ELSEVIER

Statistics & Probability Letters 48 (2000) 369–374

STATISTICS & PROBABILITY LETTERS

www.elsevier.nl/locate/stapro

An application of the Ryll-Nardzewski–Woyczyński theorem to a uniform weak law for tail series of weighted sums of random elements in Banach spaces

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Received August 1999; received in revised form October 1999

Abstract

For a sequence of Banach space valued random elements $\{V_n, n \geqslant 1\}$ (which are not necessarily independent) with the series $\sum_{n=1}^{\infty} V_n$ converging unconditionally in probability and an infinite array $a = \{a_{ni},\ i \geqslant n,\ n \geqslant 1\}$ of constants, conditions are given under which (i) for all $n \geqslant 1$, the sequence of weighted sums $\sum_{i=n}^{m} a_{ni} V_i$ converges in probability to a random element $T_n(a)$ as $m \to \infty$, and (ii) $T_n(a) \xrightarrow{\text{P}} 0$ uniformly in a as $n \to \infty$ where a is in a suitably restricted class of infinite arrays. The key tool used in the proof is a theorem of Ryll-Nardzewski and Woyczyński (1975, Proc. Amer. Math. Soc. 53, 96–98). © 2000 Elsevier Science B.V. All rights reserved

MSC: 60F05; 60B12

Keywords: Real separable Banach space; Weighted sums of random elements; Converge unconditionally in probability; Converge in probability; Tail series; Uniform weak law of large numbers

1. Introduction

In this paper, a uniform weak law of large numbers (WLLN) is established for tail series of weighted sums of Banach space valued random elements. Throughout, $\{V_n, n \geqslant 1\}$ is a sequence of random elements defined on a probability space $(\Omega, \mathscr{F}, P)$ and assuming values in a real separable Banach space $\mathscr{X}$ with norm $\|\cdot\|$. In general, it is not assumed that the $\{V_n, n \geqslant 1\}$ are independent.

Let $\Theta = \{\theta = (\theta_1, \theta_2, \ldots), \theta_n = \pm 1, n \geqslant 1\}$. The series $\sum_{n=1}^{\infty} V_n$ is assumed to *converge unconditionally in probability*; that is, for every $\theta \in \Theta$, the sequence of partial sums $S_n^{\theta} \equiv \sum_{i=1}^{n} \theta_i V_i$ converges in probability to

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PII: S0167-7152(00)00018-3

a random element S^θ in $\mathscr{X}$. Let $S_0^\theta = 0, \theta \in \Theta$. Then for all $\theta \in \Theta, \{T_n^\theta \equiv S^\theta - S_{n-1}^\theta, n \geqslant 1\}$ is a well-defined sequence of random elements (called a *tail series*) with $T_n^\theta \xrightarrow{P} 0$.

Let $b = \{b_n, n \geqslant 1\}$ be a sequence of positive constants and let $a = \{a_{ni},\ i \geqslant n,\ n \geqslant 1\}$ be an infinite array of real numbers. We will use the notation $a \preccurlyeq 1/b$ to denote that $\sup_{i \geqslant n} |a_{ni}| \leqslant 1/b_n$ for all $n \geqslant 1$. The a_{ni} comprising the array a are referred to as *weights*. For $m \geqslant n \geqslant 1$, let $S_n^m(a) = \sum_{i=n}^m a_{ni} V_i$; the $S_n^m(a)$ are referred to as *weighted sums* with weight array a.

For a given sequence b of positive constants, the main result establishes that if $T_n^\theta / b_n \xrightarrow{P} 0$ uniformly for $\theta \in \Theta$ (condition (2.2)), then for every infinite array $a \preccurlyeq 1/b$ and for all $n \geqslant 1$, the weighted sums

$S_n^m(a) \xrightarrow{P}$ a random element $T_n(a)$ as $m \to \infty$ (conclusion (2.3))

where

$T_n(a) \xrightarrow{P} 0$ uniformly in $a \preccurlyeq 1/b$ as $n \to \infty$ (conclusion (2.5)).

It is convenient to denote $T_n(a)$ by $\sum_{i=n}^\infty a_{ni} V_i,\ n \geqslant 1$. The sequence $\{T_n(a),\ n \geqslant 1\}$ is also referred to as a *tail series*. Conclusion (2.5) is thus a tail series uniform WLLN.

