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A note on the rates of convergence for weighted sums of ρ^*-mixing random variables

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Abstract. We discuss the rates of convergence for weighted sums of ρ^*-mixing random variables. We solve an open problem posed by Sung [S.H. Sung, On the strong convergence for weighted sums of ρ^*-mixing random variables, *Stat. Pap.*, 54:773–781, 2013]. In addition, the two obtained lemmas in this paper improve the corresponding ones of Sung in the above-mentioned paper and [S.H. Sung, On the strong convergence for weighted sums of random variables, *Stat. Pap.*, 52:447–454, 2011].

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1 Introduction

Let $\{X_k,\, k \in \mathbb{N}\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and define the σ-algebras

$$\mathcal{F}_n^m = \sigma(X_k,\, n \leqslant k \leqslant m), \quad n \leqslant m \leqslant \infty.$$

As usual, for a σ-algebra $\mathcal{F}$, we denote by $\mathcal{L}^2(\mathcal{F})$ the class of all $\mathcal{F}$-measurable random variables with finite second moment.

The following concept of ρ^*-mixing random variables was introduced by Moore [10].

DEFINITION 1. A sequence of random variables $\{X_n,\, n \geqslant 1\}$ is called ρ^*-mixing if for some integer $k \geqslant 1$, the mixing coefficient

$$\rho^*(k) = \sup\sup\bigl\{\operatorname{Corr}(X, Y)\colon\ X \in \mathcal{L}^2(\mathcal{F}_S),\ Y \in \mathcal{L}^2(\mathcal{F}_T)\bigr\} < 1,$$

where $\mathcal{F}_S = \sigma\{X_i,\, i \in S\}$, and the outside supremum is taken over all pairs of nonempty finite sets S, T of integers such that $\min\{|s-t|,\, s \in S,\, t \in T\} \geqslant k$.

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An array of random variables $\{X_{nk},\, k \in \mathbb{N},\, n \in \mathbb{N}\}$ is said to be rowwise ρ^*-mixing if, for every $n \in \mathbb{N}$, $\{X_{nk},\, k \in \mathbb{N}\}$ is a ρ^*-mixing sequence of random variables.

Bradley [2] was the first who introduced the concept of ρ^*-mixing random variables to limit theorems. Since the article of Bradley [2], many authors studied the convergence properties for sequences or arrays of ρ^*-mixing random variables. We refer the reader to [1, 2, 3, 4, 5, 6, 9, 11, 14, 15, 17, 18, 19, 20, 21].

A sequence of random variables $\{U_n,\, n \in N\}$ is said to converge completely to a constant a if, for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbf{P}\bigl(|U_n - a| > \varepsilon\bigr) < \infty.$$

This notion was given firstly by Hsu and Robbins [7].

In view of the Borel–Cantelli lemma, the above result implies that $U_n \to a$ almost surely. Therefore, the complete convergence is a very important tool in establishing the almost sure convergence of sums and weighted sums of random variables.

Sung [13] obtained the following complete convergence result for weighted sums of negatively associated (NA) (see [8]) random variables.

Theorem A. *Let $\{X, X_n,\, n \geqslant 1\}$ be a sequence of identically distributed NA random variables, and let $\{a_{ni},\, 1 \leqslant i \leqslant n,\, n \geqslant 1\}$ be an array of constants satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n) \tag{1.1}$$

for some $0 < \alpha \leqslant 2$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Furthermore, suppose that $\mathbf{E}X = 0$ when $1 < \alpha \leqslant 2$. If

$$\begin{aligned} &\mathbf{E}|X|^{\alpha} < \infty && \text{for } \alpha > \gamma,\\ &\mathbf{E}|X|^{\alpha}\log|X| < \infty && \text{for } \alpha = \gamma,\\ &\mathbf{E}|X|^{\gamma} < \infty && \text{for } \alpha < \gamma, \end{aligned} \tag{1.2}$$

then

$$\sum_{n=1}^{\infty} n^{-1}\mathbf{P}\left(\max_{1\leqslant m\leqslant n}\left|\sum_{i=1}^{m} a_{ni}X_i\right| > b_n\varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{1.3}$$

Zhou et al. [20] partially extended Theorem A to ρ^*-mixing random variables as follows.

