J., 1974. Some laws of large numbers for normed linear spaces. Sankhya Ser. A 36, 359–368.

Taylor, R.L., Padgett, W.J., 1976. Weak laws of large numbers in Banach spaces and their extensions. Lecture Notes in Mathematics. Vol. 526, Springer, Berlin, pp. 227–242.

Taylor, R.L., Raina, C.C., Daffer, P.Z., 1984. Stochastic convergence of randomly weighted sums of random elements. Stochastic Anal. Appl. 2, 299–321.

Wang, X., Rao, M.B., 1995. Convergence in the r-th mean and some weak laws of large numbers for random weighted sums of random elements in Banach spaces. Northeast. Math. J. 11, 113–126.

Wei, D., Taylor, R.L., 1978a. Geometrical consideration of weighted sums convergence and random weighting. Bull. Inst. Math. Acad. Sinica 6, 49–59.

Wei, D., Taylor, R.L., 1978b. Convergence of weighted sums of tight random elements. J. Multivariate Anal. 8, 282–294.