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Almost sure lim sup behavior of bootstrapped means with applications to pairwise i.i.d. sequences and stationary ergodic sequences

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Abstract

For a sequence of random variables $\{X_n,\ n\geqslant 1\}$, the exact convergence rate (i.e., an iterated logarithm-type result) is obtained for bootstrapped means. No assumptions are made concerning either the marginal or joint distributions of the random variables $\{X_n,\ n\geqslant 1\}$. As special cases, new results follow for pairwise i.i.d. sequences and stationary ergodic sequences.

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1. Introduction

The exact convergence rates in the form of iterated logarithm-type results are obtained for bootstrapped means from sequences of random variables which are not necessarily independent or identically distributed. We begin with a brief discussion of results in the literature pertaining to a sequence of independent and identically distributed (i.i.d.) random variables. Let $\{X, X_n,\ n\geqslant 1\}$ be a sequence of i.i.d. random variables defined on a probability space $(\Omega, \mathscr{F}, P)$ and write $S_n=\sum_{i=1}^n X_i,\ n\geqslant 1$. For $\omega\in\Omega$ and $n\geqslant 1$, let $P_n(\omega)=n^{-1}\sum_{i=1}^n \delta_{X_i(\omega)}$ denote the empirical measure and let $\{\hat{X}_{n,j}^{(\omega)}, 1\leqslant j\leqslant m(n)\}$ be i.i.d. random variables with law $P_n(\omega)$ where $\{m(n),\ n\geqslant 1\}$ is a sequence of positive integers. In other words, the random variables $\{\hat{X}_{n,j}^{(\omega)}, 1\leqslant j\leqslant m(n)\}$

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result by sampling $m(n)$ times with replacement from the n observations $X_1(\omega),\ldots,X_n(\omega)$ such that for each of the $m(n)$ selections, each $X_i(\omega)$ has probability n^{-1} of being chosen. For each $n\geqslant 1$, $\{\hat{X}_{n,j}^{(\omega)}, 1\leqslant j\leqslant m(n)\}$ is the so-called Efron (1979) *bootstrap sample* from $X_1,\ldots,X_n$ with bootstrap sample size $m(n)$. Let $\bar{X}_n(\omega)=S_n(\omega)/n$ denote the sample mean of $\{X_i(\omega), 1\leqslant i\leqslant n\}$, $n\geqslant 1$.

When X is nondegenerate and $EX^2<\infty$, Bickel and Freedman (1981) and Singh (1981) showed that for almost every $\omega\in\Omega$ the central limit theorem (CLT),

$$n^{1/2}\left(\frac{\hat{S}_n^{(\omega)}}{n}-\bar{X}_n(\omega)\right)\xrightarrow{\mathrm{d}}N(0,\sigma^2) \tag{1.1}$$

obtains. Here and below, $\hat{S}_n^{(\omega)}=\sum_{j=1}^{n}\hat{X}_{n,j}^{(\omega)}$, $n\geqslant 1$ and $\sigma^2=\operatorname{Var}X$. Note that by the Glivenko–Cantelli theorem $P_n(\omega)$ is close to $\mathscr{L}(X)$ for almost every $\omega\in\Omega$ and all large n, and by the classical Lévy CLT

$$n^{1/2}\left(\frac{S_n}{n}-EX\right)\xrightarrow{\mathrm{d}}N(0,\sigma^2). \tag{1.2}$$

Thus there is no major difference between the CLT (1.1) for bootstrapped means and the classical Lévy CLT (1.2) for i.i.d. random variables; for almost every $\omega\in\Omega$, the bootstrap statistic $n^{1/2}(n^{-1}\hat{S}_n^{(\omega)}-\bar{X}_n(\omega))$ is close in distribution to that of $n^{1/2}(n^{-1}S_n-EX)$ for all large n. This is, very roughly speaking, the idea behind the bootstrap. See the pioneering work of Efron (1979), where this nice idea is made explicit and where it is substantiated with several important examples. Giné and Zinn (1989) proved that in order for there to exist positive scalars $a_n\uparrow\infty$, centerings $c_n(\omega)$, and a random probability measure $v(\omega)$ nondegenerate with positive probability such that

$$\mathscr{L}\left(\frac{\hat{S}_n^{(\omega)}}{a_n}-c_n(\omega)\right)\xrightarrow{\mathrm{w}}v(\omega)\text{ for almost every }\omega\in\Omega,$$

it is necessary that $EX^2<\infty$. The limit law (1.1) tells us just the right rate at which to magnify the difference $n^{-1}\hat{S}_n^{(\omega)}-\bar{X}_n(\omega)$, which is tending to 0 in probability for almost every $\omega\in\Omega$, in order to obtain convergence in distribution to a nondegenerate law for almost every $\omega\in\Omega$. We note from (1.1) that for almost every $\omega\in\Omega$,

$$\frac{n^{1/2}}{x_n}\left(\frac{\hat{S}_n^{(\omega)}}{n}-\bar{X}_n(\omega)\right)\to 0\text{ in probability}$$

for any sequence of constants $x_n\to\infty$.

On the other hand, strong laws of large numbers (SLLNs) were proved by Athreya (1983) and Csörgő (1992) for bootstrapped means. Arenal-Gutiérrez et al. (1996) analyzed the results of Athreya (1983) and Csörgő (1992). Then, by taking into account the different growth rates for the resampling size $m(n)$, they gave new and simple proofs of those results. They also provided examples that show that the sizes of resampling required by their results to ensure almost sure (a.s.) convergence are not far from optimal.