The key tool in proving this result is the following striking theorem of Ryll-Nardzewski and Woyczyński (1975). This theorem may also be found in Vakhania et al. (1987, Theorem 4.2, p. 306) and in Kwapień and Woyczyński (1992, Theorem A.1.1, p. 308). It had been proved by Kashin (1973) and by Maurey and Pisier (1973) when $\mathscr{X}$ is the real line. An almost sure (a.s.) version of the Ryll-Nardzewski–Woyczyński theorem was obtained by Musiał et al. (1974).

Theorem 1 (Ryll-Nardzewski and Woyczyński, 1975)**.** *Let $\{V_n, n \geqslant 1\}$ be a sequence of (not necessarily independent) random elements in a real separable Banach space $\mathscr{X}$. Then the series $\sum_{n=1}^\infty V_n$ converges unconditionally in probability if and only if for every bounded real sequence $\{\lambda_n, n \geqslant 1\}$, the series $\sum_{i=1}^n \lambda_i V_i$ converges in probability to a random element in $\mathscr{X}$.*

The following lemma is used in the proof of the main result. Lemma 1 is a version with a different indexing of a result which can be found in Vakhania et al. (1987, Lemma 4.3(c), p. 307) or in Kwapień and Woyczyński (1992, p. 310).

Lemma 1. *For all $m \geqslant n \geqslant 1$ and all choices of constants $\lambda_n, \ldots, \lambda_m$ with $\max_{n \leqslant i \leqslant m} |\lambda_i| \leqslant 1$,*

$$P\left\{\left\|\sum_{i=n}^m \lambda_i V_i\right\| > t\right\} \leqslant 8 \max_{\theta_n, \ldots, \theta_m = \pm 1} P\left\{\left\|\sum_{i=n}^m \theta_i V_i\right\| > \frac{t}{8}\right\}, \quad t > 0. \tag{1.1}$$

2. The main result

With the preliminaries accounted for, the main result of this paper, which establishes a tail series uniform WLLN for weighted sums, may be stated and proved. In this section and in the next, we will use the notation introduced in Section 1. In (2.4) and (2.5) below, the notation $\sup_{a \preccurlyeq 1/b}$ signifies that the supremum is taken over all infinite arrays a with $a \preccurlyeq 1/b$.

Theorem 2. *Let $\{V_n, n \geqslant 1\}$ be a sequence of (not necessarily independent) random elements in a real separable Banach space $\mathscr{X}$ and suppose that*

$$\text{the series } \sum_{n=1}^\infty V_n \text{ converges unconditionally in probability.} \tag{2.1}$$

Let $b = \{b_n, n \geqslant 1\}$ be a sequence of positive constants such that

$$\lim_{n\to\infty} \sup_{\theta\in\Theta} P\left\{ \frac{\|T_n^\theta\|}{b_n} > \varepsilon \right\} = 0 \quad \text{for all } \varepsilon > 0. \tag{2.2}$$

Then for every infinite array $a \preccurlyeq 1/b$ and for all $n \geqslant 1$, there exists a random element $T_n(a) = \sum_{i=n}^{\infty} a_{ni} V_i$ in $\mathscr{X}$ with the weighted sums

$$S_n^m(a) = \sum_{i=n}^{m} a_{ni} V_i \xrightarrow{\text{P}} T_n(a) \quad \text{as } m \to \infty. \tag{2.3}$$

For all $n \geqslant 1$, the family of random elements $\{T_n(a), a \preccurlyeq 1/b\}$ satisfies

$$\sup_{a\preccurlyeq 1/b} P\{\|T_n(a)\| > t\} \leqslant 8 \sup_{\theta\in\Theta} P\left\{ \frac{\|T_n^\theta\|}{b_n} > \frac{t}{8} \right\} \quad \text{for all } t > 0. \tag{2.4}$$

Moreover, the tail series uniform WLLN

$$\lim_{n\to\infty} \sup_{a\preccurlyeq 1/b} P\{\|T_n(a)\| > \varepsilon\} = 0 \quad \text{for all } \varepsilon > 0 \tag{2.5}$$

is obtained.