Theorem B. *Let $\{X_n,\, n \geqslant 1\}$ be a sequence of identically distributed ρ^*-mixing random variables, and let $\{a_{ni},\, 1 \leqslant i \leqslant n,\, n \geqslant 1\}$ be an array of constants satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^{\max\{\alpha,\gamma\}} = O(n) \tag{1.4}$$

for some $0 < \alpha \leqslant 2$ and $\gamma > 0$ with $\alpha \neq \gamma$. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$. If $\mathbf{E}X_1 = 0$ for $1 < \alpha \leqslant 2$ and (1.2) *holds for $\alpha \neq \gamma$, then* (1.3) *holds.*

As Sung [14] pointed out, Theorem B extends only the case $\alpha > \gamma$ of Theorem A to ρ^*-mixing random variables. Zhou et al. [20] left an open problem whether the case $\alpha = \gamma$ of Theorem A holds for ρ^*-mixing random variables. Sung [14] solved this problem and obtained the following result.

Theorem C. *Let $\{X_n,\ n \geqslant 1\}$ be a sequence of identically distributed ρ^*-mixing random variables, and let $\{a_{ni},\ 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ be an array of constants satisfying* (1.1) *for some $0 < \alpha \leqslant 2$. Let $b_n = n^{1/\alpha}(\log n)^{1/\alpha}$. If $\mathbf{E}X_1 = 0$ for $1 < \alpha \leqslant 2$ and $\mathbf{E}|X_1|^\alpha \log(1+|X_1|) < \infty$, then* (1.3) *holds.*

Remark 1. Sung [14, Remark 2.2] presented an open problem whether the case $\alpha < \gamma$ of Theorem A remains true for ρ^*-mixing random variables?

In this work, we shall study the complete convergence result for weighted sums of ρ^*-mixing random variables and solve the above problem.

Throughout this paper, the symbol C always stands for a generic positive constant, which may differ from one place to another, and $\log x = \log_{\mathrm{e}} x$. For a finite set A, the symbol $\sharp(A)$ denotes the number of elements of A.

2 Preliminaries

We first state some lemmas, which will be used in the proofs of our main results.

Lemma 1. (*See* [15].) *Suppose that N is a positive integer, $0 \leqslant r < 1$, and $q \geqslant 2$. Then there exists a positive constant $C = C(N, r, q)$ such that the following statement holds*:

If $\{X_k,\ k \geqslant 1\}$ is a sequence of random variables such that $\rho_N^* \leqslant r$ and such that $\mathbf{E}X_k = 0$ and $\mathbf{E}|X_k|^q < \infty$ for every $k \geqslant 1$, then, for all $n \geqslant 1$,

$$\mathbf{E}\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} X_k\right|^q \leqslant C\left\{\sum_{k=1}^{n}\mathbf{E}|X_k|^q + \left(\sum_{k=1}^{n}\mathbf{E}X_k^2\right)^{q/2}\right\}.$$

Second, we present the following lemma, which improves Lemma 2.2 and Lemma 2.3 of [13].

Lemma 2. *Let $\{a_{ni},\ 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ be an array of constants satisfying* (1.1) *for some $\alpha > 0$, and X be a random variable. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. Then*

$$\sum_{n=2}^{\infty} n^{-1} b_n^{-\alpha}\sum_{i=1}^{n}\mathbf{E}|a_{ni}X|^\alpha I\big(|a_{ni}X| > b_n\big) \leqslant \begin{cases} C\mathbf{E}|X|^\alpha & \text{for } \alpha > \gamma,\\ C\mathbf{E}|X|^\alpha \log(1+|X|) & \text{for } \alpha = \gamma,\\ C\mathbf{E}|X|^\gamma & \text{for } \alpha < \gamma. \end{cases}$$