It is natural to ask about an exact convergence rate for bootstrapped means. The main finding of the current work is Theorem 1 wherein we establish the law of the iterated logarithm (LIL)-type result (3.5) for bootstrapped means from a sequence of random variables $\{X_n,\ n\geqslant 1\}$. An interesting and unusual feature of Theorem 1 is that no assumptions are made concerning either the marginal or joint distributions of the random variables $\{X_n,\ n\geqslant 1\}$; it is not assumed that these random variables are independent or that they are identically distributed. Furthermore, in general, no moment conditions are imposed on the $\{X_n,\ n\geqslant 1\}$. Pioneering work establishing a LIL-type result for bootstrapped means from a sequence of i.i.d. integrable random variables was carried out by Mikosch (1994); our method is substantially different from his.

The plan of the paper is as follows. The lemmas used in the proof of Theorem 1 will be presented in Section 2. Theorem 1 will be stated and proved in Section 3. In Section 4, special cases of Theorem 1 are presented yielding new results for pairwise i.i.d. sequences and stationary ergodic sequences. Some final comments are made in Section 5.

2. Preliminary lemmas

Two lemmas used to establish Theorem 1 are presented in this section. The first lemma may be called the *transformation principle of the bootstrap*.

Lemma 1. *Let* $\{X_i, 1\leqslant i\leqslant n\}$ *be random variables* (*which are not necessarily independent or identically distributed*) *and for* $\omega\in\Omega$ *let* $\{\hat{X}_{n,j}^{(\omega)}, 1\leqslant j\leqslant m\}$ *be i.i.d. random variables with law* $P_n(\omega)=n^{-1}\sum_{i=1}^n \delta_{X_i(\omega)}$ *and let* $f_\omega : R\to R$ *be a Borel function. Then* $\{f_\omega(\hat{X}_{n,j}^{(\omega)}), 1\leqslant j\leqslant m\}$ *are i.i.d. random variables with law* $P_{n,f_\omega}(\omega)= n^{-1}\sum_{i=1}^n \delta_{f_\omega(X_i(\omega))}$.

Proof. The i.i.d. portion of the conclusion is clear. Next, let A be a Borel set. Then

$$\begin{aligned}\text{Prob}\{f_\omega(\hat{X}_{n,1}^{(\omega)})\in A\} &= \text{Prob}\{\hat{X}_{n,1}^{(\omega)}\in f_\omega^{-1}(A)\}\\ &= P_n(\omega)\{f_\omega^{-1}(A)\}\\ &= n^{-1}\sum_{i=1}^n \delta_{X_i(\omega)}(f_\omega^{-1}(A))\\ &= n^{-1}\sum_{i=1}^n \delta_{f_\omega(X_i(\omega))}(A)\\ &= P_{n,f_\omega}(\omega)(A),\end{aligned}$$

thereby proving Lemma 1. □

The next lemma is a general result for arrays of independent random variables and is the key lemma used in the proof of Theorem 1. Lemma 2 is presented in a form somewhat more general than what is required for the proof of Theorem 1 and may be of independent interest. It should be noted that Lemma 2 extends some of the work of Sung (1996). Indeed, Lemma 2(ii) reduces to Corollary 4 of Sung (1996) in the special case where $m(n)=n,\ n\geqslant 1$.

Lemma 2. *Let* $\{Y_{n,j}, 1\leqslant j\leqslant m(n)<\infty, n\geqslant 1\}$ *be an array of independent random variables such that* $EY_{n,j}=0$ *and* $|Y_{n,j}|\leqslant c_n,\ 1\leqslant j\leqslant m(n),\ n\geqslant 1$, *where* $\{c_n,\ n\geqslant 1\}$ *is a sequence of constants in* $(0,\infty)$. *Set* $s_n^2=\sum_{j=1}^{m(n)} EY_{n,j}^2,\ n\geqslant 1$ *and suppose that* $s_n^2>0,\ n\geqslant 1$. *Let* $\{a_n,\ n\geqslant 1\}$ *be a sequence of positive constants such that* $\sum_{n=1}^{\infty}\exp\{-\beta^2 a_n^2\}<\infty$ *for some* $0<\beta<\infty$. *Then*:

(i) *If* $c_n a_n = \mathrm{o}(s_n)$,

$$\limsup_{n\to\infty}\frac{|\sum_{j=1}^{m(n)} Y_{n,j}|}{s_n a_n}=\sqrt{2}B_0 \quad a.s., \tag{2.1}$$

where

$$B_0=\inf\left\{B\in(0,\beta] : \sum_{n=1}^{\infty}\exp\{-B^2 a_n^2\}<\infty\right\}.$$

(ii) *If* $c_n(\log n)^{1/2}=\mathrm{o}(s_n)$,

$$\limsup_{n\to\infty}\frac{|\sum_{j=1}^{m(n)} Y_{n,j}|}{s_n(2\log n)^{1/2}}=1 \quad a.s. \tag{2.2}$$

Proof. Part (ii) follows immediately from (i) by taking $a_1=(\log 2)^{1/2}$, $a_n=(\log n)^{1/2}$, $n\geqslant 2$. To prove (i), note that for arbitrary $\alpha>0$, there exists a positive integer $N(\alpha)$ such that for all $n\geqslant N(\alpha)$

$$(B_0+\alpha)^2\left(1-\frac{\sqrt{2}(B_0+\alpha)a_n c_n}{s_n}\right)\geqslant\left(B_0+\frac{\alpha}{2}\right)^2.$$

