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On the weak law for randomly indexed partial sums for arrays of random elements in martingale type p Banach spaces

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Abstract

For weighted randomly indexed sums of the form $\sum_{j=1}^{N_n} a_{nj}(V_{nj} - c_{nj})$ where $\{a_{nj}, j \geqslant 1, n \geqslant 1\}$ are constants, $\{V_{nj}, j \geqslant 1, n \geqslant 1\}$ are random elements in a real separable martingale type p Banach space, $\{N_n, n \geqslant 1\}$ are positive integer-valued random variables, and $\{c_{nj}, j \geqslant 1, n \geqslant 1\}$ are suitable conditional expectations, a general weak law of large numbers is established. No conditions are imposed on the joint distributions of the $\{V_{nj}, j \geqslant 1, n \geqslant 1\}$. Also, no conditions are imposed on the joint distributions of $\{N_n, n \geqslant 1\}$. Moreover, no conditions are imposed on the joint distributions of $\{N_n, n \geqslant 1\}$. Moreover, no conditions are imposed on the joint distribution of the sequence $\{V_{nj}, j \geqslant 1, n \geqslant 1\}$ and the sequence $\{N_n, n \geqslant 1\}$. The weak law is proved under a Cesàro type condition. The sharpness of the results is illustrated by an example. The current work extends that of Gut (1992), Hong and Oh (1995), Hong (1996), Kowalski and Rychlik (1997), Adler et al. (1997) and Sung (1998).

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1. Introduction

Consider an array of constants $\{a_{nj}, j \geqslant 1, n \geqslant 1\}$ and an array of random elements $\{V_{nj}, j \geqslant 1, n \geqslant 1\}$ defined on a probability space $(\Omega, \mathscr{F}, P)$ and taking values in a real separable Banach space $\mathscr{X}$ with norm $\|\cdot\|$. Let $\{c_{nj}, j \geqslant 1, n \geqslant 1\}$ be a "centering" array consisting of (suitably selected) conditional expectations and

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$\{N_n, n\geqslant 1\}$ be a sequence of positive integer-valued random variables. In this paper, a general weak law of large numbers (WLLN) will be established. This convergence result is of the form

$$\sum_{j=1}^{N_n} a_{nj}(V_{nj}-c_{nj}) \xrightarrow{\text{P}} 0$$

as $n\to\infty$. This expression is referred to as *weighted sums* with *weights* $\{a_{nj}, j\geqslant 1, n\geqslant 1\}$.

The hypotheses to the main result impose conditions on the growth behavior of the weights $\{a_{nj}, j\geqslant 1, n\geqslant 1\}$ and on the marginal distributions of the random variables $\{\|V_{nj}\|, j\geqslant 1, n\geqslant 1\}$. The random elements in the array under consideration are not assumed to be rowwise independent. Indeed, no conditions are imposed on the joint distributions of the random elements comprising the array. Also, no conditions are imposed on the joint distributions of $\{N_n, n\geqslant 1\}$. Moreover, no conditions are imposed on the joint distribution of the sequence $\{V_{nj}, j\geqslant 1, n\geqslant 1\}$ and the sequence $\{N_n, n\geqslant 1\}$. However, the Banach space $\mathscr{X}$ is assumed to be of martingale type p. (Technical definitions such as this will be discussed below.)

The main result is an extension, generalization and improvement to a martingale type p Banach space setting and weighted sums of results of Gut (1992), Hong and Oh (1995), Hong (1996), Kowalski and Rychlik (1997) and Sung (1998) which were proved for arrays of (real-valued) random variables. Moreover, the main result is an extension on the randomly indexed sums of the results of Adler et al. (1997). The WLLN is proved assuming a Cesàro-type condition of Hong and Oh (1995) which is weaker than Cesàro uniform integrability. As will be apparent, the proofs of theorem owe much to those earlier articles. An example is provided to illustrate the sharpness of the results.

The symbol C denotes throughout a generic constant ($0<C<\infty$) which is not necessarily the same one in each appearance.

