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Addendum

Addendum to "A note on complete convergence for arrays", Statist. Probab. Lett. 38 (1) (1998) 27–31☆

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Abstract

Under some conditions on an array of rowwise-independent random variables, Hu, Szynal and Volodin obtained a complete convergence result for law of large numbers. In this addendum we mention that the convergent rate of sequence $\{c_n,\ n\geqslant 1\}$ must be bounded away from zero.

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In our paper (Hu et al., 1998), the following complete convergence theorem for arrays of rowwise-independent random variables was formulated.

Theorem. *Let* $\{X_{ni}, 1\leqslant i\leqslant k_n,\ n\geqslant 1\}$ *be an array of rowwise-independent random variables and* $\{c_n, n\geqslant 1\}$ *be a sequence of positive constants such that*

$$\sum_{n=1}^{\infty} c_n = \infty. \qquad (1)$$

Suppose that for every $\varepsilon > 0$ *and some* $\delta > 0$:
(i) $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon) < \infty$,

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(ii) *there exists* $J \geqslant 2$ *such that*

$$\sum_{n=1}^{\infty} c_n \left(\sum_{i=1}^{k_n} EX_{ni}^2 I(|X_{ni}| \leqslant \delta) \right)^J < \infty,$$

(iii) $\sum_{i=1}^{k_n} EX_{ni} I(|X_{ni}| \leqslant \delta) \to 0$ *as* $n \to \infty$.
Then $\sum_{n=1}^{\infty} c_n P(|\sum_{i=1}^{k_n} X_{ni}| > \varepsilon) < \infty$ *for all* $\varepsilon > 0$.

Instead of (1), we should give the following assumption:

the sequence $\{c_n,\ n \geqslant 1\}$ is bounded away from zero, that is, $\lim_{n\to\infty} \inf c_n > 0$. (2)

Condition (1) is not sufficient since the proof of the Theorem is based on the fact that $\sum_{i=1}^{k_n} X_{ni} \to 0$ in probability as $n \to \infty$. We mention that this does not necessarily follow from the conditions of the Theorem if $\{c_n,\ n \geqslant 1\}$ is not bounded away from zero. We shall give such an example provided by Professors Berty and Rigo.

Example 1. Define sequences $\{c_n,\ n \geqslant 1\}$ and $\{k_n,\ n \geqslant 1\}$ by $c_n = 1/n$ and $k_n = n$. Define an array $\{X_{ni}, 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ by $X_{ni} = 0, 1 \leqslant i \leqslant n$, if $\sqrt{n} \notin N$, and, if $\sqrt{n} \in N$, let $\{X_{ni}, 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ be i.i.d. with $P(X_{n1} = 0) = (n-1)/n$ and $P(X_{n1} = n) = 1/n$. Then, $\{X_{ni}, 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ are rowwise independent, conditions (ii) and (iii) trivially hold, and

$$\sum_n c_n \sum_{k=1}^{n} P(|X_{nk}| > \varepsilon) = \sum_{\sqrt{n} \in N} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

for all $0 < \varepsilon < 1$ so that (i) holds too. But S_n does not converge to 0 in probability, since $P(|S_n| > \varepsilon) = 1 - ((n-1)/n)^n \to 1 - 1/e$ as $n \to \infty$ with $\sqrt{n} \in N$.

Nevertheless, we have to mention that $\sum_{n=1}^{\infty} c_n P(|\sum_{i=1}^{k_n} X_{ni}| > \varepsilon) < \infty$ for all $\varepsilon > 0$. So, this is a counterexample to the *proof* of the Theorem, but not to the result. It is an interesting project to investigate whether the Theorem is true for general sequences.

There are some other places in (Hu et al., 1998) that need correction.

1. Remark 1 on p. 28 should be read in the following way.

Let $\{X_{ni}, 1 \leqslant i \leqslant k_n,\ n \geqslant 1\}$ be an *infinitesimal* array (that is, $\lim_{n\to\infty} \max_{i \leqslant k_n} P\{|X_{ni}| > \varepsilon\} = 0$ for every $\varepsilon > 0$) of rowwise-independent random variables and let $\{c_n,\ n \geqslant 1\}$ be a sequence of positive constants. Suppose that $\{c_n,\ n \geqslant 1\}$ is bounded away from 0. Then $\sum_{n=1}^{\infty} c_n P(|\sum_{i=1}^{k_n} X_{ni}| > \varepsilon) < \infty$ for all $\varepsilon > 0$ implies $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{k_n} P(|X_{ni}| > \varepsilon) < \infty$.

The following example, provided again by Professors Berti and Rigo shows that infinitesimality is necessary.

Example 2. For each $n \geqslant 1$ let $c_n = 1$, $k_n = n$ and $X_{ni} = 1$ for $1 \leqslant i \leqslant n-1$ and $X_{nn} = 1 - n$. Then the array is rowwise-independent, conditions (ii) and (iii) trivially hold for $\delta < 1$. Next, we note that $\sum_{n=1}^{\infty} c_n P(|S_n| > \varepsilon) = 0 < \infty$ for all $\varepsilon > 0$ since $S_n = 0$ for all n. However $\sum_{n=1}^{\infty} c_n \sum_{i=1}^{n} P(|X_{ni}| > \varepsilon) = \sum_{n=2}^{\infty} n = \infty$.

2. In Remark 2 on pp. 28 and 29 the $\{c_n,\ n \geqslant 1\}$ in condition (ii′) is missed. Remark 2 should be read in the following way:

A special case is when all variables have mean zero and conditions (i) and (ii′) there exist $J \geqslant 2$ and $1 \leqslant q \leqslant 2$ such that

$$\sum_{n=1}^{\infty} c_n \left(\sum_{k=1}^{k_n} E|X_{nk}|^q \right)^J < \infty$$

are satisfied. Then

$$\sum_{n=1}^{\infty} c_n P\{|S_n| > \varepsilon\} < \infty \quad \text{for all } \varepsilon > 0.$$

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Reference

Hu, T.-C., Szynal, D., Volodin, A.I., 1998. A note on complete convergence for arrays. Statist. Probab. Lett. 38, 27–31.