Employing a Kolmogorov exponential bounds inequality (see, e.g., Theorem 5.2.2(i) of Stout (1974), p. 262) we have for $n\geqslant N(\alpha)$

$$\begin{aligned}P\left\{\frac{|\sum_{j=1}^{m(n)} Y_{n,j}|}{s_n a_n}\geqslant\sqrt{2}(B_0+\alpha)\right\} &\leqslant 2\exp\left\{-(B_0+\alpha)^2 a_n^2\left(1-\frac{\sqrt{2}(B_0+\alpha)a_n c_n}{2s_n}\right)\right\}\\ &\leqslant 2\exp\left\{-\left(B_0+\frac{\alpha}{2}\right)^2 a_n^2\right\}.\end{aligned}$$

By definition of B_0, it follows that

$$\sum_{n=1}^{\infty}P\left\{\frac{|\sum_{j=1}^{m(n)} Y_{n,j}|}{s_n a_n}\geqslant\sqrt{2}\left(B_0+\frac{\alpha}{2}\right)\right\}<\infty.$$

Then by the Borel–Cantelli lemma,

$$P\left\{\frac{|\sum_{j=1}^{m(n)} Y_{n,j}|}{s_n a_n}\geqslant\sqrt{2}\left(B_0+\frac{\alpha}{2}\right)\text{ i.o. }(n)\right\}=0$$

and, hence,

$$\limsup_{n\to\infty} \frac{|\sum_{j=1}^{m(n)} Y_{n,j}|}{s_n a_n} \leqslant \sqrt{2}\left(B_0 + \frac{\alpha}{2}\right) \quad \text{a.s.}$$

Since α is arbitrary,

$$\limsup_{n\to\infty} \frac{|\sum_{j=1}^{m(n)} Y_{n,j}|}{s_n a_n} \leqslant \sqrt{2}B_0 \quad \text{a.s.} \tag{2.3}$$

To prove the reverse inequality, note that we can assume $B_0 > 0$. (For if $B_0 = 0$, then (2.1) coincides with (2.3).) Note that $a_n \to \infty$ by $\sum_{n=1}^{\infty} \exp\{-\beta^2 a_n^2\} < \infty$. Then by employing another Kolmogorov exponential bounds inequality (see, e.g., Stout, 1974, Theorem 5.2.2(iii), p. 262), it follows that for arbitrary $\alpha \in (0, B_0)$, there exists an integer $N(\alpha)$ such that for all $n \geqslant N(\alpha)$

$$P\left\{\frac{|\sum_{j=1}^{m(n)} Y_{n,j}|}{s_n a_n} \geqslant \sqrt{2}\,(B_0 - \alpha)\right\} \geqslant 2\exp\{-(B_0-\alpha)^2 a_n^2 (1+\gamma)\},$$

where

$$\gamma = \frac{(B_0 - \alpha/2)^2}{(B_0-\alpha)^2} - 1.$$

Then

$$\sum_{n=1}^{\infty} P\left\{\frac{|\sum_{j=1}^{m(n)} Y_{n,j}|}{s_n a_n} \geqslant \sqrt{2}\,(B_0 - \alpha)\right\} \geqslant \sum_{n=N(\gamma)}^{\infty} \exp\left\{-\left(B_0 - \frac{\alpha}{2}\right)^2 a_n^2\right\} = \infty$$

by definition of B_0. Since $\left\{\sum_{j=1}^{m(n)} Y_{n,j}, n \geqslant 1\right\}$ is a sequence of independent random variables, by the Borel–Cantelli lemma,

$$P\left\{\frac{|\sum_{j=1}^{m(n)} Y_{n,j}|}{s_n a_n} \geqslant \sqrt{2}(B_0 - \alpha) \text{ i.o. } (n)\right\} = 1$$

and hence

$$\limsup_{n\to\infty} \frac{|\sum_{j=1}^{m(n)} Y_{n,j}|}{s_n a_n} \geqslant \sqrt{2}\,(B_0 - \alpha) \quad \text{a.s.}$$

Since α is arbitrary,

$$\limsup_{n\to\infty} \frac{|\sum_{j=1}^{m(n)} Y_{n,j}|}{s_n a_n} \geqslant \sqrt{2}B_0 \quad \text{a.s.} \qquad \square$$

Remark 1. If the independence hypothesis to Lemma 2 is weakened to $\{Y_{n,j}, 1 \leqslant j \leqslant m(n) < \infty, n \geqslant 1\}$ being an array of rowwise independent random variables, then Lemma 2 still holds with conclusions (2.1) and (2.2) weakened to

$$\limsup_{n\to\infty} \frac{|\sum_{j=1}^{m(n)} Y_{n,j}|}{s_n a_n} \leqslant \sqrt{2}B_0 \quad \text{a.s.}$$

and

$$\limsup_{n\to\infty} \frac{|\sum_{j=1}^{m(n)} Y_{n,j}|}{s_n(2\log n)^{1/2}} \leqslant 1 \quad \text{a.s.,}$$

respectively. This follows by noting that nothing beyond rowwise independence was used to prove (2.3).