Proof. Let

$$L = \sum_{n=2}^{\infty} n^{-1} b_n^{-\alpha}\sum_{i=1}^{n}\mathbf{E}|a_{ni}X|^\alpha I\big(|a_{ni}X| > b_n\big).$$

Without loss of generality, we may assume that $\sum_{i=1}^n |a_{ni}|^\alpha \leqslant n$. Then we have

$$\begin{aligned} L &= \sum_{n=2}^{\infty} n^{-2}(\log n)^{-\alpha/\gamma}\sum_{i=1}^{n}\mathbf{E}|a_{ni}X|^\alpha I\big(|a_{ni}|^\alpha|X|^\alpha > n(\log n)^{\alpha/\gamma}\big)\\ &\leqslant \sum_{n=2}^{\infty} n^{-2}(\log n)^{-\alpha/\gamma}\sum_{i=1}^{n}\mathbf{E}|a_{ni}X|^\alpha I\left(\left(\sum_{i=1}^{n}|a_{ni}|^\alpha\right)|X|^\alpha > n(\log n)^{\alpha/\gamma}\right)\\ &= \sum_{n=2}^{\infty} n^{-2}(\log n)^{-\alpha/\gamma}\sum_{i=1}^{n}\mathbf{E}|a_{ni}X|^\alpha I\left(|X|^\alpha > n(\log n)^{\alpha/\gamma}\left(\sum_{i=1}^{n}|a_{ni}|^\alpha\right)^{-1}\right) \end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{n=2}^{\infty} n^{-2}(\log n)^{-\alpha/\gamma} \sum_{i=1}^{n} \mathbf{E}|a_{ni}X|^{\alpha} I\bigl(|X| > (\log n)^{1/\gamma}\bigr) \\
&\leqslant \sum_{n=2}^{\infty} n^{-1}(\log n)^{-\alpha/\gamma} \mathbf{E}|X|^{\alpha} I\bigl(|X| > (\log n)^{1/\gamma}\bigr) \\
&= \sum_{n=2}^{\infty} n^{-1}(\log n)^{-\alpha/\gamma} \sum_{m=n}^{\infty} \mathbf{E}|X|^{\alpha} I\bigl(\log m < |X|^{\gamma} \leqslant \log(m+1)\bigr) \\
&= \sum_{m=2}^{\infty} \mathbf{E}|X|^{\alpha} I\bigl(\log m < |X|^{\gamma} \leqslant \log(m+1)\bigr) \sum_{n=2}^{m} n^{-1}(\log n)^{-\alpha/\gamma}.
\end{aligned}
$$

Observing that

$$
\sum_{n=2}^{m} n^{-1}(\log n)^{-\alpha/\gamma} \leqslant \begin{cases} C & \text{for } \alpha > \gamma, \\ C \log\log m & \text{for } \alpha = \gamma, \\ C(\log m)^{1-\alpha/\gamma} & \text{for } \alpha < \gamma, \end{cases}
$$

we can get

$$
L \leqslant \begin{cases} C\mathbf{E}|X|^{\alpha} & \text{for } \alpha > \gamma, \\ C\mathbf{E}|X|^{\alpha} \log|X| & \text{for } \alpha = \gamma, \\ C\mathbf{E}|X|^{\gamma} & \text{for } \alpha < \gamma. \end{cases}
$$

The proof of Lemma 2 is completed. □

Remark 2. Noting that

$$
\sum_{n=2}^{\infty} n^{-1} \sum_{i=1}^{n} \mathbf{P}\bigl(|a_{ni}X| > b_n\bigr) \leqslant \sum_{n=2}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} \mathbf{E}|a_{ni}X|^{\alpha} I\bigl(|a_{ni}X| > b_n\bigr).
$$

Therefore, we know that this lemma improves Lemma 2.2 and Lemma 2.3 of Sung [13]. In addition, the method used in this paper is novel and much simpler than that in [13].