A real separable Banach space $\mathscr{X}$ is said to be of *martingale type* p $(1\leqslant p\leqslant 2)$ if there exists a finite constant C such that for all martingales $\{S_n, n\geqslant 1\}$ with values in $\mathscr{X}$,

$$\sup_{n\geqslant 1} E\|S_n\|^p \leqslant C\sum_{n=1}^{\infty} E\|S_n-S_{n-1}\|^p,$$

where $S_0\equiv 0$. It can be shown, using classical methods from martingale theory, that if $\mathscr{X}$ is of martingale type p, then for all $1\leqslant r<\infty$ there exists a finite constant C' such that for all $\mathscr{X}$-valued martingales $\{S_n, n\geqslant 1\}$

$$E\sup_{n\geqslant 1}\|S_n\|^r \leqslant C'E\left(\sum_{n=1}^{\infty}\|S_n-S_{n-1}\|^p\right)^{r/p}.$$

Clearly, every real separable Banach space is of martingale type 1. It follows from the Hoffmann-Jørgensen and Pisier (1976) characterization of Rademacher type p Banach spaces that if a Banach space is of martingale type p, then it is of Rademacher type p. But the notion of martingale type p is only superficially similar to that of Rademacher type p and has a geometric characterization in terms of smoothness. For proofs and more details, the reader may refer to Pisier (1975, 1986).

2. Mainstream

Now, the main result can be stated and proved.

Theorem. *Let* $\{V_{nj}, j\geqslant 1, n\geqslant 1\}$ *be an array of random elements in a real separable, martingale type* p $(1\leqslant p\leqslant 2)$ *Banach space, and* $\{N_n, n\geqslant 1\}$ *be a sequence of positive integer-valued random variables such that for some nonrandom sequence of positive integers* $k_n\to\infty$ *we have*

$$P\{N_n>k_n\}=\mathrm{o}(1) \quad \textit{as } n\to\infty. \tag{1}$$

Let $\{a_{nj}, j\geqslant 1, n\geqslant 1\}$ *be an array of constants and let the sequence* $\{f(n), n\geqslant 1\}$, *where* $f(n) = 1/\max_{1\leqslant j\leqslant k_n}|a_{nj}|$, *satisfying*

$$k_n f^{-p}(n) = o(1) \quad \text{as } n\to\infty. \tag{2}$$

Suppose that there exists a positive nondecreasing sequence $\{g(m), m\geqslant 0\}$, $g(0)=0$, *such that*

$$\sum_{m=1}^{k_n-1}\frac{g^p(m+1)-g^p(m)}{m} = O(f^p(n)/k_n) \quad \text{as } n\to\infty. \tag{3}$$

Suppose the uniform Cesàro-type condition

$$\lim_{m\to\infty}\sup_{n\geqslant 1}\frac{1}{k_n}\sum_{j=1}^{k_n} mP\{\|V_{nj}\| > g(m)\} = 0 \tag{4}$$

holds. Then the WLLN

$$\sum_{j=1}^{N_n} a_{nj}(V_{nj} - E(V'_{nj}|\mathscr{F}_{n,j-1})) \xrightarrow{P} 0 \quad \text{as } n\to\infty \tag{5}$$

follows where $V'_{nj} = V_{nj}I(\|V_{nj}\|\leqslant g(k_n))$, $\mathscr{F}_{nj} = \sigma(V_{ni}, 1\leqslant i\leqslant j), j\geqslant 1,\ n\geqslant 1$, *and* $\mathscr{F}_{n0} = \{\emptyset, \Omega\},\ n\geqslant 1$.

Proof. Note that (1) and (4) imply for arbitrary $\varepsilon > 0$ and $n\geqslant 1$

$$P\left\{\left\|\sum_{j=1}^{N_n} a_{nj}V_{nj} - \sum_{j=1}^{N_n} a_{nj}V'_{nj}\right\| > \varepsilon\right\} \leqslant P\{N_n > k_n\} + P\left\{\bigcup_{j=1}^{k_n}[V_{nj}\neq V'_{nj}]\right\}$$

$$\leqslant o(1) + \frac{1}{k_n}\sum_{j=1}^{k_n} k_n P\{\|V_{nj}\| > g(k_n)\} = o(1).$$

Thus

$$\sum_{j=1}^{N_n} a_{nj}V_{nj} - \sum_{j=1}^{N_n} a_{nj}V'_{nj} \xrightarrow{P} 0$$

and it suffices to show that

$$\sum_{j=1}^{k_n} a_{nj}V'_{nj} - \sum_{j=1}^{k_n} a_{nj}c_{nj} \xrightarrow{P} 0,$$

where $c_{nj} = E(V'_{nj}|\mathscr{F}_{n,j-1})$.