3. Mainstream

With the preliminaries accounted for, the main result will now be established. For an arbitrary sequence of random variables $\{X_n, n \geqslant 1\}$ defined on a probability space $(\Omega, \mathscr{F}, P)$, let $\bar{X}_n$ and $\hat{X}_{n,j}^{(\omega)}$ be defined as in Section 1 (even though the $\{X_n, n \geqslant 1\}$ are not assumed to be i.i.d.) and set

$$\tilde{\sigma}_n = \left(\frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2}{n} \right)^{1/2} = \left(\frac{\sum_{i=1}^n X_i^2}{n} - \left(\frac{\sum_{i=1}^n X_i}{n} \right)^2 \right)^{1/2}, \quad n \geqslant 1. \tag{3.1}$$

It is assumed that the bootstrap samples

$$\{\hat{X}_{n,j}^{(\omega)}, 1 \leqslant j \leqslant m(n)\}, \quad n \geqslant 1 \text{ are independent for almost every } \omega \in \Omega. \tag{3.2}$$

Theorem 1. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of random variables (which are not necessarily independent or identically distributed) and let $\{m(n),\ n \geqslant 1\}$ be a sequence of positive integers. Suppose that*

$$\lim_{n\to\infty} \frac{(\log n) \max_{1 \leqslant i \leqslant n} (X_i - \bar{X}_n)^2}{m(n)} = 0 \quad \textit{a.s.} \tag{3.3}$$

and for almost every $\omega \in \Omega$ the limit

$$\lim_{n\to\infty} \tilde{\sigma}_n(\omega) \equiv \tilde{\sigma}(\omega) > 0 \textit{ exists.} \tag{3.4}$$

If the bootstrap samples $\{\hat{X}_{n,j}^{(\omega)}, 1 \leqslant j \leqslant m(n)\}$, $n \geqslant 1$ satisfy (3.2), *then for almost every $\omega \in \Omega$*

$$\limsup_{n\to\infty} \left(\frac{m(n)}{2\log n} \right)^{1/2} \left| \frac{\sum_{j=1}^{m(n)} \hat{X}_{n,j}^{(\omega)}}{m(n)} - \bar{X}_n(\omega) \right| = \tilde{\sigma}(\omega) \quad \textit{a.s.} \tag{3.5}$$

Proof. Conclusion (3.5) is equivalent to: for almost every $\omega \in \Omega$

$$\limsup_{n\to\infty} \frac{|\sum_{j=1}^{m(n)} (\hat{X}_{n,j}^{(\omega)} - \bar{X}_n(\omega))|}{(2m(n)\log n)^{1/2}} = \tilde{\sigma}(\omega) \quad \text{a.s.} \tag{3.6}$$

To prove (3.6), set for $\omega \in \Omega$

$$Y_{n,j}^{(\omega)} = \hat{X}_{n,j}^{(\omega)} - \bar{X}_n(\omega), \quad 1 \leqslant j \leqslant m(n),\ n \geqslant 1.$$

Note that for $\omega \in \Omega$,

$$EY_{n,j}^{(\omega)} = 0, |Y_{n,j}^{(\omega)}| \leqslant \max_{1 \leqslant i \leqslant n} |X_i(\omega) - \bar{X}_n(\omega)|, \quad 1 \leqslant j \leqslant m(n),\ n \geqslant 1$$

and by Lemma 1

$$\sum_{j=1}^{m(n)} E(Y_{n,j}^{(\omega)})^2 = m(n)E(Y_{n,1}^{(\omega)})^2 = m(n)\tilde{\sigma}_n^2(\omega). \tag{3.7}$$

Now for almost every $\omega \in \Omega$, by (3.7), (3.3), and (3.4)

$$\frac{(\max_{1 \leqslant i \leqslant n} |X_i(\omega) - \bar{X}_n(\omega)|)(\log n)^{1/2}}{(\sum_{j=1}^{m(n)} E(Y_{n,j}^{(\omega)})^2)^{1/2}} = \frac{(\max_{1 \leqslant i \leqslant n} |X_i(\omega) - \bar{X}_n(\omega)|)(\log n)^{1/2}}{(m(n))^{1/2}\tilde{\sigma}_n} \to 0$$

and so for almost every $\omega \in \Omega$, by (3.2) and Lemma 2(ii)

$$\limsup_{n\to\infty} \frac{|\sum_{j=1}^{m(n)} Y_{n,j}^{(\omega)}|}{((2\log n)\sum_{j=1}^{m(n)} E(Y_{n,j}^{(\omega)})^2)^{1/2}} = 1 \quad \text{a.s.} \tag{3.8}$$

Then for almost every $\omega \in \Omega$

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{|\sum_{j=1}^{m(n)} (\hat{X}_{n,j}^{(\omega)} - \bar{X}_n(\omega))|}{(2m(n)\log n)^{1/2}} \\
&\quad = \limsup_{n\to\infty} \left(\frac{|\sum_{j=1}^{m(n)} Y_{n,j}^{(\omega)}|}{((2\log n)\sum_{j=1}^{m(n)} E(Y_{n,j}^{(\omega)})^2)^{1/2}} \left(\frac{\sum_{j=1}^{m(n)} E(Y_{n,j}^{(\omega)})^2}{m(n)} \right)^{1/2} \right) \\
&\quad = \limsup_{n\to\infty} \left(\frac{|\sum_{j=1}^{m(n)} Y_{n,j}^{(\omega)}|}{\left((2\log n)\sum_{j=1}^{m(n)} E(Y_{n,j}^{(\omega)})^2\right)^{1/2}} \tilde{\sigma}_n(\omega) \right) \quad \text{(by (3.7))} \\
&\quad = \tilde{\sigma}(\omega) \quad \text{a.s. (by(3.8) and (3.4))}. \qquad \square
\end{aligned}$$

Remark 2. (i) If the sequence of random variables $\{X_n,\ n \geqslant 1\}$ satisfies $\sup_{n \geqslant 1} |X_n| < \infty$ a.s. (*a fortiori* if $\{X_n,\ n \geqslant 1\}$ is uniformly bounded), then (3.3) can be replaced by the structurally simpler condition $\log n = \mathrm{o}(m(n))$.