The following lemma plays an essential role in the proof of our main results, which complements Lemma 2.3 of Sung [14].

Lemma 3. *Let $\{a_{ni},\ i \geqslant 1,\ n \geqslant 1\}$ be an array of constants satisfying* (1.1) *for some $\alpha > 0$, and X be a random variable. Let $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$ for some $\gamma > 0$. If $q > \max\{\alpha, \gamma\}$, then*

$$
\sum_{n=2}^{\infty} n^{-1} b_n^{-q} \sum_{i=1}^{n} \mathbf{E}|a_{ni}X|^{q} I\bigl(|a_{ni}X| \leqslant b_n\bigr) \leqslant \begin{cases} C\mathbf{E}|X|^{\alpha} & \text{for } \alpha > \gamma, \\ C\mathbf{E}|X|^{\alpha} \log(1+|X|) & \text{for } \alpha = \gamma, \\ C\mathbf{E}|X|^{\gamma} & \text{for } \alpha < \gamma. \end{cases} \tag{2.1}
$$

Proof. Let

$$
I = \sum_{n=2}^{\infty} n^{-1} b_n^{-q} \sum_{i=1}^{n} \mathbf{E}|a_{ni}X|^{q} I\bigl(|a_{ni}X| \leqslant b_n\bigr).
$$

For $j \geqslant 1$ and $n \geqslant 2$, let

$$I_{nj} = \{1 \leqslant i \leqslant n\colon\ n^{1/\alpha}(j+1)^{-1/\alpha} < |a_{ni}| \leqslant n^{1/\alpha} j^{-1/\alpha}\}.$$

Then $\{I_{nj},\ j \geqslant 1\}$ are disjoint, and $\bigcup_{j\geqslant 1} I_{nj} = \{1 \leqslant i \leqslant n\colon\ 0 < |a_{ni}| \leqslant n^{1/\alpha}\}$ since $\sum_{i=1}^{n} |a_{ni}|^\alpha \leqslant n$. Note that, for all $k \geqslant 1$, we have

$$\begin{aligned} n &\geqslant \sum_{i=1}^{n} |a_{ni}|^\alpha = \sum_{j=1}^{\infty} \sum_{i\in I_{nj}} |a_{ni}|^\alpha \geqslant \sum_{j=1}^{\infty} \sharp(I_{nj})\, n\, (j+1)^{-1} \\ &\geqslant \sum_{j=k}^{\infty} \sharp(I_{nj})\, n\, (j+1)^{-1} = \sum_{j=k}^{\infty} \sharp(I_{nj})\, n\, (j+1)^{-q/\alpha} (j+1)^{q/\alpha - 1} \\ &\geqslant \sum_{j=k}^{\infty} \sharp(I_{nj})\, n\, (j+1)^{-q/\alpha} (k+1)^{q/\alpha - 1}. \end{aligned}$$

Hence, for all $k \geqslant 1$, we have

$$\sum_{j=k}^{\infty} \sharp(I_{nj}) j^{-q/\alpha} \leqslant C(k+1)^{1-q/\alpha}. \tag{2.2}$$