For $n\geqslant 1$ and $m\geqslant 1$ denote

$$B_m^n = \left\{\left\|\sum_{j=1}^{m} a_{nj}V'_{nj} - \sum_{j=1}^{m} a_{nj}c_{nj}\right\| > \varepsilon\right\}$$

and $D_n = \bigcup_{m=1}^{k_n} B_m^n$. Then by (1)

$$P\{B_{N_n}^n\}\leqslant P\{B_{N_n}^n, N_n\leqslant k_n\} + P\{N_n > k_n\}\leqslant P\{D_n\} + o(1)$$

and it is suffices to show that $P\{D_n\} = o(1)$.

Note that Markov's inequality yields

$$P\{D_n\} = P\left\{ \max_{1 \leqslant m \leqslant k_n} \left\| \sum_{j=1}^{m} a_{nj}\,(V'_{nj} - c_{nj}) \right\| > \varepsilon \right\}$$

$$\leqslant CE \max_{1 \leqslant m \leqslant k_n} \left\| \sum_{j=1}^{m} a_{nj}(V'_{nj} - c_{nj}) \right\|^p.$$

Since for every $n \geqslant 1$ the sequence $\{V'_{nj} - c_{nj}, 1 \leqslant j \leqslant k_n\}$ is a martingale difference sequence and since the underlying Banach space is of martingale type p,

$$P\{D_n\} \leqslant C \sum_{j=1}^{k_n} E(|a_{nj}|\ \|V'_{nj} - c_{nj}\|)^p$$

$$\leqslant C2^{p-1} f^{-p}(n) \sum_{j=1}^{k_n} E(\|V'_{nj}\|^p + E\|c_{nj}\|^p)$$

$$\leqslant C2^p f^{-p}(n) \sum_{j=1}^{k_n} E\|V'_{nj}\|^p$$

(by applying Jensen's inequality for conditional expectations (see, e.g., Chow and Teicher, 1988, p. 209) to the convex function $|\cdot|^p$).

Moreover,

$$\sum_{j=1}^{k_n} E\|V'_{nj}\|^p = \sum_{j=1}^{k_n} \sum_{m=1}^{k_n} E\|V_{nj}\|^p I(g(m-1) < \|V_{nj}\| \leqslant g(m))$$

$$\leqslant \sum_{j=1}^{k_n} \sum_{m=1}^{k_n} g^p(m)(P\{\|V_{nj}\| > g(m-1)\} - P\{\|V_{nj}\| > g(m)\})$$

$$= \sum_{j=1}^{k_n} \Bigg[g^p(1)P\{\|V_{nj}\| > g(0)\} - g^p(k_n)P\{\|V_{nj}\| > g(k_n)\}$$

$$+ \sum_{m=1}^{k_n-1} (g^p(m+1) - g^p(m))P\{\|V_{nj}\| > g(m)\} \Bigg]$$

$$\leqslant g^p(1) \sum_{j=1}^{k_n} P\{\|V_{nj}\| > g(0)\}$$

$$+ \sum_{j=1}^{k_n} \sum_{m=1}^{k_n-1} (g^p(m+1) - g^p(m))P\{\|V_{nj}\| > g(m)\}]$$

$$= R_1 + R_2 \quad \text{say.}$$

Now taking into account (2) we can estimate the first sum

$$f^{-p}(n)R_1 = Cf^{-p}(n)\sum_{j=1}^{k_n} P\{\|V_{nj}\| > g(0)\}$$

$$\leqslant Ck_n f^{-p}(n) = \mathrm{o}(1).$$

In order to estimate the second sum, for every $n \geqslant 1$ and $m \geqslant 1$ denote

$$b_{nm} = \frac{1}{k_n}\sum_{j=1}^{k_n} mP\{\|V_{nj}\| > g(m)\}.$$

Then, by the uniform Cesàro-type condition (4), $\sup_{n\geqslant 1} b_{nm} = \mathrm{o}(1)$ as $m \to \infty$. Furthermore,