(ii) Since

$$\max_{1 \leqslant i \leqslant n} |X_i - \bar{X}_n| \leqslant 2 \max_{1 \leqslant i \leqslant n} |X_i|, \quad n \geqslant 1,$$

(3.3) will hold if

$$\lim_{n\to\infty} \frac{(\log n)\ \max_{1 \leqslant i \leqslant n} X_i^2}{m(n)} = 0 \quad \text{a.s.} \tag{3.9}$$

(iii) If

$$\frac{m(n)}{\log n} \uparrow \infty, \tag{3.10}$$

then (3.9) is equivalent to the apparently weaker and structurally simpler condition

$$\lim_{n\to\infty} \frac{(\log n) X_n^2}{m(n)} = 0 \quad \text{a.s.} \tag{3.11}$$

Proof. Assume (3.10) and (3.11). For arbitrary $n \geqslant k \geqslant 2$,

$$\begin{aligned}
\frac{(\log n)\max_{1\leqslant i\leqslant n} X_i^2}{m(n)} &\leqslant \frac{(\log n)\max_{1\leqslant i\leqslant k-1} X_i^2}{m(n)} + \frac{\log n}{m(n)} \max_{k\leqslant i\leqslant n} X_i^2 \\
&\leqslant \frac{(\log n)\ \max_{1\leqslant i\leqslant k-1} X_i^2}{m(n)} + \max_{k\leqslant i\leqslant n} \frac{\log i}{m(i)} X_i^2 \quad \text{(by (3.10))} \\
&\leqslant \frac{(\log n)\max_{1\leqslant i\leqslant k-1} X_i^2}{m(n)} + \sup_{i\geqslant k} \frac{\log i}{m(i)} X_i^2 \\
&\to 0
\end{aligned}$$

as first $n \to \infty$ and then $k \to \infty$ by (3.10) and (3.11). The reverse implication is obvious. □

(iv) Conclusion (3.5) still holds with $\tilde{\sigma}(\omega) = \lim_{n\to\infty} \tilde{\sigma}_n(\omega)$ when $P\{\tilde{\sigma} = 0\} > 0$ provided (3.3) is replaced by the condition

$$\lim_{n\to\infty} \frac{(\log n)\max_{1\leqslant i\leqslant n}(X_i - \bar{X}_n)^2}{m(n)\tilde{\sigma}_n^2} = 0 \quad \text{a.s.}$$

The details are left to the reader. Note that for $\omega \in \Omega, \tilde{\sigma}_n^2(\omega) = 0$ for all $n \geqslant 1$ if and only if $\max_{1\leqslant i\leqslant n}(X_i(\omega) - \bar{X}_n(\omega))^2 = 0$ for all $n \geqslant 1$. For any such $\omega \in \Omega, \max_{1\leqslant i\leqslant n}$ $(X_i(\omega) - \bar{X}_n(\omega))^2/\tilde{\sigma}_n^2(\omega)$ should be interpreted as 0 and Conclusion (3.5) trivially holds.

(v) It follows as a special case of Theorem 2.1 of Li et al. (1999) by taking $a_n = (\log n/m(n))^{1/2}$, $n \geqslant 1$ that if (3.9) holds and if $\sum_{i=1}^n X_i^2/n \to 0$ a.s., then for every real number r, every $\varepsilon > 0$, and almost every $\omega \in \Omega$ that the complete convergence result

$$\sum_{n=1}^{\infty} n^r P\left\{ \left(\frac{m(n)}{\log n}\right)^{1/2} \left| \frac{\sum_{j=1}^{m(n)} \hat{X}_{n,j}^{(\omega)}}{m(n)} - \bar{X}_n(\omega) \right| > \varepsilon \right\} < \infty$$

obtains. This of course implies by the Borel–Cantelli lemma that for almost every $\omega \in \Omega$, the SLLN

$$\lim_{n\to\infty} \left(\frac{m(n)}{\log n}\right)^{1/2} \left(\frac{\sum_{j=1}^{m(n)} \hat{X}_{n,j}^{(\omega)}}{m(n)} - \bar{X}_n(\omega) \right) = 0 \quad \text{a.s.}$$

holds.

(vi) If either

(a) Condition (3.4) is weakened to $\liminf_{n\to\infty} \tilde{\sigma}_n(\omega) > 0$ for almost every $\omega \in \Omega$, or

(b) Condition (3.2) is dispensed with,

then setting $\sigma_n^*(\omega) \equiv \limsup_{n\to\infty} \tilde{\sigma}_n(\omega)$, $\omega \in \Omega$, a slight modification of the proof of Theorem 1 yields the following upper bound result: for almost every $\omega \in \Omega$

$$\limsup_{n\to\infty} \left(\frac{m(n)}{2\log n}\right)^{1/2} \left| \frac{\sum_{j=1}^{m(n)} \hat{X}_{n,j}^{(\omega)}}{m(n)} - \bar{X}_n(\omega) \right| \leqslant \sigma^*(\omega) \quad \text{a.s.}$$

The details are left to the reader. (For case (b), refer to Remark 1 and also note that under (3.4), $\sigma^*(\omega)=\tilde{\sigma}(\omega)$ for almost every $\omega\in\Omega$.)