Proof. Let a be an infinite array with $a \preccurlyeq 1/b$. Then setting $a_{ni} = 0, 1 \leqslant i \leqslant n-1, n \geqslant 2$,

$$\sup_{i\geqslant 1} |a_{ni} b_n| \leqslant 1 \quad \text{for all } n \geqslant 1 \tag{2.6}$$

and so by (2.1) and Theorem 1, for all $n \geqslant 1$ the series $\sum_{i=n}^{m} (a_{ni} b_n) V_i = \sum_{i=1}^{m} (a_{ni} b_n) V_i$ converges in probability as $m \to \infty$ to a random element in $\mathscr{X}$. Consequently, for all $n \geqslant 1$, there exists a random element $T_n(a) = \sum_{i=n}^{\infty} a_{ni} V_i$ in $\mathscr{X}$ with

$$S_n^m(a) = \sum_{i=n}^{m} a_{ni} V_i = \frac{1}{b_n} \sum_{i=n}^{m} (a_{ni} b_n) V_i \xrightarrow{\text{P}} T_n(a) \quad \text{as } m \to \infty$$

and so (2.3) holds.

To prove (2.4), note that for all $n \geqslant 1, t > 0$, and $\varepsilon > 0$

$$\begin{aligned}
P\{\|T_n(a)\| > t + \varepsilon\} &\leqslant \liminf_{m\to\infty} P\{\|S_n^m(a)\| > t + \varepsilon\} \\
&\qquad \text{(by Chow and Teicher, 1997, Theorem 8.1.3, p. 278)} \\
&= \liminf_{m\to\infty} P\left\{ \left\| \sum_{i=n}^{m} (a_{ni} b_n) V_i \right\| > (t+\varepsilon) b_n \right\} \\
&\leqslant \liminf_{m\to\infty} 8 \max_{\theta_n,\ldots,\theta_m = \pm 1} P\left\{ \frac{1}{b_n} \left\| \sum_{i=n}^{m} \theta_i V_i \right\| > \frac{t+\varepsilon}{8} \right\} \quad \text{(by (2.6) and Lemma 1)} \\
&= \liminf_{m\to\infty} 8 \sup_{\theta\in\Theta} P\left\{ \frac{1}{b_n} \|(S^\theta - S_{n-1}^\theta) - (S^\theta - S_m^\theta)\| > \frac{t+\varepsilon}{8} \right\} \\
&\leqslant \liminf_{m\to\infty} 8 \left[\sup_{\theta\in\Theta} P\left\{ \frac{1}{b_n} \|T_n^\theta\| > \frac{t}{8} \right\} + \sup_{\theta\in\Theta} P\left\{ \|T_{m+1}^\theta\| > \frac{\varepsilon b_n}{8} \right\} \right]
\end{aligned}$$

$$= 8 \sup_{\theta\in\Theta} P\left\{\frac{1}{b_n}\|T_n^\theta\| > \frac{t}{8}\right\}$$

(by (2.1); see Vakhania et al., 1987, p. 304 and 310).

Thus, for all $n\geqslant 1$, $t > 0$, and $\varepsilon > 0$

$$P\{\|T_n(a)\| > t+\varepsilon\} \leqslant 8 \sup_{\theta\in\Theta} P\left\{\frac{1}{b_n}\|T_n^\theta\| > \frac{t}{8}\right\}.$$

Let $\varepsilon \downarrow 0$. Thus, for all $n\geqslant 1$ and $t > 0$

$$P\{\|T_n(a)\| > t\} \leqslant 8 \sup_{\theta\in\Theta} P\left\{\frac{1}{b_n}\|T_n^\theta\| > \frac{t}{8}\right\}$$

implying (2.4) since a is arbitrary. Finally, (2.5) follows immediately from (2.2) and (2.4). □

Remark 1. (i) Under (2.1), condition (2.2) will automatically hold if $\inf_{n\geqslant 1} b_n > 0$. (See Vakhania et al., 1987, p. 304 and 310.) However, the condition $\inf_{n\geqslant 1} b_n > 0$ is rather restrictive and Theorem 2 is most interesting when $b_n \to 0$.