$$R_2 \leqslant k_n \sum_{m=1}^{k_n-1} \frac{g^p(m+1) - g^p(m)}{m} b_{nm}.$$

Thus by (3) and the Toeplitz lemma (see, e.g., Loève, 1977, p. 250) the expression

$$P\{D_n\} \leqslant \mathrm{o}(1) + Ck_n f^{-p}(n) \sum_{m=1}^{k_n-1} \frac{g^p(m+1) - g^p(m)}{m} b_{nm}$$

is o(1) thereby completing the proof of Theorem. □

The same example (which was inspired by another one due to Beck (1963) and Ordóñez Cabrera (1994)) as in Adler et al. (1997, Remarks 3(ii)) shows that the martingale type p hypotheses cannot be dispensed with.

Example. Consider the real separable martingale type r Banach space ℓ_r, where $1 \leqslant r < 2$, of absolutely rth power summable real sequences $v = \{v_i,\ i \geqslant 1\}$ with norm $\|v\| = (\sum_{i=1}^{\infty} |v_i|^r)^{1/r}$. Let $v^{(n)}$ denote the element having 1 in its nth position and 0 elsewhere, $n \geqslant 1$. Define a sequence $\{V_n,\ n \geqslant 1\}$ of independent random elements in ℓ_r by requiring the $\{V_n,\ n \geqslant 1\}$ to be independent with

$$P\{V_n = v^{(n)}\} = P\{V_n = -v^{(n)}\} = \tfrac{1}{2}, \quad n \geqslant 1.$$

Let $N_n = k_n = n,\ n \geqslant 1$ be nonrandom and $V_{nj} = V_j,\ 1 \leqslant j \leqslant n,\ n \geqslant 1$. Moreover, let $g(m) = m^{1/r}$. Now for all $m > 1$, $P[\|V_{nj}\|^r > m] = 0$ and $V_{nj} = V'_{nj},\ 1 \leqslant j \leqslant n,\ n \geqslant 1$ and so uniform Cesàro-type condition (4) holds. Let $a_{nj} = n^{-1/r},\ 1 \leqslant j \leqslant n,\ n \geqslant 1$. Then for any $p \in (r, 2]$

$$\left\|\sum_{j=1}^{n} a_{nj}(V_{nj} - E(V_{nj}|\mathscr{F}_{n,j-1}))\right\|^p = \frac{\|\sum_{j=1}^{n} V_j\|^p}{n^{p/r}} = 1 \quad \text{almost certainly}, \quad n \geqslant 1.$$

Now the hypotheses of the Theorem are satisfied except for the Banach space being of martingale type p. From above the conclusion of Theorem fails, thereby showing that the martingale type p hypotheses cannot be dispensed with.

3. Additional results

First of all, it is worth mentioning that condition (3) can be difficult to check. Now we present two simple propositions which can be useful in order to check (3).

Let $\{g(m),\ m\geqslant 0\}$ and $\{f(n),\ n\geqslant 1\}$ be two sequences of positive constants, $p>0$ and $k_n\to\infty$ be a sequence of integers. The monotonity of sequence $\{g(m),\ m\geqslant 0\}$ is not necessary now as it was in the Theorem.

Proposition 1. *If* $g(k_n)=\mathrm{O}(f(n))$ *as* $n\to\infty$ *and*

$$\sum_{m=1}^{k_n} g^p(m)/m^2=\mathrm{O}(f^p(n)/k_n)\quad \text{as } n\to\infty \tag{6}$$

then (3) *holds.*

Proof. Denote $a_m=g^p(m),\ m\geqslant 1$ and $b_n=f^p(n),\ n\geqslant 1$. Then the statement of Proposition 1 will be rewritten as follows. If $a_{k_n}=\mathrm{O}(b_n)$ and $\sum_{m=1}^{k_n} a_m/m^2=\mathrm{O}(b_n/k_n)$ then $\sum_{m=1}^{k_n-1}(a_{m+1}-a_m)/m=\mathrm{O}(b_n/k_n)$.