The first example shows apropos of Theorem 1 that $\tilde{\sigma}(\omega)$, $\omega\in\Omega$ need not be a constant a.s.

Example 1. Let $X_n=\varepsilon_n X$, $n\geqslant 1$ where X is a nondegenerate random variable with $P\{X=0\}=0$ and $\{\varepsilon_n,\ n\geqslant 1\}$ is a sequence of independent random variables with $P\{\varepsilon_n=1\}=p$, $P\{\varepsilon_n=-1\}=1-p$, $n\geqslant 1$ where $0<p<1$. Now by the Kolmogorov SLLN $\sum_{i=1}^n \varepsilon_i/n\to 2p-1$ a.s. whence $\{\tilde{\sigma}_n, n\geqslant 1\}$ (as defined by (3.1)) satisfies

$$\tilde{\sigma}_n=\left(X^2-\left(X\frac{\sum_{i=1}^n\varepsilon_i}{n}\right)^2\right)^{1/2}\to 2(p(1-p))^{1/2}|X|>0 \quad \text{a.s.}$$

For any sequence of positive integers $\{m(n),\ n\geqslant 1\}$ satisfying $\log n=\mathrm{o}(m(n))$, Condition (3.3) holds by Remark 2(i). Thus by Theorem 1, for almost every $\omega\in\Omega$, (3.5) holds with $\tilde{\sigma}(\omega)=2(p(1-p))^{1/2}|X(\omega)|$, $\omega\in\Omega$. It is interesting to note that no moment condition has been imposed on X.

The following corollary is an application of Theorem 1 to finite population sampling.

Corollary 1. *Let $\{x_n,\ n\geqslant 1\}$ be a sequence of real numbers and let $\{m(n),\ n\geqslant 1\}$ be a sequence of positive integers and suppose that*

$$(\log n)\max_{1\leqslant i\leqslant n}(x_i-\bar{x}_n)^2=\mathrm{o}(m(n))$$

and that the limit

$$\lim_{n\to\infty}\left(\frac{\sum_{i=1}^n x_i^2}{n}-\bar{x}_n^2\right)^{1/2}=\sigma>0 \text{ exists,}$$

where $\bar{x}_n=\sum_{i=1}^n x_i/n$, $n\geqslant 1$. For each $n\geqslant 1$, let $X_{n,1},\ldots,X_{n,m(n)}$ be random variables each uniformly distributed on $\{x_1,\ldots,x_n\}$ and suppose that the random variables $\{X_{n,j}, 1\leqslant j\leqslant m(n),\ n\geqslant 1\}$ are independent. Then

$$\limsup_{n\to\infty}\left(\frac{m(n)}{2\log n}\right)^{1/2}\left|\frac{\sum_{j=1}^{m(n)}X_{n,j}}{m(n)}-\bar{x}_n\right|=\sigma \quad \text{a.s.}$$

4. Pairwise i.i.d. and stationary ergodic specializations

In this section, new results are obtained by specializing Theorem 1 to the cases where $\{X_n,\ n\geqslant 1\}$ is either a sequence of pairwise i.i.d. random variables or a stationary ergodic sequence of random variables. The next theorem is an extension of a result of Mikosch (1994) which was obtained for a sequence $\{X_n,\ n\geqslant 1\}$ of i.i.d. random variables. The overall argument in Theorem 2 is substantially different and considerably simpler than that of Mikosch (1994).

Theorem 2. *Let $\{X_n,\ n\geqslant 1\}$ be a sequence of nondegenerate random variables such that either*

(i) *$\{X_n,\ n\geqslant 1\}$ is a sequence of pairwise i.i.d. random variables*

or

(ii) *$\{X_n,\ n\geqslant 1\}$ is a stationary ergodic sequence of random variables.*

Let $\{m(n),\ n\geqslant 1\}$ be a sequence of positive integers such that

$$\frac{m(n)}{\log n}\uparrow. \tag{4.1}$$

Suppose that there exists a constant $\theta\geqslant 1$ such that

$$n^{1/\theta}\log n = \mathcal{O}(m(n)) \tag{4.2}$$

and

$$E|X_1|^{2\theta}<\infty. \tag{4.3}$$

Set $\sigma^2=\mathrm{Var}\,X_1$. Then for almost every $\omega\in\Omega$

$$\limsup_{n\to\infty}\left(\frac{m(n)}{2\log n}\right)^{1/2}\left|\frac{\sum_{j=1}^{m(n)}\hat{X}_{n,j}^{(\omega)}}{m(n)}-\bar{X}_n(\omega)\right|=\sigma\quad a.s. \tag{4.4}$$

Proof. Under case (i), $\{X_n^2,\ n\geqslant 1\}$ is also a sequence of pairwise i.i.d. random variables. Now in view of (4.3), by a double application of the Etemadi (1981) SLLN

$$\tilde{\sigma}_n=\left(\frac{\sum_{i=1}^n X_i^2}{n}-\left(\frac{\sum_{i=1}^n X_i}{n}\right)^2\right)^{1/2}\to(EX_1^2-(EX_1)^2)^{1/2}=\sigma>0\quad \text{a.s.} \tag{4.5}$$

Under case (ii), $\{X_n^2,\ n\geqslant 1\}$ is also a stationary ergodic sequence by Theorem 3.5.8 of Stout (1974), p. 182. Again in view of (4.3), by a double application of the pointwise ergodic theorem for stationary sequences (see, e.g., Stout, 1974, p. 181), (4.5) holds.