(ii) Theorem 3.1 of Rosalsky and Rosenblatt (1997) furnishes conditions under which (2.1) and (2.2) will hold. If $\{g_n(x), n\geqslant 1\}$ is a sequence of nondecreasing functions defined on $[0,\infty]$ with

$$0\leqslant g_n(0)\uparrow,\ 0 < g_n(x)\uparrow \quad \text{as } n\uparrow \text{ for each } x > 0$$

and for all $n\geqslant 1$

$$\lim_{x\to\infty} g_n(x) = \infty, \quad \frac{x}{g_n(x)} \uparrow \text{ on } (0,\infty), \tag{2.7}$$

it follows from Theorem 3.1 of Rosalsky and Rosenblatt (1997) and the second-half of (2.7) that the condition

$$\sum_{i=n}^{\infty} Eg_i(\|V_i\|) = \mathrm{o}(g_n(b_n))$$

ensures that assumptions (2.1) and (2.2) hold. In particular, taking $g_n(x) = x^p, x\geqslant 0, n\geqslant 1$ where $0 < p\leqslant 1$, assumptions (2.1) and (2.2) will hold if

$$\sum_{i=n}^{\infty} E\|V_i\|^p = \mathrm{o}(b_n^p). \tag{2.8}$$

(iii) Theorem 3.2 of Rosalsky and Rosenblatt (1997) also provides conditions under which (2.8) (when $1 < p\leqslant 2$) implies (2.1) and (2.2). However, additional conditions are imposed on the sequence of random elements $\{V_n, n\geqslant 1\}$ and on the Banach space $\mathscr{X}$. Specifically, suppose that $\{V_n, n\geqslant 1\}$ is a sequence of independent, mean 0 random elements and that the Banach space $\mathscr{X}$ is of Rademacher type p where $1 < p\leqslant 2$. Then it follows from Theorem 3.2 of Rosalsky and Rosenblatt (1997) that (2.8) ensures that assumptions (2.1) and (2.2) hold.

3. Two counterexamples

The main result of Nam et al. (1999) establishes for an a.s. convergent series $\sum_{n=1}^{\infty} V_n$ of independent random elements with tail series $T_n = \sum_{i=n}^{\infty} V_i, n\geqslant 1$ and a sequence of positive constants $\{b_n, n\geqslant 1\}$ satisfying the quasi-monotonicity condition

$$b_m \leqslant \text{Const. } b_n \quad \text{whenever } m\geqslant n\geqslant 1 \tag{3.1}$$

that the tail series WLLN

$$\frac{T_n}{b_n} \xrightarrow{P} 0 \tag{3.2}$$

and the apparently stronger limit law

$$\frac{\sup_{j\geqslant n} \|T_j\|}{b_n} \xrightarrow{P} 0 \tag{3.3}$$

are indeed equivalent. The example of Nam and Rosalsky (1996) reveals that (3.2) does not necessarily imply (3.3) without the quasi-monotonicity condition (3.1). We will now present two examples which show that the equivalence between (3.2) and (3.3) does not have a direct generalization for the type of weighted sums being considered in the current work. In the first example, we present a sequence of independent (real-valued) random variables $\{V_n, n\geqslant 1\}$ such that the series $\sum_{n=1}^{\infty} \theta_n V_n$ converges a.s. for every $\theta \in \Theta$, a sequence $b = o(1)$ of positive constants satisfying (3.1), and an infinite array a with $a \preccurlyeq 1/b$ such that (2.2) holds, $T_n(a) \xrightarrow{P} 0$, but

$$\sup_{j\geqslant n} |T_j(a)| \xrightarrow{P} 0 \tag{3.4}$$

fails. As is well known, assertion (3.4) is equivalent to $T_n(a) \to 0$ a.s.