Indeed,

$$\sum_{m=1}^{k_n-1}\frac{a_{m+1}-a_m}{m}=\sum_{m=1}^{k_n-2}\frac{a_{m+1}}{m(m+1)}+\frac{a_{k_n}}{k_n-1}-a_1$$
$$\leqslant 2\sum_{m=1}^{k_n-2}\frac{a_{m+1}}{(m+1)^2}+\frac{k_n}{k_n-1}\frac{a_{k_n}}{k_n}\leqslant 2\sum_{m=2}^{k_n-1}\frac{a_m}{m^2}+2\frac{a_{k_n}}{k_n}\leqslant Cb_n/k_n. \qquad\square$$

Proposition 2. *If* (6) *holds and the sequence* $\{g^p(m)/m,\ m\leqslant 1\}$ *is nonincreasing then* (3) *holds.*

Proof. Using the notation as in the proof of Proposition 1, the statement of Proposition 2 will be rewritten as follows. If the sequence $\{a_m/m,\ m\geqslant 1\}$ is nonincreasing and $\sum_{m=1}^{k_n} a_m/m^2=\mathrm{O}(b_n/k_n)$ then $\sum_{m=1}^{k_n-1}(a_{m+1}-a_m)/m=\mathrm{O}(b_n/k_n)$. Indeed,

$$\sum_{m=1}^{k_n-1}\frac{a_{m+1}-a_m}{m}\leqslant\sum_{m=2}^{k_n-2}\frac{a_m}{m^2}+\frac{a_{k_n}}{k_n-1}-a_1\leqslant Cb_n/k_n$$

since

$$\frac{a_{k_n}}{k_n-1}-a_1\leqslant\frac{a_{k_n}}{k_n-1}-\frac{a_{k_n}}{k_n}\leqslant\frac{a_{k_n-1}}{(k_n-1)^2}. \qquad\square$$

Remark. Propositions 1 and 2 can also be proved using summation by parts.

In Hong and Oh (1995), Hong (1996), Kowalski and Rychlik (1997), Adler et al.(1997) and Sung (1998) the sequence $\{g(m),\ m\geqslant 1\}$ is taken such that $g(k_n)$ is equal to $f(n),\ n\geqslant 1$. In this case condition (6) will be rewritten as

$$\sum_{m=1}^{k_n} g^p(m)/m^2=\mathrm{O}(g^p(k_n)/k_n).$$

In connection with this the following proposition, which is a slight modification of a result reported to us by Professor Thomas Mikosch, may be useful.

A sequence $\{a_n,\ n\geqslant 1\}$ of positive constants is said to be *quasi*-increasing if there exists $C>0$ such that for all positive integer $n\geqslant m$ we have that $a_n\geqslant Ca_m$.

Proposition 3. *If the sequence* $\{g^p(m)/m,\ m\geqslant 1\}$ *is quasi-increasing and* $\{k_n,\ n\geqslant 1\}$ *is a sequence of positive integers such that* $k_n\to\infty$, *then the following conditions are equivalent.*

(a) $\sum_{m=1}^{k_n} g^p(m)/m^2 = \mathrm{O}(g^p(k_n)/k_n)$ *as* $n \to \infty$,
(b) $\sum_{m=k_n}^{\infty} g^{-p}(m) = \mathrm{O}(k_n/g^p(k_n))$ *as* $n \to \infty$,
(c) *There exists a positive integer* $r_0 > 1$ *such that for all* $r > r_0$

$$\liminf_{n\to\infty} g^p(k_r k_n)/g^p(k_n) > k_r,$$

(d) *There exists a positive integer* $r > 1$ *such that* $\liminf_{n\to\infty} g^p(k_r k_n)/g^p(k_n) > k_r$.

Proof. Using the same notation as in the proof of Proposition 1 the statement of Proposition 3 will be rewritten as follows. If the sequence $\{a_m/m,\ m \geqslant 1\}$ is quasi-increasing, then the following conditions are equivalent.
(a) $\sum_{m=1}^{k_n} a_m/m^2 = \mathrm{O}(a_{k_n}/k_n)$,
(b) $\sum_{m=k_n}^{\infty} 1/a_m = \mathrm{O}(k_n/a_{k_n})$,
(c) There exists a positive integer $r_0 > 1$ such that for all $r > r_0$ $\liminf_{n\to\infty} a_{k_r k_n}/a_{k_n} > k_r$.
(d) There exists a positive integer $r > 1$ such that $\liminf_{n\to\infty} a_{k_r k_n}/a_{k_n} > k_r$.