Then

$$\begin{aligned} I &= \sum_{n=2}^{\infty} n^{-1-q/\alpha} (\log n)^{-q/\gamma} \sum_{i=1}^{n} |a_{ni}|^q \mathbf{E}|X|^q I\big(|a_{ni}X| \leqslant n^{1/\alpha} (\log n)^{1/\gamma}\big) \\ &= \sum_{n=2}^{\infty} n^{-1-q/\alpha} (\log n)^{-q/\gamma} \sum_{j=1}^{\infty} \sum_{i\in I_{nj}} |a_{ni}|^q \mathbf{E}|X|^q I\big(|a_{ni}X| \leqslant n^{1/\alpha} (\log n)^{1/\gamma}\big) \\ &\leqslant \sum_{n=2}^{\infty} n^{-1-q/\alpha} (\log n)^{-q/\gamma} \sum_{j=1}^{\infty} \sharp(I_{nj}) n^{q/\alpha} j^{-q/\alpha} \mathbf{E}|X|^q I\big(|X| \leqslant (j+1)^{1/\alpha} (\log n)^{1/\gamma}\big) \\ &\leqslant \sum_{n=2}^{\infty} n^{-1} (\log n)^{-q/\gamma} \sum_{j=1}^{\infty} \sharp(I_{nj}) j^{-q/\alpha} \mathbf{E}|X|^q I\big(|X| \leqslant (\log n)^{1/\gamma}\big) \\ &\quad + \sum_{n=2}^{\infty} n^{-1} (\log n)^{-q/\gamma} \sum_{j=1}^{\infty} \sharp(I_{nj}) j^{-q/\alpha} \\ &\quad \times \sum_{k=1}^{j} \mathbf{E}|X|^q I\big(k^{1/\alpha} (\log n)^{1/\gamma} < |X| \leqslant (k+1)^{1/\alpha} (\log n)^{1/\gamma}\big) \\ &=: I^* + I^{**}. \end{aligned}$$

If $\alpha > \gamma$, by (2.2) and $q > \alpha$ we have

$$I^* \leqslant C \sum_{n=2}^{\infty} n^{-1} (\log n)^{-q/\gamma} \mathbf{E}|X|^q I\big(|X|^\gamma \leqslant \log n\big)$$

$$\leqslant C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-\alpha/\gamma}\mathbf{E}|X|^\alpha I\big(|X|^\gamma \leqslant \log n\big)$$
$$\leqslant C\mathbf{E}|X|^\alpha \sum_{n=2}^{\infty} n^{-1}(\log n)^{-\alpha/\gamma} \leqslant C\mathbf{E}|X|^\alpha.$$

If $\alpha \leqslant \gamma$, by (2.2) and $q > \gamma$ we have

$$\begin{aligned}
I^* &\leqslant C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma}\mathbf{E}|X|^q I\big(|X|^\gamma \leqslant \log n\big)\\
&= C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma}\sum_{m=2}^{n}\mathbf{E}|X|^q I\big(\log(m-1) < |X|^\gamma \leqslant \log m\big)\\
&= C\sum_{m=2}^{\infty}\mathbf{E}|X|^q I\big(\log(m-1) < |X|^\gamma \leqslant \log m\big)\sum_{n=m}^{\infty} n^{-1}(\log n)^{-q/\gamma}\\
&\leqslant C\sum_{m=2}^{\infty}(\log m)^{1-q/\gamma}\mathbf{E}|X|^q I\big(\log(m-1) < |X|^\gamma \leqslant \log m\big) \leqslant C\mathbf{E}|X|^\gamma.
\end{aligned}$$

Next, we consider I^{**}. By (2.2) we have

$$\begin{aligned}
I^{**} &= \sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma}\sum_{k=1}^{\infty}\mathbf{E}|X|^q I\big(k^{1/\alpha}(\log n)^{1/\gamma} < |X| \leqslant (k+1)^{1/\alpha}(\log n)^{1/\gamma}\big)\sum_{j=k}^{\infty}\sharp(I_{nj})\, j^{-q/\alpha}\\
&\leqslant C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma}\sum_{k=1}^{\infty}(k+1)^{1-q/\alpha}\mathbf{E}|X|^q I\big(k^{1/\alpha}(\log n)^{1/\gamma} < |X| \leqslant (k+1)^{1/\alpha}(\log n)^{1/\gamma}\big)\\
&\leqslant C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-\alpha/\gamma}\sum_{k=1}^{\infty}\mathbf{E}|X|^\alpha I\big(k^{1/\alpha}(\log n)^{1/\gamma} < |X| \leqslant (k+1)^{1/\alpha}(\log n)^{1/\gamma}\big)\\
&= C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-\alpha/\gamma}\mathbf{E}|X|^\alpha I\big(|X| > (\log n)^{1/\gamma}\big) \quad \text{(similarly to the proof of Lemma 2)}\\
&\leqslant \begin{cases} C\mathbf{E}|X|^\alpha & \text{for } \alpha > \gamma,\\ C\mathbf{E}|X|^\alpha \log(1+|X|) & \text{for } \alpha = \gamma,\\ C\mathbf{E}|X|^\gamma & \text{for } \alpha < \gamma. \end{cases}
\end{aligned}$$