Example 1. Let $\{X_n, n\geqslant 1\}$ be a sequence of independent and identically distributed random variables with $EX_1 = 0$, $EX_1^2 = 1$. Let $V_n = n^{-1}X_n, n\geqslant 1$ and $t_n^2 = \sum_{i=n}^{\infty} i^{-2}, n\geqslant 1$. Now for every $\theta \in \Theta$, by the Khintchine–Kolmogorov convergence theorem, the series $\sum_{n=1}^{\infty} \theta_n V_n$ converges a.s. and $\{T_n^{\theta} = \sum_{i=n}^{\infty} \theta_i V_i, n\geqslant 1\}$ is a well-defined sequence of random variables with $ET_n^{\theta} = 0$, $\operatorname{Var} T_n^{\theta} = t_n^2, n\geqslant 1$. Define the sequence b and infinite array a by

$$b_n = t_n(\log\log(t_n^{-2} \vee e^e))^{1/2}, \quad n\geqslant 1, \qquad a_{ni} = b_n^{-1}, \quad i\geqslant n,\ n\geqslant 1.$$

Then $b = o(1)$ and satisfies (3.1) and $a \preccurlyeq 1/b$. Now, by Chebyshev's theorem, for all $\varepsilon > 0$

$$\sup_{\theta\in\Theta} P\left\{\frac{|T_n^{\theta}|}{b_n} > \varepsilon\right\} \leqslant \frac{t_n^2}{\varepsilon^2 b_n^2} = o(1) \tag{3.5}$$

and so (2.2) holds. Now, by Theorem 2 of Rosalsky (1983), the tail series law of the iterated logarithm

$$\limsup_{n\to\infty} T_n(a) = \limsup_{n\to\infty} \frac{\sum_{i=n}^{\infty} V_i}{b_n} = \sqrt{2} \quad \text{a.s.}$$

holds. Hence, it is *false* that $T_n(a) \to 0$ a.s. and so $\sup_{j\geqslant n} |T_j(a)| \overset{P}{\nrightarrow} 0$. Taking $\theta = (1, 1, \ldots)$, it follows from (3.5) that $T_n(a) \xrightarrow{P} 0$. Finally, it may be mentioned that it follows from Corollary 4 of Nam and Rosalsky (1995) or Theorem 7 of Nam and Rosalsky (1996) (which are distinctly different results) that

$$\frac{\sup_{j\geqslant n} |\sum_{i=j}^{\infty} V_i|}{b_n} \xrightarrow{P} 0.$$

Remark 2. Note in the preceding example that if $P\{X_1 = -1\} = P\{X_1 = 1\} = \frac{1}{2}$, then $\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} n^{-1}X_n$ is the "harmonic series" with the random choice of sign for each term being independent and equally likely to be positive or negative.

The next example concerns any series $\sum_{n=1}^{\infty} V_n$ of random elements which converges unconditionally in probability but which does not converge a.s. A sequence b of positive constants and an infinite array a are specified such that (3.1) holds, $a \preccurlyeq 1/b$, (2.2) holds, $T_n(a) \xrightarrow{P} 0$, but $\sup_{j\geqslant n} \|T_j(a)\| \xrightarrow{P} 0$ *fails*.

Example 2. Let $\{V_n, n\geqslant 1\}$ be any sequence of random elements such that $\sum_{n=1}^{\infty} V_n$ converges unconditionally in probability but does not converge a.s. Let the random element S be such that $\sum_{i=1}^{n} V_i \xrightarrow{\text{P}} S$. Define the sequence b and infinite array a by

$$b_n = 1, \quad n\geqslant 1, \qquad a_{ni} = 1, \quad i\geqslant n, \ n\geqslant 1.$$

Then b satisfies (3.1) and $a\preccurlyeq 1/b$. Now (2.2) holds by Remark 1(i). Then $T_1(a) = S$ and for $n\geqslant 2$,

$$T_n(a) = S - \sum_{i=1}^{n-1} a_{ni} V_i \xrightarrow{\text{P}} 0,$$

but $P\{T_n(a) \to 0\} < 1$ since $\sum_{n=1}^{\infty} V_n$ does not converge a.s. Hence

$$\sup_{j\geqslant n} \|T_j(a)\| \xrightarrow{\text{P}} 0$$

fails.

Acknowledgements

The authors are grateful to the referee for carefully reading the manuscript and for providing some comments which helped them improve the presentation. Part of the research of Tien-Chung Hu was conducted during his visit to the Department of Statistics at the University of Florida and he thanks the Department for the kind hospitality extended to him. Part of the research of A.I. Volodin was conducted during his visit to the Mathematical Institute at Copenhagen University. He wishes to express his gratitude to the Institute and especially to Professor F. Topsøe for their exceptionally warm hospitality and for the use of their facilities.

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