The proposition can be proved if we show the following: (a) ⇒ (c) ⇒ (d) ⇒ (a) and (b) ⇒ (c) ⇒ (d) ⇒ (b). Since both parts can be proved analogously, we only prove the second one.

(**b**) ⇒ (**c**): Let $r > 1$, then

$$\sum_{m=k_n}^{k_r k_n} 1/a_m \leqslant \sum_{m=k_n}^{\infty} 1/a_m \leqslant C k_n/a_{k_n}.$$

Moreover, as the sequence $\{a_m/m,\ m \geqslant 1\}$ is quasi-increasing

$$\frac{k_r k_n}{a_{k_r k_n}} \sum_{m=k_n}^{k_r k_n} \frac{1}{m} \leqslant C \sum_{m=k_n}^{k_r k_n} \frac{m}{a_m}\frac{1}{m} \leqslant C k_n/a_{k_n}.$$

Hence,

$$\frac{a_{k_r k_n}}{a_{k_n}} \geqslant C k_r \sum_{m=k_n}^{k_r k_n} 1/m > C k_r \int_{m=k_n}^{k_r k_n} \frac{\mathrm{d}x}{x-1} > C k_r \ln(k_r - 1) > k_r$$

for all sufficiently large r.

(**c**) ⇒ (**d**) is trivial.

(**d**) ⇒ (**b**) First of all notice that since the sequence $\{a_m/m,\ m \geqslant 1\}$ is quasi-increasing,

$$\sum_{m=L}^{M} \frac{1}{a_m} = \frac{1}{L}\sum_{m=L}^{M} \frac{L}{a_m} \leqslant \frac{1}{L}\sum_{m=L}^{M} \frac{m}{a_m} \leqslant \frac{C}{L}\sum_{m=L}^{M} \frac{L}{a_L} \leqslant \frac{CM}{a_L}.$$

Then, for all integers $p \geqslant 0$, we have that

$$\sum_{m=k_r^p}^{\infty} 1/a_m = \sum_{l=p}^{\infty} \sum_{m=k_r^l}^{k_r^{l+1}} 1/a_m \leqslant C \sum_{l=p}^{\infty} \frac{k_r^{l+1}-1}{a_{k_r^l}} \leqslant C(r) \sum_{l=p}^{\infty} \frac{k_r^l}{a_{k_r^l}}.$$

Let $r = r_0$. By (d) there exists $\varepsilon > 0$ and positive integer n_0 such that for all $n \geqslant n_0$ we have $a_{k_r k_n}/a_{k_n} > k_r + \varepsilon$. If $r^p > n_0$ then for all $l \geqslant p > 1$ $a_{k_r^l} k_r^p > (k_r+\varepsilon)^{p-l}$ or $k_r^l/a_{k_r^l} \leqslant k_r^l (k_r+\varepsilon)^{p-l}/a_{k_r^p}$. Hence

$$\sum_{m=k_r^p}^{\infty} 1/a_m \leqslant C \frac{(k_r+\varepsilon)^p}{k_r^p} \sum_{l=p}^{\infty} \left(\frac{k_r}{k_r+\varepsilon}\right)^l \leqslant C(r,\varepsilon) k_r^p/a_{k_r^p}.$$

Now if $n \geqslant n_0$ we can choose p such that $k_r^{p-1} < k_n \leqslant k_r^p$. Consequently,

$$\sum_{m=k_n}^{\infty} 1/a_m \leqslant \sum_{m=k_n}^{k_r^p} 1/a_m + \sum_{m=k_r^p}^{\infty} 1/a_m \leqslant Ck_r^p/a_{k_n} + Ck_r^p/a_{k_r^p} \leqslant Ck_n/a_{k_n},$$

where $C = C(r, \varepsilon)$. □

Now we obtain a modification of Theorem 2 of Adler et al. (1997) for randomly indexed partial sums.