Next, by (4.2) there exists a constant $M<\infty$ such that $n^{1/\theta}\log n\leqslant Mm(n)$, $n\geqslant 1$. Then for arbitrary $\varepsilon>0$

$$\sum_{n=1}^{\infty}P\left\{\frac{(\log n)X_n^2}{m(n)}>\varepsilon\right\}\leqslant\sum_{n=1}^{\infty}P\left\{|X_1|>\left(\frac{\varepsilon}{M}\right)^{1/2}n^{1/2\theta}\right\}<\infty\quad(\text{by (4.3)}).$$

Thus by the Borel–Cantelli lemma

$$\lim_{n\to\infty}\frac{(\log n)X_n^2}{m(n)}=0\quad\text{a.s.}$$

Since (4.1) and (4.2) ensure that (3.10) holds, (3.3) then follows from Remarks 2(ii) and 2(iii). The conclusion (4.4) results directly from Theorem 1. □

Remark 3. (i) Theorem 2(i) gains added interest in light of the fact that the classical Hartman–Wintner (1941) LIL does not hold in general for sequences of pairwise i.i.d. random variables. (See Révész and Wschebor (1964) or, for a more thorough discussion, Cuesta and Matrán (1991).)

(ii) Theorem 2(ii) can fail for a stationary sequence $\{X_n,\ n\geqslant 1\}$ if the hypothesis that $\{X_n,\ n\geqslant 1\}$ is ergodic is dispensed with. To see this, let X be a random variable with $0<\sigma^2\equiv \mathrm{Var}\, X<\infty$ and set $X_n=X,\ n\geqslant 1$. Then $\{X_n,\ n\geqslant 1\}$ is a stationary sequence which is not ergodic. It is clear for every sequence of positive integers $\{m(n),\ n\geqslant 1\}$ that for almost every $\omega\in\Omega$,

$$\frac{\sum_{j=1}^{m(n)}\hat{X}_{n,j}^{(\omega)}}{m(n)}-\bar{X}_n(\omega)=X(\omega)-X(\omega)=0 \quad \text{a.s.}$$

and so (4.4) fails.

(iii) The simple example in (ii) also shows that for a sequence of identically distributed random variables $\{X_n,\ n\geqslant 1\}$ with $0<\mathrm{Var}\, X_1<\infty$ that the limit $\tilde{\sigma}(\omega)$ (in (3.4)) can be 0 for almost every $\omega\in\Omega$. Conclusion (3.5) of Theorem 1 trivially holds for this example.

(iv) Theorem 2 also holds for an M-dependent process $\{X_n,\ n\geqslant 1\}$ of nondegenerate identically distributed random variables. Verification is left to the reader.

When $\{X_n,\ n\geqslant 1\}$ is a sequence of i.i.d. random variables, the next corollary provides conditions so that $\bar{X}_n(\omega)$ can be replaced by EX_1 in the conclusion (4.4) of Theorem 2.

Corollary 2. *Let $\{X_n,\ n\geqslant 1\}$ be a sequence of nondegenerate i.i.d. random variables and let $\{m(n),\ n\geqslant 1\}$ be a sequence of positive integers such that* (4.1) *and*

$$m(n)=\mathrm{o}\left(\frac{n\log n}{\log\log n}\right) \tag{4.6}$$

hold. Suppose that there exists a constant $\theta>1$ such that (4.2) *and* (4.3) *hold. Set $\sigma^2=\mathrm{Var}\, X_1$. Then for almost every $\omega\in\Omega$*

$$\limsup_{n\to\infty}\left(\frac{m(n)}{2\log n}\right)^{1/2}\left|\frac{\sum_{j=1}^{m(n)}\hat{X}_{n,j}^{(\omega)}}{m(n)}-EX_1\right|=\sigma \quad a.s. \tag{4.7}$$

Proof. By Theorem 2, (4.4) holds. Now for almost every $\omega\in\Omega$,

$$\begin{aligned}
&\left(\frac{m(n)}{2\log n}\right)^{1/2}\left|\left|\frac{\sum_{j=1}^{m(n)}\hat{X}_{n,j}^{(\omega)}}{m(n)}-EX_1\right|-\left|\frac{\sum_{j=1}^{m(n)}\hat{X}_{n,j}^{(\omega)}}{m(n)}-\bar{X}_n(\omega)\right|\right|\\
&\quad\leqslant\left(\frac{m(n)}{2\log n}\right)^{1/2}|\bar{X}_n(\omega)-EX_1|\\
&\quad=\left(\frac{m(n)\log\log n}{n\log n}\right)^{1/2}\frac{|\sum_{i=1}^{n}(X_i(\omega)-EX_1)|}{(2n\log\log n)^{1/2}}\\
&\quad\to 0 \text{ a.s.}
\end{aligned}$$

by (4.6) and the Hartman–Wintner (1941) LIL. This combined with (4.4) yields conclusion (4.7). $\square$

5. Some final comments

To conclude, some final comments are in order concerning bootstrapped means (with $m(n) = n, n \geqslant 1$) formed from a sequence $\{X_n, n \geqslant 1\}$ of nondegenerate i.i.d. random variables. Set $L(x) = \log_e(e \vee x),\ x \geqslant 0$.