Therefore,

$$I \leqslant I^* + I^{**} \leqslant \begin{cases} C\mathbf{E}|X|^\alpha & \text{for } \alpha > \gamma,\\ C\mathbf{E}|X|^\alpha \log(1+|X|) & \text{for } \alpha = \gamma,\\ C\mathbf{E}|X|^\gamma & \text{for } \alpha < \gamma. \end{cases}$$

The proof is completed. □

Remark 3. Clearly, Lemma 2.3 of Sung [14] is a special case of Lemma 3 in this paper ($\alpha = \gamma$). It is worth pointing out that, in [14], it is required that $a_{ni} = 0$ or $|a_{ni}| > 1$. Here, we do not require these extra conditions.

3 Main result

Now we state the following result, which solves the open problem described in Remark 1.

Theorem 1. *Let* $\{X_n,\ n \geqslant 1\}$ *be a sequence of identically distributed* ρ^**-mixing random variables, and let* $\{a_{ni},\ 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ *be an array of constants satisfying* (1.1) *for some* $0 < \alpha \leqslant 2$. *Let* $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$. *If* $\mathbf{E}X_1 = 0$ *for* $1 < \alpha \leqslant 2$ *and* (1.2) *holds for* $\alpha < \gamma$, *then* (1.3) *holds.*

Proof. Without loss of generality, we may assume that $\sum_{i=1}^n |a_{ni}|^\alpha \leqslant n$. For fixed $n \geqslant 1$, let

$$Y_{ni} = a_{ni}X_i I\big(|a_{ni}X_i| \leqslant b_n\big), \quad Z_{ni} = a_{ni}X_i - Y_{ni}.$$

Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1}\mathbf{P}\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > b_n\varepsilon\right) \\
&\quad \leqslant 1 + \sum_{n=2}^{\infty} n^{-1}\sum_{i=1}^{n}\mathbf{P}\big(|a_{ni}X_i| > b_n\big) + \sum_{n=2}^{\infty} n^{-1}\mathbf{P}\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} Y_{ni}\right| > b_n\varepsilon\right) \\
&\quad =: 1 + J_1 + J_2.
\end{aligned}$$

To prove (1.3), we need only to show that $J_1 < \infty$ and $J_2 < \infty$. By Lemma 2 we get

$$J_1 \leqslant C\sum_{n=2}^{\infty} n^{-1} b_n^{-\alpha}\sum_{i=1}^{n} |a_{ni}|^\alpha \mathbf{E}|X_1|^\alpha I\big(|a_{ni}X_1| > b_n\big) \leqslant C\mathbf{E}|X_1|^\gamma < \infty.$$

Next, we prove $J_2 < \infty$. We first show that

$$b_n^{-1}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}\mathbf{E}Y_{ni}\right| \to 0 \quad \text{as } n \to \infty. \tag{3.1}$$

Clearly, $\mathbf{E}|X|^\gamma < \infty$ and $\alpha < \gamma$ imply $\mathbf{E}|X|^\alpha < \infty$. If $0 < \alpha \leqslant 1$, by $\sum_{i=1}^n |a_{ni}|^\alpha \leqslant n$ we have

$$\begin{aligned}
b_n^{-1}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}\mathbf{E}Y_{ni}\right| &\leqslant b_n^{-1}\sum_{i=1}^{n}|a_{ni}|\mathbf{E}|X_i|I\big(|a_{ni}X_i| \leqslant b_n\big) \leqslant b_n^{-\alpha}\sum_{i=1}^{n}|a_{ni}|^\alpha\mathbf{E}|X_i|^\alpha I\big(|a_{ni}X_i| \leqslant b_n\big) \\
&\leqslant (\log n)^{-\alpha/\gamma}\mathbf{E}|X_1|^\alpha \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