Corollary. *Let $\{V_{nj},\ j \geqslant 1,\ n \geqslant 1\}$ be an array of random elements in a real separable, martingale type p $(1 \leqslant p \leqslant 2)$ Banach space, and $\{N_n,\ n \geqslant 1\}$ be a sequence of positive integer-valued random variables such that* (1) *holds for some nonrandom sequence of positive integers $k_n \to \infty$.*

Let $\{a_{nj},\ j \geqslant 1,\ n \geqslant 1\}$ be an array of constants such that for some $0 < r < p$

$$\max_{1 \leqslant j \leqslant k_n} |a_{nj}| = \mathrm{O}(k_n^{-1/r}). \tag{7}$$

Suppose the uniform Cesàro-type condition

$$\lim_{a \to \infty} \sup_{n \geqslant 1} \frac{1}{k_n} \sum_{j=1}^{k_n} aP\{\|V_{nj}\|^r > a\} = 0 \tag{8}$$

holds. Then the WLLN (5) *follows.*

Proof. Consider $g(m) = m^{1/r}$ and check the conditions (2), (3) of the Theorem and (6) of Proposition 1. Condition (3) follows from Proposition 1, since

$$g(k_n) = k_n^{1/r} \leqslant C \left(\max_{1 \leqslant j \leqslant k_n} |a_{nj}| \right)^{-1} = Cf(n)$$

if and only if $\max_{1 \leqslant j \leqslant k_n} |a_{nj}| = \mathrm{O}(k_n^{-1/r})$.

Condition (6) is rewritten $\sum_{m=1}^{k_n} m^{p/r-2} = \mathcal{O}(f^p(n)/k_n)$. Note that since $p > r$, $p/r - 2 > -1$ and

$$k_n f^{-p}(n) \sum_{m=1}^{k_n} m^{p/r-2} \leqslant C k_n^{1-p/r} k_n^{p/r-1} = C.$$

Condition (2) holds since

$$\lim_{n \to \infty} k_n f^{-p}(n) = \lim_{n \to \infty} k_n \left(\max_{1 \leqslant j \leqslant k_n} |a_{nj}| \right)^p \leqslant \lim_{n \to \infty} C k_n^{1-p/r} = 0.$$

Condition (4) is rewritten as (8). □

Remark. The prior Example also shows that Corollary can fail if $r = p$. Moreover, the Example (with r replaced by r') shows that Corollary can fail if (8) holds for all $0 < r < p$ but (7) fails for all $0 < r < p$.

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References

Adler, A., Rosalsky, A., Volodin, A.I., 1997. A mean convergence theorem and weak law for arrays of random elements in martingale type p Banach spaces. Statist. Probab. Lett. 32, 167–174.

Beck, A., 1963. On the strong law of large numbers. In: Wright, F.B. (Ed.), Ergodic Theory: Proceedings of an International Symposium held at Tulane University, New Orleans, Louisiana, October 1961. Academic Press, New York, pp. 21–53.

Chow, Y.S., Teicher, H., 1988. Probability Theory: Independence, Interchangeability, Martingales, 2nd Edition. Springer, New York.

Gut, A., 1992. The weak law of large numbers for arrays. Statist. Probab. Lett. 14, 49–52.

Hoffmann-Jørgensen, J., Pisier, G., 1976. The law of large numbers and the central limit theorem in Banach spaces. Ann. Probab. 4, 587–599.

Hong, D.H., 1996. On the weak law of large numbers for randomly indexed partial sums for arrays. Statist. Probab. Lett. 28, 127–130.

Hong, D.H., Oh, K.S., 1995. On the weak law of large numbers for arrays. Statist. Probab. Lett. 22, 55–57.

Kowalski, P., Rychlik, Z., 1997. On the weak law of large numbers for randomly indexed partial sums for arrays. Ann. Univ. Mariae Curie–Sklodovska Sect. A LI.1, 109–119.

Loève, M., 1977. Probability Theory, vol. I, 4th Edition. Springer, New York.

Ordóñez Cabrera, M., 1994. Convergence of weighted sums of random variables and uniform integrability concerning the weights. Collect. Math. 45, 121–132.

Pisier, G., 1975. Martingales with values in uniformly convex spaces. Israel J. Math. 20, 326–350.

Pisier, G., 1986. Probabilistic methods in the geometry of Banach spaces, In: Lette, G., Pratelli, M. (Eds.), Probability and Analysis, Lectures given at the 1st 1985 Session of the Centro Internazionale Matematico Estivo (C.I.M.E.) held at Varenna (Como), Italy, May 31–June 8, 1985, Lecture Notes in Mathematics, vol. 1206. Springer, Berlin, pp. 167–241.

Sung, S.H., 1998. Weak law of large numbers for arrays. Statist. Probab. Lett. 38, 101–105.