(i) Suppose that $E|X_1|^p < \infty$ for some $p > 2$. Set $\sigma^2 = \mathrm{Var}\, X_1$. Now by the Hartman–Wintner (1941) LIL

$$\limsup_{n\to\infty} \left(\frac{n}{2\log\log n}\right)^{1/2} \left|\frac{\sum_{i=1}^n X_i}{n} - EX_1\right| = \sigma \quad \text{a.s.} \tag{5.1}$$

However, for bootstrapped means, it follows from Corollary 2 that for almost every $\omega \in \Omega$

$$\limsup_{n\to\infty} \left(\frac{n}{2\log n}\right)^{1/2} \left|\frac{\sum_{j=1}^n \hat{X}_{n,j}^{(\omega)}}{n} - EX_1\right| = \sigma \quad \text{a.s.} \tag{5.2}$$

Obviously (5.1) and (5.2) have different orders of convergence; an iterated logarithm appears in (5.1) whereas a single logarithm appears in (5.2).

(ii) Suppose that $E(|X_1|^p (L(|X_1|)^p) < \infty$ for some $p \in [1,2)$. Now by the classical Marcinkiewicz–Zygmund SLLN

$$\lim_{n\to\infty} n^{1-p^{-1}} \left(\frac{\sum_{i=1}^n X_i}{n} - EX_1\right) = 0 \quad \text{a.s.} \tag{5.3}$$

and by Proposition 3.3 of Mikosch (1994) for almost every $\omega \in \Omega$

$$\lim_{n\to\infty} n^{1-p^{-1}} \left(\frac{\sum_{j=1}^n \hat{X}_{n,j}^{(\omega)}}{n} - \frac{\sum_{i=1}^n X_i(\omega)}{n}\right) = 0 \quad \text{a.s.}$$

implying for almost every $\omega \in \Omega$

$$\lim_{n\to\infty} n^{1-p^{-1}} \left(\frac{\sum_{j=1}^n \hat{X}_{n,j}^{(\omega)}}{n} - EX_1\right) = 0 \quad \text{a.s.} \tag{5.4}$$

Thus, in contrast to the LIL situation discussed in (i) above, there is no difference between the order of convergence for the classical Marcinkiewicz–Zygmund SLLN (5.3) and the Marcinkiewicz–Zygmund-type SLLN (5.4) for bootstrapped means.

(iii) If $E|X_1|^p < \infty$ for some $p > 2$, then by Theorem 2 and Corollary 2 for almost every $\omega \in \Omega$

$$\begin{aligned}
&\limsup_{n\to\infty} \left(\frac{n}{2\log n}\right)^{1/2} \left|\frac{\sum_{j=1}^n \hat{X}_{n,j}^{(\omega)}}{n} - \bar{X}_n(\omega)\right| \\
&\quad = \limsup_{n\to\infty} \left(\frac{n}{2\log n}\right)^{1/2} \left|\frac{\sum_{j=1}^n \hat{X}_{n,j}^{(\omega)}}{n} - EX_1\right| \\
&\quad = \sigma \text{ a.s.}
\end{aligned}$$

where $\sigma^2 = \mathrm{Var}\, X_1$. It will now be shown that

$$\frac{\sum_{j=1}^{n} \hat{X}_{n,j}^{(\omega)}}{n} - \bar{X}_n(\omega) \quad \text{and} \quad \frac{\sum_{j=1}^{n} \hat{X}_{n,j}^{(\omega)}}{n} - EX_1, \; n \geqslant 1$$

do not exhibit similar behavior to each other in the CLT sense. When $EX_1^2 < \infty$, recalling (1.1), for almost every $\omega \in \Omega$

$$n^{1/2}\left(\frac{\sum_{j=1}^{n} \hat{X}_{n,j}^{(\omega)}}{n} - \bar{X}_n(\omega)\right) \xrightarrow{\mathrm{d}} \mathrm{N}(0,1).$$

However, for almost every $\omega \in \Omega$

$$n^{1/2}\left(\frac{\sum_{j=1}^{n} \hat{X}_{n,j}^{(\omega)}}{n} - EX_1\right) \text{ does } \textit{not} \text{ converge in distribution.} \tag{5.5}$$

To see this, note that

$$n^{1/2}\left(\frac{\sum_{j=1}^{n} \hat{X}_{n,j}^{(\omega)}}{n} - EX_1\right) = n^{1/2}\left(\frac{\sum_{j=1}^{n} \hat{X}_{n,j}^{(\omega)}}{n} - \bar{X}_n(\omega)\right) + n^{1/2}(\bar{X}_n(\omega) - EX_1).$$

Since either (1.2) or (5.1) ensures that for almost every $\omega \in \Omega$,

$$\limsup_{n\to\infty} n^{1/2}(\bar{X}_n(\omega) - EX_1) = \infty,$$

the assertion (5.5) follows.

(iv) Let $\{X_{n,j}, 1 \leqslant j \leqslant n, n \geqslant 1\}$ be a triangular array of nondegenerate i.i.d. random variables with $EX_{1,1}^2 < \infty$. Set $\sigma^2 = \mathrm{Var}\, X_{1,1}$. Li et al. (1995) showed that

$$\limsup_{n\to\infty} \left(\frac{n}{2\log n}\right)^{1/2} \left|\frac{\sum_{j=1}^{n} X_{n,j}}{n} - EX_{1,1}\right| = \sigma \quad \text{a.s.} \tag{5.6}$$

if and only if

$$E\left(\frac{X_{1,1}^4}{(L(|X_{1,1}|))^2}\right) < \infty.$$

Note that (5.2) and (5.6) have the same order of convergence. We thus pose the question: Do conditions exist which are both necessary and sufficient for (5.2)? The authors hope that this problem will be investigated by the interested reader.

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