If $1 < \alpha \leqslant 2$, by $\mathbf{E}X_1 = 0$ and $\sum_{i=1}^n |a_{ni}|^\alpha \leqslant n$ we also have

$$\begin{aligned}
b_n^{-1}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}\mathbf{E}Y_{ni}\right| &\leqslant b_n^{-1}\sum_{i=1}^{n}|a_{ni}|\mathbf{E}|X_i|I\big(|a_{ni}X_i| > b_n\big) \leqslant b_n^{-\alpha}\sum_{i=1}^{n}|a_{ni}|^\alpha\mathbf{E}|X_i|^\alpha I\big(|a_{ni}X_i| > b_n\big) \\
&\leqslant (\log n)^{-\alpha/\gamma}\mathbf{E}|X_1|^\alpha \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

To sum up, we prove that (3.1) holds for $0 < \alpha \leqslant 2$. Hence, when n is sufficiently large, we have

$$\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}\mathbf{E}Y_{ni}\right| < \frac{b_n\varepsilon}{2}.$$

Let $q > \max\{2, 2\gamma/\alpha\}$. By the Markov inequality and Lemma 1 we have

$$J_2 \leqslant \sum_{n=2}^{\infty} n^{-1}\mathbf{P}\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}(Y_{ni}-\mathbf{E}Y_{ni})\right| > \frac{b_n\varepsilon}{2}\right) \leqslant C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\mathbf{E}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}(Y_{ni}-\mathbf{E}Y_{ni})\right|^q$$
$$\leqslant C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\sum_{i=1}^{n}\mathbf{E}|Y_{ni}|^q + C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\left(\sum_{i=1}^{n}\mathbf{E}Y_{ni}^2\right)^{q/2} =: J_3 + J_4.$$

By Lemma 3 and $\alpha < \gamma$ we get

$$J_3 \leqslant C\sum_{n=2}^{\infty} n^{-1}b_n^{-q}\sum_{i=1}^{n}|a_{ni}|^q\mathbf{E}|X_i|^q I\big(|a_{ni}X_i| \leqslant b_n\big) \leqslant C\mathbf{E}|X_1|^\gamma \leqslant \infty.$$

Finally, we prove $J_4 < \infty$. By $\alpha \leqslant 2$, $\sum_{i=1}^{n}|a_{ni}|^\alpha \leqslant n$, and $q > 2\gamma/\alpha$ we have

$$J_4 \leqslant C\sum_{n=2}^{\infty} n^{-1}\left(b_n^{-\alpha}\sum_{i=1}^{n}|a_{ni}|^\alpha\mathbf{E}|X_i|^\alpha I\big(|a_{ni}X_i| \leqslant b_n\big)\right)^{q/2}$$
$$\leqslant C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-\alpha q/(2\gamma)}\big(\mathbf{E}|X_1|^\alpha\big)^{q/2} < \infty.$$

The proof is completed. □

Remark 4. As Sung [14] pointed out, the crucial tool of the proof of Theorem 1 is the Rosenthal-type inequality for maximum partial sums of ρ^*-mixing random variables. For m-dependent random variables and φ-mixing random variables satisfying $\sum_{n=1}^{\infty}\varphi^{1/2}(n) < \infty$, the Rosenthal-type inequality for maximum partial sums also holds (see [12] and [16], respectively). Therefore, Theorem 1 also holds for m-dependent random variables and φ-mixing random variables satisfying $\sum_{n=1}^{\infty}\varphi^{1/2}(n) < \infty$.

References

1. J. An and D.M. Yuan, Complete convergence of weighted sums for $\tilde{\rho}$-mixing sequence of random variables, *Stat. Probab. Lett.*, **78**:1466–1472, 2008.

2. R.C. Bradley, Equivalent mixing conditions for random fields, Technical Report No. 336, Center for Stochastic Processes, Department of Statistics, University of North Carolina, Chapel Hill, 1990.

3. G.H. Cai, Marcinkiewicz strong laws for linear statistics of $\tilde{\rho}$-mixing sequences of random variables, *An. Acad. Bras. Ciênc.*, **78**:615–621, 2006.

4. G.H. Cai, Moment inequality and complete convergence of $\tilde{\rho}$-mixing sequences, *J. Syst. Sci. Math. Sci.*, **28**:251–256, 2008 (in Chinese).

5. S.X. Gan, Almost sure convergence for $\tilde{\rho}$-mixing random variable sequences, *Stat. Probab. Lett.*, **67**:289–298, 2004.

6. M.L. Guo and D.J. Zhu, Equivalent conditions of complete moment convergence of weighted sums for ρ^*-mixing sequence of random variables, *Stat. Probab. Lett.*, **83**:13–20, 2013.

7. P.L. Hsu and H. Robbins, Complete convergence and the law of large numbers, *Proc. Natl. Acad. Sci. USA*, **33**:25–31, 1947.

8. K. Joag-Dev and F. Proschan, Negative association of random variables with applications, *Ann. Stat.*, **11**:286–295, 1983.

9. A. Kuczmaszewska, On complete convergence for arrays of rowwise dependent random variables, *Stat. Probab. Lett.*, **77**:1050–1060, 2007.

10. C.C. Moore, The degree of randomness in a stationary time series, *Ann. Math. Stat.*, **34**:1253–258, 1963.

11. M. Peligrad and A. Gut, Almost-sure results for a class of dependent random variables, *J. Theor. Probab.*, **12**:87–104, 1999.

12. Q.M. Shao, A moment inequality and its application, *Acta Math. Sin., Chin. Ser.*, **31**:736–747, 1988 (in Chinese).

13. S.H. Sung, On the strong convergence for weighted sums of random variables, *Stat. Pap.*, **52**:447–454, 2011.

14. S.H. Sung, On the strong convergence for weighted sums of ρ^*-mixing random variables, *Stat. Pap.*, **54**:773–781, 2013.

15. S. Utev and M. Peligrad, Maximal inequalities and an invariance principle for a class of weakly dependent random variables, *J. Theor. Probab.*, **16**:101–115, 2003.

16. X.J. Wang, S.H. Hu, W.Z. Yang, and Y. Shen, On complete convergence for weighted sums of φ-mixing random variables, *J. Inequal. Appl.*, 2010, doi:10.1155/2010/372390.

17. X.J. Wang, X.Q. Li, W.Z. Yang, and S.H. Hu, On complete convergence for arrays of rowwise weakly dependent random variables, *Appl. Math. Lett.*, **25**:1916–1920, 2012.

18. Q.Y. Wu and Y.Y. Jiang, Some strong limit theorems for $\tilde{\rho}$-mixing sequences of random variables, *Stat. Probab. Lett.*, **78**:1017–1023, 2008.

19. Y.F. Wu, C.H. Wang, and A. Volodin, Limiting behavior for arrays of rowwise ρ^*-mixing random variables, *Lith. Math. J.*, **52**:214–221, 2012.

20. X.C. Zhou, C.C. Tan, and J.G. Lin, On the strong laws for weighted sums of ρ^*-mixing random variables, *J. Inequal. Appl.*, 2011, doi:10.1155/2011/157816.

21. M.H. Zhu, Strong laws of large numbers for arrays of rowwise ρ^*-mixing random variables, *Discrete Dyn. Nat. Soc.*, **2007**, Article ID 74296, 6pp., 2007.